IOP Publishing — Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 10557–10587 doi:10.1088/1751-8113/40/34/013

Exact propagators for SUSY partners

Andrey M Pupasov[1], **Boris F Samsonov**[1] and **Uwe Günther**[2]

[1] Physics Department, Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia
[2] Research Center Dresden-Rossendorf, PO Box 510119, D-01314 Dresden, Germany

E-mail: pupasov@phys.tsu.ru, samsonov@phys.tsu.ru and u.guenther@fzd.de

Received 26 February 2007, in final form 8 June 2007
Published 7 August 2007
Online at stacks.iop.org/JPhysA/40/10557

Abstract

Pairs of SUSY partner Hamiltonians are studied which are interrelated by usual (linear) or polynomial supersymmetry. Assuming the model of one of the Hamiltonians as exactly solvable with a known propagator, the expressions for propagators of partner models are derived. The corresponding general results are applied to 'a particle in a box', the harmonic oscillator and a free particle (i.e. to transparent potentials).

PACS numbers: 03.65.Ge, 11.30.Pb

1. Introduction

The spacetime evolution of a quantum–mechanical system is governed by its Schrödinger equation and in its most complete form it is encoded in the propagator. The propagator defines the probability amplitude for a particle to move from one point of the space to another in a given time. Similar to propagators in relativistic field theories, it provides a global picture of the causal structure of a quantum system which goes beyond the information contained in a single wavefunction. Moreover, it plays the essential role in solving the probability related Cauchy problem of quantum mechanics (QM).

The vast literature on QM propagators, summarized, e.g. in [1], lists mainly explicit expressions of propagators for stationary Schrödinger equations in one space dimension which are reducible to hypergeometric differential equations (DEs) or their confluent forms. This is in strong contrast to the available methods of supersymmetric quantum mechanics (SUSY QM) [2, 3] which usually lead to much broader classes of exactly solvable Schrödinger equations with solutions, in particular, expressible in terms of linear combinations of hypergeometric functions (see, e.g. [4]). From the structure of the SUSY-induced relations between superpartner Hamiltonians, it is clear that via the corresponding Schrödinger equations these relations should extend to relations between the associated propagators. The main goal of the present paper is to analyze these relations between SUSY partner propagators and to reshape them into user-friendly general recipes for the construction of new propagator classes.

1751-8113/07/3410557+31$30.00

Our interest in such new exact propagators is less motivated by their mere existence or their technical subtleties, but rather it is in their applicability to concrete physical problems (see, e.g. [5]) and here especially to models with well-tailored new properties and to setups which up to now were not related to SUSY techniques at all (see, e.g. [6]). In order to derive the corresponding technical tools, we concentrate in this paper on the general approach which allows to establish the link between the propagators of any two SUSY partner Hamiltonians.

Our main idea is the following. For a given Hamiltonian h_0, whose Schrödinger equation as a second-order differential equation is exactly solvable, the general solutions of the Schrödinger equation of its SUSY partner Hamiltonian h_N can be explicitly constructed. Knowing the relations between the solutions of the Schrödinger equations for h_0 and h_N, one can expect to derive transformation operators relating the associated propagators. At a first glance, this approach seems to be technically trivial. But when one recalls that the calculation of a closed expression for a propagator is usually more difficult than the derivation of a corresponding wavefunction, one may expect the problem to be connected with rather nontrivial technical subtleties. Moreover, if one wants to establish the explicit link between the propagators one may imagine that sometimes the problem may even become unsolvable.

Probably the first indication that the problem may have a solution was given by Jauslin [7] who constructed a general integral transformation scheme simultaneously for Schrödinger equations and for heat equations, but who did not provide a discussion of convergency and divergency of the derived expressions. For the sake of convergency, he applied his technique to the heat-equation-type Fokker–Planck equation only. In general, this result may be extended via Wick rotation to propagators for Schrödinger equations of a free particle and a particle moving through transparent potentials. But the question of convergency and with it the question of solvability remains to be clarified. Another indication that the problem may be solvable has been provided in [8, 9] where a similar model has been analyzed at the level of Green functions of stationary Schrödinger equations. In the present paper, we carefully analyze the problem along the ideas first announced in [10].

