# The MHD $\alpha^2$-dynamo, $\mathbb{Z}_2$-graded pseudo-Hermiticity, level crossings and exceptional points of branching type[*]

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The spectral branching behavior of the $2 \times 2$ operator matrix of the magneto-hydro-dynamic $\alpha^2$-dynamo is analyzed numerically. Some qualitative aspects of level crossings are briefly discussed with the help of a simple toy model which is based on a $\mathbb{Z}_2$-graded-pseudo-Hermitian $2 \times 2$ matrix. The considered issues comprise: the underlying $SU(1,1)$ symmetry and the Krein space structure of the system, exceptional points of branching type and diabolic points, as well as the algebraic and geometric multiplicity of corresponding degenerate eigenvalues.

## 1 The $2 \times 2$-operator matrix of the MHD $\alpha^2$-dynamo

The magnetic fields of planets, stars and galaxies are maintained by homogeneous dynamo effects, which can be successfully described within magnetohydrodynamics (MHD) [1]. This is achieved by appropriately combining the Maxwell equations of electrodynamics with the Navier–Stokes equations of hydrodynamics — for certain topologically non-trivial, helical flow fields. The resulting highly complicated equation systems are subject of large and cost-intensive computer simulations (see, e.g., Ref. [2]). Recently, the homogeneous dynamo effect has been demonstrated in large scale liquid sodium experiments in Riga/Latvia [3] and Karlsruhe/Germany [4]. Next generation experiments are currently planned at 7 sites around the world.

One of the simplest dynamo toy models, which can be regarded as important for MHD dynamo theory as the harmonic oscillator for quantum mechanics, is the spherically symmetric $\alpha^2$-dynamo [1] in its kinematic regime. Its operator matrix has the form [5]

$$\hat{H}_l[\alpha] = \begin{pmatrix} -Q[1] & \alpha \\ Q[\alpha] & -Q[1] \end{pmatrix} \tag{1}$$

and consists of formally selfadjoint blocks

$$Q[\alpha] := p\alpha p + \alpha \frac{l(l+1)}{r^2} . \tag{2}$$

($p = -\mathrm{i}(\partial_r + 1/r)$ is the radial momentum operator). It describes the coupled $l$-modes of the poloidal and toroidal magnetic field components in a mean-field

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dynamo model with helical turbulence function ($\alpha$-profile) $\alpha(r)$. The differential expression (1) has the symmetry property [5]

$$\hat{H}_l[\alpha] = J\hat{H}_l^\dagger[\alpha]J, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3}$$

and shows that — depending on its domain $\mathcal{D}(\hat{H}_l[\alpha])$ and the chosen boundary conditions for the two-component eigenfunctions $\psi$ — the corresponding operator can be pseudo-Hermitian ($J$-selfadjoint). Modulo $J$-selfadjoint extensions, this will hold for functions $\psi$ with idealized boundary conditions at $r = 1$:

$$\mathcal{D}(\hat{H}_l[\alpha]) := \left\{ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \ \psi \in \tilde{\mathcal{H}} \equiv \mathcal{H} \oplus \mathcal{H}, \ \mathcal{H} = L_2(\Omega, r^2 \mathrm{d}r), \right.$$
$$\left. \Omega = [0,1], \ \psi(1) = 0, \ r\psi(r)|_{r \to 0} \to 0 \right\}, \tag{4}$$

whereas the physically realistic boundary conditions

$$\hat{B}_l\psi\Big|_{r=1} = 0, \qquad \hat{B}_l = \mathrm{diag}[\partial_r + (l+1)/r, 1] \tag{5}$$

lead to a non-$J$-Hermitian operator.

The dynamo effect starts when the first eigenvalue $\lambda$ of $\hat{H}_l[\alpha]$ enters the right half-plane $\Re(\lambda) > 0$. This leads to an exponential growth $\sim \mathrm{e}^{\lambda t}$ of the solutions of the time dependent kinematic dynamo problem — and a corresponding growth of the magnetic field components. In Ref. [6] it was demonstrated numerically that this can happen even on a complex-valued branch of the spectrum, so that the dynamo can start in an oscillating regime.

In Figs. 1–5 some typical spectral branches are shown for an $\alpha^2$-dynamo operator with physically realistic boundary conditions (5). The real and imaginary components of the eigenvalues for modes with angular mode number $l = 1$ and radial mode numbers $n = 1, \ldots, 9$ are depicted over a scaling parameter $C$ of an $\alpha(r)$-profile, which is chosen as quartic polynomial $\alpha(r) = C \times [a_0 + a_2 r^2 + a_3 r^3 + a_4 r^4]$. Of special physical interest are the critical value $C_\mathrm{c}$, where the spectrum enters the right half-plane $\Re(\lambda) > 0$, as well as the location of those level crossing points, where two real-valued branches (non-oscillating regime) of the spectrum meet and continue to evolve as two complex conjugate branches (oscillating dynamo regime). Within the considered range of $C$ such transitions occur locally only pairwise — there are always only two spectral branches which are locally involved in such transitions. As shown in Fig. 3, globally, there are more branches involved in mutual transitions. A detailed and rigorous non-numerical study is still missing. Below, we collect some few building blocks for such an analysis.