The paper is organized as follows. In section 2, we briefly recall some main facts and notations from SUSY QM necessary for our subsequent analysis and sketch the definition of the propagator. Section 3 is devoted to the basic tool of our approach—the interrelation between propagators for models with partner Hamiltonians which are linked by first-order SUSY transformations. In section 4, we generalize these results to polynomial supersymmetry. The general technique developed in these sections is afterwards used in section 5 to derive the propagators for SUSY partner models of 'a particle in a box', of the harmonic oscillator and of the free particle (i.e. for models with transparent potentials). Section 6 concludes the paper.

2. Preliminaries

The subject of our analysis will be a polynomial generalization[3] of the simplest two-component system of Witten's non-relativistic supersymmetric quantum mechanics [2, 3, 13] described by the Schrödinger equation

$$(\mathrm{i}I\partial_t - H)\Psi(x,t) = 0, \qquad x \in (a,b), \tag{1}$$

where H is a diagonal super-Hamiltonian consisting of the two super-partners h_0 and h_N as components

$$H = \begin{pmatrix} h_0 & 0 \\ 0 & h_N \end{pmatrix}, \qquad h_{0,N} = -\partial_x^2 + V_{0,N}(x). \tag{2}$$

[3] For recent reviews, see, e.g. [11, 12].

The interval (a, b) may be both finite or infinite. For simplicity we restrict our consideration to a stationary setup with h_0 and h_N not explicitly depending on time so that the evolution equation (1) reduces via standard substitution $\mathbf{\Psi}(x,t) = \mathbf{\Psi}(x)\,\mathrm{e}^{-\mathrm{i}Et}$ and properly chosen boundary conditions to the spectral problem

$$H\mathbf{\Psi}(x) = E\mathbf{\Psi}(x). \tag{3}$$

The time evolution of the system may be described in terms of a corresponding propagator.

Furthermore, we assume that the partner Hamiltonians h_0 and h_N are intertwined by an Nth-order differential operator L with the following properties:

(1) *Intertwining relations*

$$Lh_0 = h_N L, \qquad h_0 L^+ = L^+ h_N. \tag{4}$$

(2) *Factorization rule*

$$\begin{aligned} &L^+L = P_N(h_0), \qquad LL^+ = P_N(h_N),\\ &P_N(x) = (x-\alpha_0)\cdots(x-\alpha_{N-1}),\\ &\mathrm{Im}(\alpha_i) = 0, \qquad \alpha_i \neq \alpha_{k\neq i}, \qquad i,k = 0,\ldots,N-1. \end{aligned} \tag{5}$$

Here, the adjoint operation is understood in the sense of Laplace (i.e. as formally adjoint with the property $\partial_x^+ = -\partial_x$, $(AB)^+ = B^+A^+$ and $\mathrm{i}^+ = -\mathrm{i}$) and the roots α_i of the polynomial P_N play the role of factorization constants. For simplicity we assume that the polynomial P_N has only simple roots. The intertwining relations together with the factorization rule can be represented in terms of a polynomial super-algebra (for details, see, e.g. [3, 11, 12, 14]).

Although the component Hamiltonians h_0 and h_N enter the super-Hamiltonian (2) in an algebraically symmetric way, we consider h_0 as the given Hamiltonian with the known spectral properties and h_N as the derived Hamiltonian with still undefined spectrum. More precisely, we assume $V_0(x)$ to be real-valued, continuous and bounded from below[4] so that the differential expression $h_0 = -\partial_x^2 + V_0(x)$ defines a Sturm–Liouville operator which is symmetric with respect to the usual $\mathcal{L}^2(a,b)$ inner product. The corresponding functions $\psi \in \mathcal{L}^2(a,b)$ are additionally assumed sufficiently smooth[5], e.g. $\psi \in C^2(a,b)$, over the interval $(a,b) \subseteq \mathbb{R}$. Moreover, we assume Dirichlet boundary conditions (BCs) for the bound state eigenfunctions of h_0, i.e. a domain $\mathcal{D}(h_0) := \{\psi : \psi \in \mathcal{L}^2(a,b) \cap C^2(a,b), \psi(a) = \psi(b) = 0\}$ (see, e.g. [15]) and the operator h_0 itself being at least essentially self-adjoint (with a closure that we denote by the same symbol h_0). As usual, eigenfunctions which correspond to the continuous spectrum of h_0 are supposed to have an oscillating asymptotic behavior at spatial infinity. Concentrating on physically relevant cases, we restrict our attention to the following three types of setups:

(i) The interval (a,b) is finite $|a|, |b| < \infty$ so that h_0 has a non-degenerate purely discrete spectrum (see, e.g. [16]).