Some first and rough qualitative aspects of the behavior of the dynamo system at a level crossing point can be easily understood, e.g., by passing from the eigenvalue problem for the linear pencil

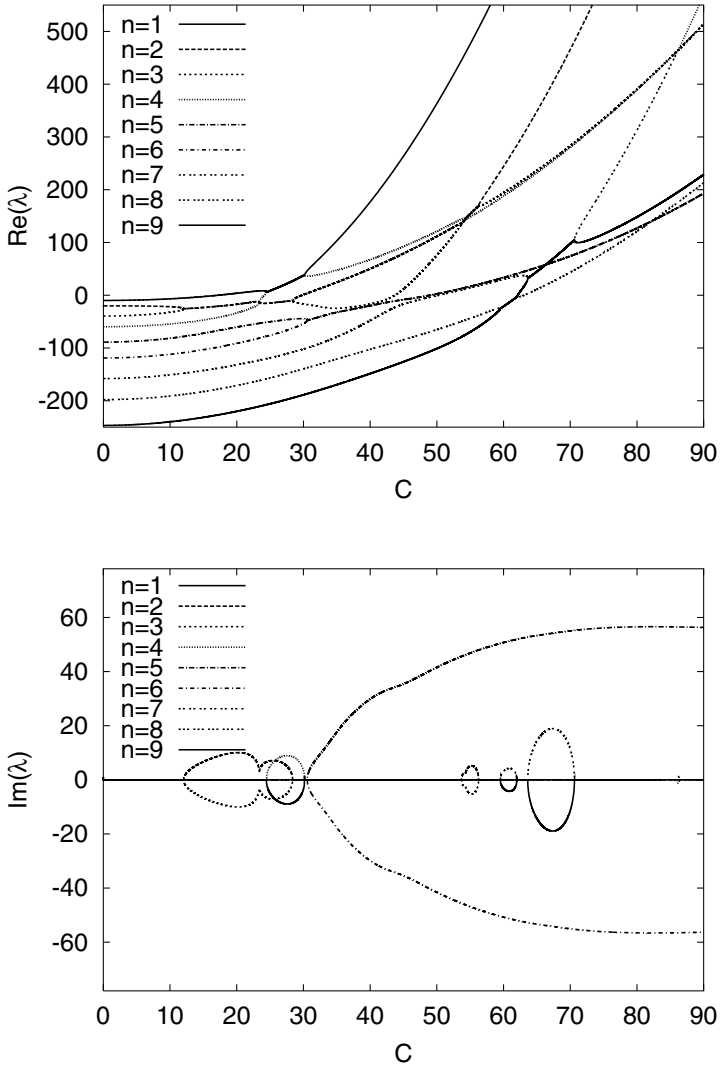$$\hat{L}_l[\alpha, \lambda]\psi := \left( \hat{H}_l[\alpha] - \lambda \right) \psi = 0 \tag{6}$$