(ii) For spectral problems on the half-line $(a = 0, b = \infty)$, we consider the so-called scattering (or short-ranged) potentials which decrease at infinity faster than any finite power of x and have a continuous spectrum filling the positive semi-axis and a finite number of discrete levels; the whole spectrum is non-degenerate.

[4] In addition, we assume the scattering potential $V_0(x)$ short-ranged, so that the corresponding Hamiltonian h_0 has a finite number of bound states. For spectral problems on the half-line, a repulsive singularity at the origin not stronger than $\ell(\ell+1)x^{-2}$, $\ell = 0, 1, \ldots,$ is also assumed.

[5] As usual, $C^2(a,b)$ denotes the space of twice continuously differentiable functions.

(iii) For spectral problems on the whole real line ($a = -\infty, b = \infty$), we consider confining as well as scattering potentials. Confining potentials produce purely discrete non-degenerate spectra (see, e.g. [17]), whereas scattering potentials lead to two-fold degenerate continuous spectra filling the positive real half-line and to a finite number of non-degenerate discrete levels (see, e.g. [16]).

Everywhere in the text we choose real-valued solutions of the differential equation $(h_0 - E)\psi = 0$. This is always possible since $V_0(x)$ is supposed to be a real-valued function and we always restrict ourselves to real values of the parameter E.

The intertwiner L is completely described by a set of N transformation functions $u_n(x)$, which may be both 'physical' and 'unphysical'[6] solutions of the stationary Schrödinger equation with h_0 as Hamiltonian:

$$h_0 u_n = \alpha_n u_n, \qquad n = 0, \dots, N-1. \tag{6}$$

In our case of a polynomial $P_N(x)$ with simple roots (i.e. $\alpha_i \neq \alpha_k$), the action of the intertwiner L on a function f is given by the Crum–Krein formula [18, 19]

$$Lf = \frac{W(u_0, u_1, \dots, u_{N-1}, f)}{W(u_0, u_1, \dots, u_{N-1})}, \tag{7}$$

with W denoting the Wronskian[7]

$$W(u_0, u_1, \dots, u_{N-1}) = \begin{vmatrix} u_0 & u_1 & \cdots & u_{N-1} \\ u_0' & u_1' & \cdots & u_{N-1}' \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{(N-1)} & u_1^{(N-1)} & \cdots & u_{N-1}^{(N-1)} \end{vmatrix}. \tag{8}$$

Below we often use the shortened notation $W_N(x) := W(u_0, u_1, \dots, u_{N-1})$ for this Wronskian indicating only the transformation functions $u_n = u_n(x), n = 0, \dots, N-1$, and the total number N of such functions which are supposed to be solutions of (6). The Wronskian (8) links the solutions ϕ and ψ of the Schrödinger equations with h_N and h_0 as Hamiltonians by the relation $\phi = L\psi$ with L as given in (7). In particular, for $N = 1$ (a first-order transformation) (7) reads

$$\phi = Lf = (-u_{0x}/u_0 + \partial_x)f = \frac{W(u_0, f)}{u_0}, \tag{9}$$

with $W(u_0) \equiv u_0$. Furthermore, the determinant structure (7) of the operator L leads to the immediate implication that it has a nontrivial kernel space Ker L spanned by the set of transformation functions u_n:

$$\text{Ker}\, L = \text{span}\{u_0, \dots, u_{N-1}\}, \qquad \dim(\text{Ker}\, L) = \dim(\text{Ker}\, L^+) = N.$$

The solutions v_n of the equation $h_N v_n = \alpha_n v_n$ are elements of the kernel space of the adjoint operator L^+ and can be obtained as[8]

$$\begin{aligned} v_n &= \frac{W_n(u_0, u_1, \dots, u_{N-1})}{W(u_0, u_1, \dots, u_{N-1})}, \qquad n = 0, \dots, N-1, \\ \text{Ker}\, L^+ &= \text{span}\{v_0, \dots, v_{N-1}\}, \end{aligned} \tag{10}$$

[6] By 'physical' solutions we mean solutions belonging to the domain $\mathcal{D}(h_0) = \{\psi : \psi \in \mathcal{L}^2(a, b) \cap C^2(a, b), \psi(a) = \psi(b) = 0\}$. All solutions $\psi \notin \mathcal{D}(h_0)$ corresponding to a spectral parameter E outside the continuous spectrum are interpreted as 'unphysical'. In the present paper, eigenfunctions corresponding to the continuous spectrum are not used as transformation functions and, although physically meaningful, we excluded them from our classification scheme of 'physical' and 'unphysical' solutions.