Fig. 1. Large-scale behavior of real and imaginary parts $\Re(\lambda)$, $\Im(\lambda)$ of the $\alpha^2$-dynamo spectrum (for physically realistic boundary conditions (5)) as function of the scaling parameter $C$ in the concrete $\alpha$-profile $\alpha(r) = C \times \left(1 - 26.09 \times r^2 + 53.64 \times r^3 - 28.22 \times r^4\right)$.
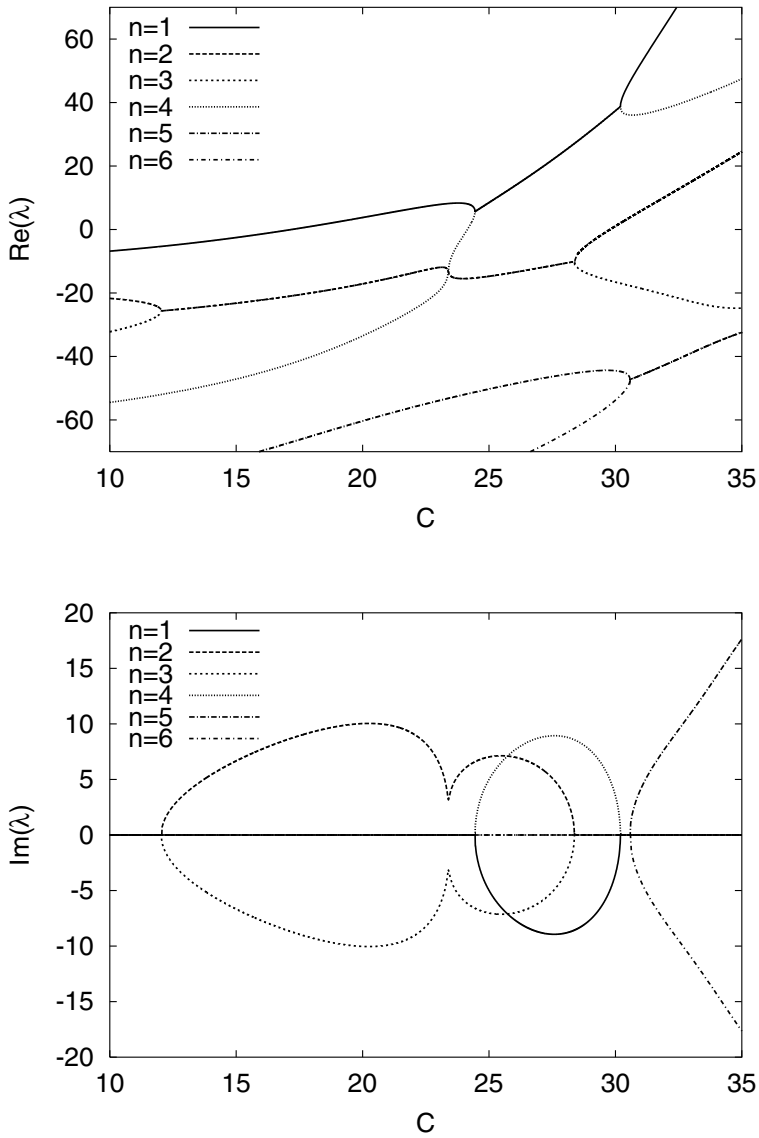
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Fig. 2. Zooming into a scaling region with multiple level-crossings and branches which "mutually influence" each other.
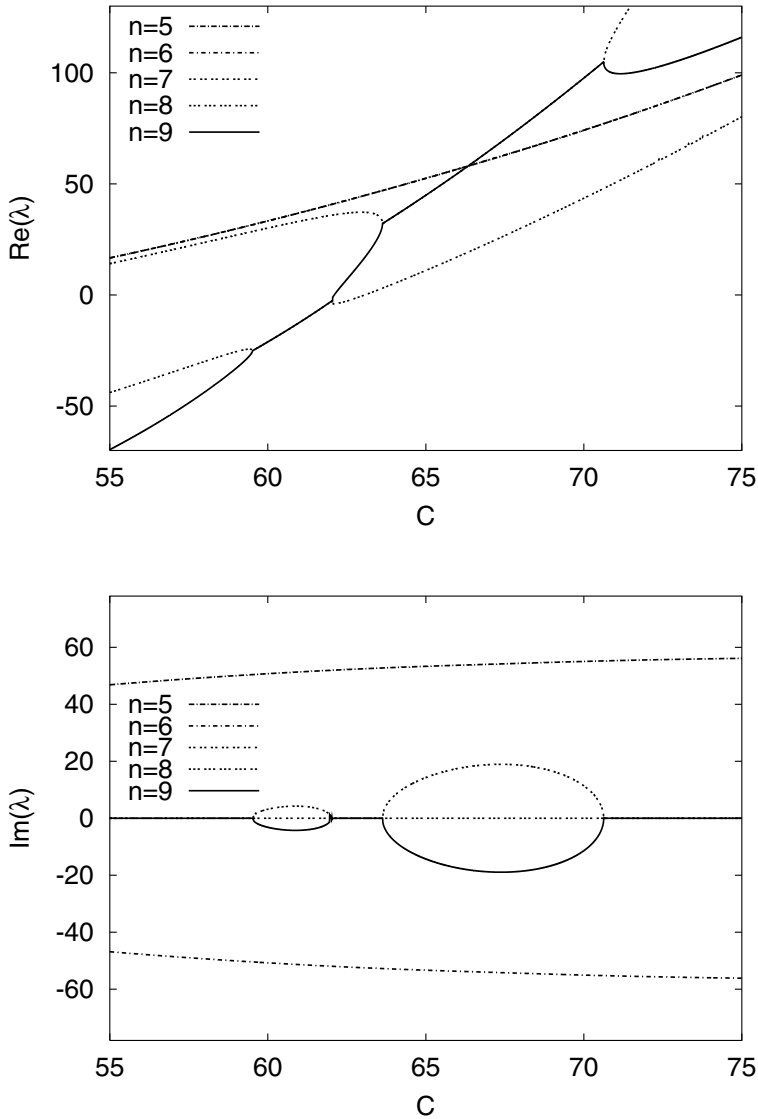
Fig. 3. Zooming into another scaling region with multiple level crossings. Interestingly, although locally only two branches undergo a transition from pairwise real-valued eigenvalues to pairwise complex-conjugate ones, such transitions globally occur also for more branches. In the depicted scaling region three branches participate in mutual transitions. This is a natural indication of the underlying Riemann surface structure of the operator spectrum (see, e.g., [7, 8]).
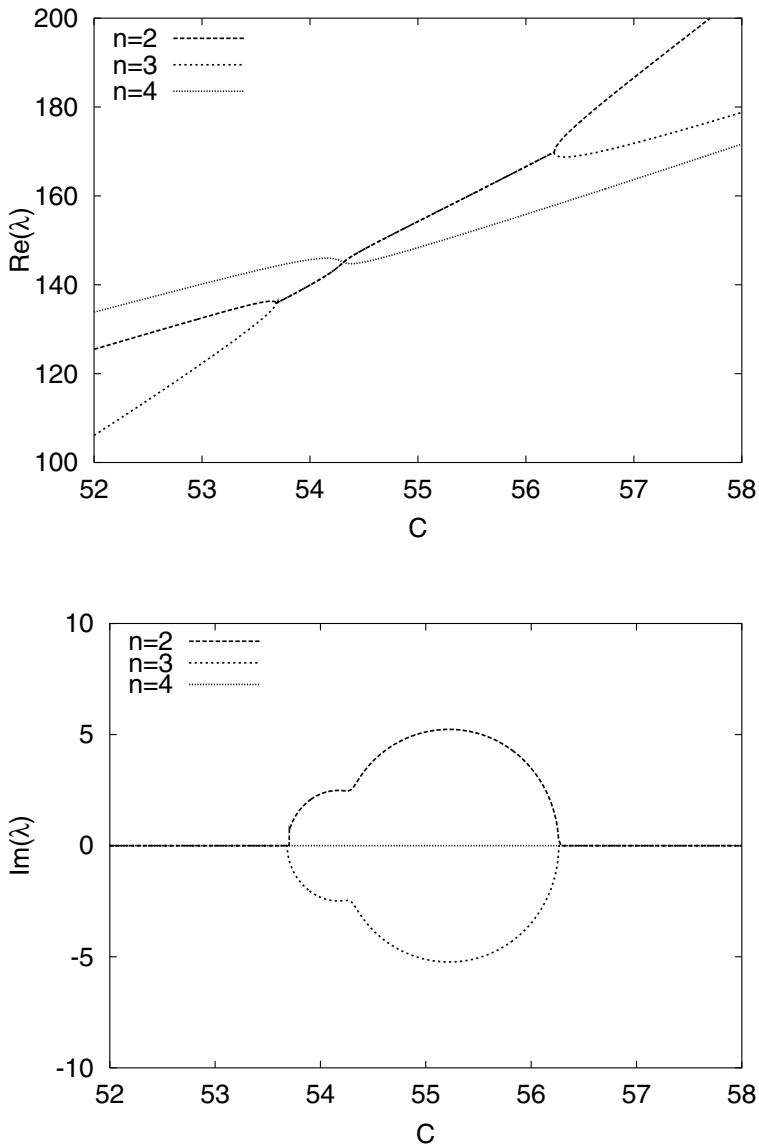
Fig. 4. The real and imaginary parts of crossing spectral branches without transition from real to complex eigenvalues "feel" each other.
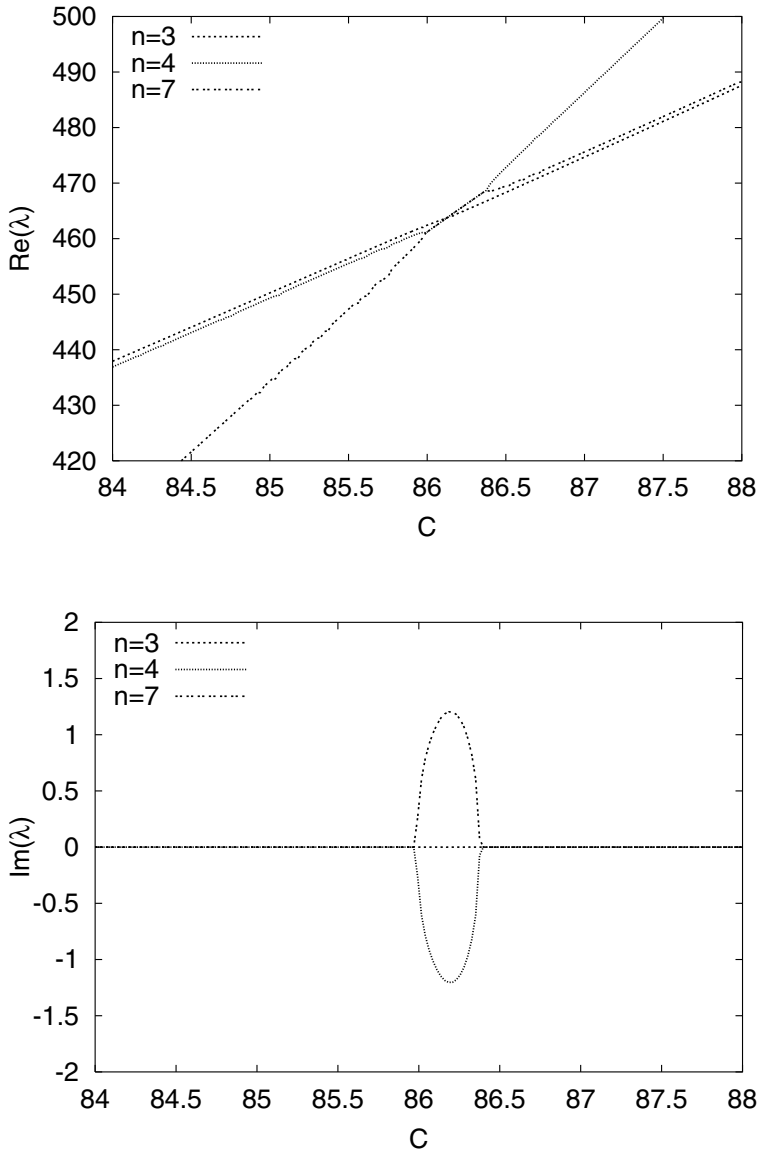
Fig. 5. The long-term parallel location (repelling) of different branches without real-to-complex transition is not a numerical artefact. The branches can be faithfully identified as different ones.

of the $2 \times 2$-operator matrix via substitution

$$\psi = \begin{pmatrix} \psi_1 \\ \dfrac{1}{\alpha}[Q(1) + \lambda]\psi_1 \end{pmatrix}, \qquad \alpha(r) \neq 0, \tag{7}$$

to the equivalent eigenvalue problem of the associated quadratic operator pencil

$$L_l[\alpha, \lambda]\psi_1 \equiv \left\{ [Q[1] + \lambda] \frac{1}{\alpha} [Q[1] + \lambda] - Q[\alpha] \right\} \psi_1 = 0$$
$$= (A_2\lambda^2 + A_1\lambda + A_0)\psi_1 = 0. \tag{8}$$

Solving the quadratic (functional) equation

$$M_l[\alpha, \lambda] := (L_l[\alpha, \lambda]\psi_1, \psi_1) = a_2\lambda^2 + a_1\lambda + a_0 = 0, \qquad a_j := (A_j\psi_1, \psi_1), \tag{9}$$

for $\lambda$ we see from the solutions

$$\lambda_\pm = \frac{1}{2a_2} \left( -a_1 \pm \sqrt{a_1^2 - 4a_0a_2} \right) \tag{10}$$

that a level crossing occurs when the discriminant $\Delta = a_1^2 - 4a_0a_2$ of Eq. (9) vanishes. At the level crossing points it holds

$$M_l[\alpha, \lambda] = 0, \qquad \partial_\lambda M_l[\alpha, \lambda] = 0 \tag{11}$$

and the operator pencil has a Jordan–Keldysh chain [9–11] $\{\psi_1, \chi_1, \phi_1\}$, consisting of the eigenfunction $\psi_1$ and the associated functions $\chi_1, \phi_1$ which satisfy the relations

$$L_l[\alpha, \lambda_0]\psi_1 = 0, \tag{12}$$
$$L_l[\alpha, \lambda_0]\chi_1 + \partial_\lambda L_l[\alpha, \lambda]|_{\lambda_0} \psi_1 = 0, \tag{13}$$
$$L_l[\alpha, \lambda_0]\phi_1 + \partial_\lambda L_l[\alpha, \lambda]|_{\lambda_0} \chi_1 + \tfrac{1}{2} \partial_\lambda^2 L_l[\alpha, \lambda]|_{\lambda_0} \psi_1 = 0. \tag{14}$$

Due to the lack of rigorous analytical and non-numerical operator theoretic studies of the $\alpha^2$-dynamo, we will collect in the next section some few qualitative facts about a highly simplified $2 \times 2$ matrix model with some rough structural analogies to the operator matrix (1).

## 2   $\mathbb{Z}_2$-graded pseudo-Hermiticity

In this section[1]) we start from the simple model of an $\alpha^2$-dynamo with idealized boundary conditions (4) so that the corresponding operator $\hat{H}_l[\alpha]$ is $J$-selfadjoint ($J$-pseudo-Hermitian) [see relation (3)]. We will try to get a rough intuitive insight

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[1]) The few issues presented in this and the next section are part of a detailed study given in: U. Günther, $\mathbb{Z}_2$-*graded pseudo-Hermitian systems: exceptional points, pseudo-unitary fibrations and nontrivial holonomy*, in preparation.

into some of the basic properties of such a system in the vicinity of a level crossing point.[2])

General pseudo-Hermitian operators $H$ were defined in Refs. [14] as operators which satisfy a relation

$$H = \eta H^\dagger \eta^{-1}. \tag{15}$$

Because of the involution property of $J$, i.e. $J^2 = I$, $J = J^{-1}$, the operator $\hat{H}_l[\alpha]$ is part of a narrow subclass of operators with $H = \mu H^\dagger \mu$, $\mu^2 = I, \mu^{-1} = \mu$. To the same subclass belong the $\mathcal{PT}$-symmetric Hamiltonians of Refs. [15–18], which are $\mathcal{P}$-pseudo-Hermitian and for which necessarily holds $\mathcal{P}^2 = I$, as well as the Schrödinger Hamiltonian associated to a Wheeler–DeWitt equation of minisuperspace quantum cosmology considered in Ref. [14]. A general feature of systems with involutive pseudo-Hermiticity operators $\mu$ is an underlying $\mathbb{Z}_2$-graded structure of their Hilbert space: Every state (vector) of the Hilbert space can be naturally split into (projected onto) $\mu$-even and $\mu$-odd components (for $\mathcal{PT}$-symmetric Hamiltonians into parity-even and parity-odd states, see, e.g., [14, 19])

$$x = P_+ x + P_- x = x_+ + x_-, \qquad P_\pm := \tfrac{1}{2}(I \pm \mu), \tag{16}$$

with

$$\mu x_\pm = \pm x_\pm. \tag{17}$$

Explicitly, this leads to a natural $\mathbb{Z}_2$-grading of the Hilbert space $\tilde{\mathcal{H}} \hookrightarrow \mathcal{H}_+ \oplus \mathcal{H}_-$ and a Krein space structure[3]) [21]: Beside the usual inner product $(.,.)$ of the Hilbert space $\tilde{\mathcal{H}}$, which induces a non-negative norm $(x, x) = (x_+, x_+) + (x_-, x_-) \geq 0$ for $x_\pm \in \mathcal{H}_\pm$, one can consider a Krein space $(\mathfrak{K}_\mu, [.,.]_\mu)$ with indefinite inner product $[x, y]_\mu = (\mu x, y) = (x_+, y_+) - (x_-, y_-)$ (and a corresponding indefinite "norm"). One naturally distinguishes states (vectors) of positive type $[x, x]_\mu > 0$, of negative type $[x, x]_\mu < 0$, and isotropic states $[x, x]_\mu = 0$. (In rough analogy to Minkowski space this corresponds to time-like, space-like, and light-like vectors.) Because of $\mu H = H^\dagger \mu$ a $\mu$-selfadjoint ($\mu$-pseudo-Hermitian) operator is selfadjoint in the Krein space $\mathfrak{K}_\mu$:

$$[Hx, y]_\mu = (\mu H x, y) = (x, H^\dagger \mu y) = (x, \mu H y) = [x, Hy]_\mu. \tag{18}$$

A natural representation of a $\mathbb{Z}_2$-graded system can be given in terms of 2-component vectors and $2 \times 2$-operator matrices

$$x = \begin{pmatrix} x_+ \\ x_- \end{pmatrix}, \qquad H = \begin{pmatrix} H_{++} & H_{+-} \\ H_{-+} & H_{--} \end{pmatrix},$$
$$\mu = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad P_+ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \tag{19}$$

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[2]) After finishing this proceedings contribution we became aware that some aspects of level crossings in $\mathbb{Z}_2$-graded pseudo-Hermitian systems had been briefly discussed earlier — with the help of different techniques — in Refs. [12, 13]. The present analysis overlaps only marginally with those results and provides a different view on the subject.

[3]) For a detailed introduction into the operator theory over Krein spaces see, e.g., Refs. [11, 20].

where $H_{++} = P_+ H P_+$, etc. The $\mu$-pseudo-Hermiticity implies

$$H_{++} = H_{++}^\dagger, \qquad H_{--} = H_{--}^\dagger, \qquad H_{+-} = -H_{-+}^\dagger. \tag{20}$$

The operator $\hat{H}_l[\alpha]$ of the $\alpha^2$-dynamo with idealized boundary conditions can be transformed into this representation by first diagonalizing the involution operator $J$

$$J \mapsto \mu = S^{-1} J S, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}. \tag{21}$$

Applying then the same transformation $S$ to $\hat{H}_l[\alpha]$ and the elements of the Hilbert space $\tilde{\mathcal{H}}$ one obtains the equivalent operator [5]

$$\check{H}_l[\alpha] = S^{-1} \hat{H}_l[\alpha] S = \frac{1}{2} \begin{pmatrix} Q[\alpha - 2] + \alpha & -Q[\alpha] + \alpha \\ Q[\alpha] - \alpha & Q[-\alpha - 2] - \alpha \end{pmatrix} \tag{22}$$

which acts on 2-vectors

$$\check{\psi} = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_2 + \psi_1 \\ \psi_2 - \psi_1 \end{pmatrix}. \tag{23}$$

We leave a detailed analysis of this highly non-trivial operator to future studies.

Instead we try to get a rough qualitative understanding of the level-crossing in a $\mathbb{Z}_2$-graded system. For this purpose, we consider its simplest example[4]) — the eigenvalue crossing of a $2 \times 2$ matrix with the basic symmetry properties (20):

$$\begin{aligned} H &= \begin{pmatrix} a & b \\ -b^* & d \end{pmatrix} = \begin{pmatrix} e_0 & 0 \\ 0 & e_0 \end{pmatrix} + h, \qquad h := \begin{pmatrix} f & b \\ -b^* & -f \end{pmatrix}, \\ e_0 &= \tfrac{1}{2}(a + d), \quad f = \tfrac{1}{2}(a - d), \quad b = b_1 + i b_2, \qquad a, d, b_1, b_2 \in \mathbb{R}. \end{aligned} \tag{24}$$

For the eigenvalues holds

$$\det\left(H - E I\right) = 0 \qquad \Longrightarrow \qquad E = e_0 \pm \sqrt{f^2 - b_1^2 - b_2^2} \tag{25}$$

and we see that a level crossing occurs at

$$\Delta(f, b_1, b_2) := f^2 - b_1^2 - b_2^2 = 0. \tag{26}$$

$\Delta(f, b_1, b_2) = 0$ defines a two-dimensional variety (a double-cone) in the three dimensional parameter space $\mathcal{M} \ni (f, b_1, b_2)$ so that for $f \neq 0 \neq |b|$ the degeneracy has co-dimension one. This is in obvious contrast to a Hermitian $2 \times 2$ (spin) matrix problem with

$$h_s = \begin{pmatrix} f & b \\ b^* & -f \end{pmatrix} \tag{27}$$

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[4]) This gives a level crossing with transition from a pair of real-valued eigenvalues to a pair of complex conjugate eigenvalues. Apart from such level crossing with real-to-complex transitions, there occur level crossings without such transitions. They are present in all Figs. 1–5 and can be easily understood in a matrix setup as crossings of spectral branches which belong to different $\mathbb{Z}_2$-blocks.

and eigenvalues

$$\det (H_{\mathrm{s}} - E_{\mathrm{s}} I) = 0 \qquad \Longrightarrow \qquad E_{\mathrm{s}} = e_0 \pm \sqrt{f^2 + b_1^2 + b_2^2}. \qquad (28)$$

Here a degeneracy $\Delta_{\mathrm{s}}(f, b_1, b_2) := f^2 + b_1^2 + b_2^2 = 0$ occurs only in the single (diabolic) point $f = b_1 = b_2 = 0$ and the crossing has co-dimension three [22, 23].

Another difference between the two models is the underlying symmetry of "iso-energetic" (adiabatic) deformations. Obviously, the eigenvalues $E$ and $E_{\mathrm{s}}$ in (25) and (28) are invariant, respectively, under $SO(1,2)$ and $SO(3)$ transformations in the parameters $f, b_1, b_2$ [24]. This is in natural correspondence with the generators of "iso-energetic" (adiabatic) transformations which are defined by the matrices $H$ and $H_{\mathrm{s}}$ themselves. Modulo the Abelian $U(1)$ transformations induced by the elements $\mathrm{i} e_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ these generators are Lie algebra elements of the type

$$\mathrm{i} h = -b_2 \sigma_1 - b_1 \sigma_2 + \mathrm{i} f \sigma_3 \in su(1,1) \sim so(1,2) \sim sl(2, \mathbb{R}), \qquad (29)$$
$$\mathrm{i} h_s = \mathrm{i}\, (b_1 \sigma_1 + b_2 \sigma_2 + b_3 \sigma_3) \in su(2) \sim so(3). \qquad (30)$$

($\sigma_i$ are the Pauli matrices.) Of course, this interlinks[5]) also nicely with the invariance transformations of the two-dimensional Krein space[6]) $(\mathfrak{K}_\mu, [.,.]_\mu)$, $\mu = \sigma_3$ and the 2-vector Hilbert space $\mathcal{H}_s$ of the spin system. The inner product $[.,.]_\mu$ is invariant under pseudo-unitary $U(1,1) \sim U(1) \times SU(1,1)$ transformations, whereas the inner product $(.,.)$ in $\mathcal{H}_s$ is invariant under unitary $U(2) \sim U(1) \times SU(2)$ transformations.

Further insight into the structure of the simple pseudo-Hermitian $2 \times 2$ matrix eigenvalue problem can be gained by diagonalizing the matrix $H$. Before we do this explicitly, we note that any $2 \times 2$ involution matrix $\eta$:

$$\eta^2 = I, \qquad \eta = \eta^{-1} = \eta^\dagger \qquad (31)$$

belongs to one of the following classes $\eta_+$ or $\eta_-$

$$\det(\eta_+) = 1, \qquad \eta_+ \in \{I, -I\}, \qquad (32)$$
$$\det(\eta_-) = -1, \qquad \eta_- = a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3, \quad a_1^2 + a_2^2 + a_3^2 = 1. \qquad (33)$$

This means that two involution matrices $\eta_{1-}$, $\eta_{2-}$ are connected by an $SU(2) \sim \mathbb{Z}_2 \times SO(3)$ rotation.

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[5]) Because of its underlying pseudo-unitary symmetry pseudo-Hermitian ($\mathcal{PT}$-symmetric) quantum mechanics is not a special variant of quaternionic quantum mechanics [25]. One sees this immediately from the simplest models with $H$ and $H_{\mathrm{s}}$ as Hamiltonians. Quaternionic structures are naturally connected with $SU(2)$ symmetries [24, 25] and $H_s$, whereas the pseudo-Hermitian model with Hamiltonian $H$ has invariants which are related to $SU(1,1)$ symmetry transformations. Additionally, we note that $SU(2)$ is a compact group whereas $SU(1,1)$ is a non-compact one.

[6]) Because of $\dim(\mathcal{H}_\pm) = 1 < \infty$ for the considered $2 \times 2$ matrix model, this Krein space is a so-called Pontryagin space [11].

For non-degenerate eigenvalues $E$ the diagonalization is done as

$$H = SDS^{-1}, \; D = \begin{pmatrix} e_0 - \epsilon & 0 \\ 0 & e_0 + \epsilon \end{pmatrix}, \; S = \begin{pmatrix} \dfrac{-f + \epsilon}{b^*}\gamma_1 & \dfrac{-f - \epsilon}{b^*}\gamma_2 \\ \gamma_1 & \gamma_2 \end{pmatrix}, \quad (34)$$

$$\epsilon := \Delta^{1/2}, \quad \gamma_{1,2} \in \mathbb{C}.$$

The complex constants $\gamma_{1,2}$ are still arbitrary and we can fix them by requiring that the diagonalization preserves the pseudo-Hermitian structure, i.e., that there exists an involution matrix $\eta$ so that

$$D = \eta D^\dagger \eta. \tag{35}$$

From the subtitution chain

$$H = \mu H^\dagger \mu \implies SDS^{-1} = \mu \left(S^{-1}\right)^\dagger D^\dagger S^\dagger \mu$$

$$\implies D = S^{-1}\mu \left(S^{-1}\right)^\dagger D^\dagger S^\dagger \mu S \tag{36}$$

and relation (35) one identifies

$$\eta = S^\dagger \mu S = S^{-1}\mu \left(S^{-1}\right)^\dagger = \mu^{-1} = \mu^\dagger \tag{37}$$

so that

$$\mu = SS^\dagger \mu SS^\dagger, \qquad \eta = S^\dagger S\eta S^\dagger S. \tag{38}$$

From the latter equations one concludes that $|\det(S)|^2 = 1$. Explicit calculations[7] show that one can set $\det(S) = 1$ and that $S$ is in general neither unitary, $S^\dagger \neq S^{-1}$, nor pseudo-unitary. Complementary information can be gained from the explicit structure of the matrix $D$. It holds

$$\Delta > 0 : \implies \epsilon = \Delta^{1/2} > 0, \quad D = \begin{pmatrix} e_0 - \epsilon & 0 \\ 0 & e_0 + \epsilon \end{pmatrix} = D^\dagger, \tag{39}$$

$$\Delta < 0 : \implies \epsilon = \mathrm{i}|\epsilon|, \quad D = \begin{pmatrix} e_0 - \mathrm{i}|\epsilon| & 0 \\ 0 & e_0 + \mathrm{i}|\epsilon| \end{pmatrix},$$

$$D^\dagger = \begin{pmatrix} e_0 + \mathrm{i}|\epsilon| & 0 \\ 0 & e_0 - \mathrm{i}|\epsilon| \end{pmatrix} \tag{40}$$

and one obtains from (35) and the explicit form of $S$ in (34) (by appropriately tuning the constants $\gamma_{1,2}$) that

$$\Delta > 0 : \; f > 0 : \; \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{41}$$

$$f < 0 : \; \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{42}$$

$$\Delta < 0 : \qquad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{43}$$

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[7] See footnote 1.

We see that smooth changes of the starting $\mu$-pseudo-Hermitian matrix $H$ lead to qualitative "switchings" (discontinuities in the mapping $\mu \mapsto \eta$ which correspond to rotations in the space of $2 \times 2$ involution matrices $\eta_-$ — see Eq. (33)) in the pseudo-Hermitian structure of the diagonal matrix $D$. These "switchings" occur when the system intersects critical surfaces in the parameter space: (1) the surface $f = 0$ and (2) the degeneration double cone $\Delta = 0$. In the latter case, the "switching" in the pseudo-Hermiticity matrices corresponds to a "switching" from a Hermitian diagonal matrix $D = D^\dagger$ for $\Delta > 0$ to a complex-valued diagonal matrix $D \neq D^\dagger$ for $\Delta < 0$. A "switching" occurs also in the properties of the eigenvectors of the diagonal matrix $D$. Choosing these eigenvectors in the simplest form as

$$D|\pm\rangle = (e_0 \pm \epsilon)\,|\pm\rangle, \qquad |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\langle\pm|\pm\rangle = 1, \qquad \langle+| = (0,1), \quad \langle-| = (1,0), \tag{44}$$

one finds for the Krein space inner products

$$\Delta > 0: \qquad [\pm,\pm]_\eta = \pm\mathrm{sign}(f), \quad [\pm,\mp]_\eta = 0\,, \tag{45}$$
$$\Delta < 0: \qquad [\pm,\pm]_\eta = 0, \quad [\pm,\mp]_\eta = 1\,. \tag{46}$$

This means that the eigenvectors $|\pm\rangle$ are of positive or negative type for real-valued eigenvalues, $\Delta > 0$, and of isotropic type for pair-wise complex conjugate eigenvalues, $\Delta < 0$. We explicitly reproduced a basic result of Krein space theory which is discussed in [20] and which is also implicitly present, e.g., in Refs. [14, 17, 18].

## 3 Exceptional points

From the structure of $S$ in (34) one sees that in the degeneration limit $\epsilon \to 0$ the determinant $\det(S)$ vanishes, $\det(S) \to 0$, $S$ becomes singular and the diagonalization (34) breaks down. Instead the matrix $D$ turns into a Jordan block. For $\epsilon = 0$, $f^2 = |b|^2 \neq 0$, $f = \pm|b|$, $a = d \pm 2|b|$ one finds the explicit relation

$$H = SDS^{-1}, \quad D = \begin{pmatrix} E & 1 \\ 0 & E \end{pmatrix}, \quad S = \begin{pmatrix} \mp\dfrac{|b|}{b^*} & \dfrac{1}{b^*} \\ 1 & 0 \end{pmatrix}, \quad E = d \pm |b|. \tag{47}$$

Additionally, one observes[8]) that only $|-\rangle$ survives as an (geometric) eigenvector of $D$, whereas $|+\rangle$ is now an associated vector (algebraic eigenvector)

$$(D - EI)|-\rangle = 0, \qquad (D - EI)|+\rangle = |-\rangle, \qquad (D - EI)^2|+\rangle = 0. \tag{48}$$

Hence, the degenerate eigenvalues on the double cone have algebraic multiplicity two and geometric multiplicity one. The degeneracy is a branching point degeneracy

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[8]) The existence of a Jordan block structure with a single geometric eigenvector is a generic feature of systems at level-crossing points and was observed for an 1D pseudo-Hermitian Hamiltonian, e.g., in [26].

[27] — a double cone of exceptional points of branching type [28] in the sense of Kato [29].

A singularity of higher order occurs at the center of the double cone, i.e., in the diabolic point [30] at the origin $f = |b| = 0$ of the parameter space $\mathcal{M} \ni (f, b_1, b_2)$. There the transformation matrix $S$ in (47) becomes singular and instead of the Jordan block (47) the matrix $H$ has the form

$$H = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \qquad E = a = d. \tag{49}$$

This is the same type of co-dimension three degeneracy as in the case of the spin matrix $H_{\mathrm{s}}$ — with two (geometric) eigenvectors, i.e., the algebraic and the geometric multiplicity of this eigenvalue coincide and equal two.

The question of whether nontrivial holonomy (geometric phases [23]) and chirality properties of eigenfunctions [31] in the vicinity of level-crossing points can play a role in the dynamics of unstable magnetic field configurations with field reversals [2] remains as one of the interesting open issues.

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