[7] If there is only one spatial variable in an equation (x in the present case), we often will not indicate it explicitly for the sake of brevity and also to keep notations simple.

[8] See also (A.11) in the appendix.

where $W_n(u_0, u_1, \dots, u_{N-1})$ denotes the Wronskian of the functions $u_0, u_1, \dots, u_{N-1}$ with the function u_n omitted, i.e.

$$W_n(u_0, u_1, \dots, u_{N-1}) = W(u_0, u_1, \dots, u_{n-1}, u_{n+1}, \dots, u_{N-1}).$$

When we need to explicitly distinguish a spatial variable, we will use for this Wronskian the shortened notation $W_{N,n}(x) := W_n(u_0, u_1, \dots, u_{N-1})$ indicating explicitly the argument of the transformation functions $u_j = u_j(x)$, $j = 0, \dots, N-1$, satisfying equation (6). The potential V_N of the Hamiltonian h_N can be expressed as [18, 19]

$$V_N = V_0 - 2[\ln W(u_0, u_1, \dots, u_{N-1})]''. \tag{11}$$

In general, it is not excluded that the transformation operator L may move a solution of h_0 out of $\mathcal{H}_0 = \mathcal{L}^2(a, b)$ thus transforming a physical solution of h_0 into an unphysical solution of h_N. Moreover, the inverse scenario is also possible, i.e. L may transform an unphysical solution of h_0 into a physical solution of h_N. In such cases, the point spectrum of h_N will differ from that of h_0. Subsequently, we will concentrate on mild transformations L, which leave most of the original spectrum invariant with exception of a finite number of spectral points—a characteristic feature of differential intertwining operators L leaving the boundary behavior of the solutions of the Schrödinger equation unchanged. Recently in [20] a conjecture has been proven, which was originally formulated in [21] and which states that any Nth-order mild differential transformation L can be constructed as a superposition from only first- and second-order mild transformations. In this case, it is possible to show [3] that for problems formulated over the whole $\mathbb{R}$ (for infinite values of a and b) V_N behaves asymptotically like V_0. Therefore, the operator h_N is also essentially self-adjoint and 'lives' in the same Hilbert space $\mathcal{H}$ as h_0. Moreover, since the point spectrum of the self-adjoint Sturm–Liouville problems that we consider is non-degenerate there is no way to create a new discrete level at the position of an already existing discrete level and by this means to increase the geometric multiplicity of that level[9].

According to [19] the necessary condition for an Nth-order transformation to produce an essentially self-adjoint operator h_N is[10]

$$(E - \alpha_0) \cdots (E - \alpha_{N-1}) \geqslant 0, \qquad \forall E \in \operatorname{spec}(h_0). \tag{12}$$

This criterion ensures the mildness of the transformation L leading only to changes in maximally N spectral points (of the point spectrum). Specifically, the spectrum of h_N may contain p points more and q points less than $\operatorname{spec}(h_0)$, where necessarily $p + q \leqslant N$.

In the present paper, we will consider the following possibilities.

- The spectrum of h_0 is a subset of the spectrum of h_N. A new energy level may be created in the spectrum of h_N if and only if the corresponding transformation function $u(x)$ is such that $1/u(x) \in \mathcal{D}(h_N)$, i.e., in particular, that $1/u(x)$ is $\mathcal{L}^2$-integrable and satisfies the Dirichlet BCs.
- The spectrum of h_N is a subset of the spectrum of h_0. An energy level may be removed from the spectrum of h_0 if and only if the corresponding transformation function $u(x)$

[9] This is in contrast to SUSY-intertwined non-self-adjoint operators h_0, h_2 for which a second-order intertwiner L can map two distinct discrete levels of h_0 into a second-order Jordan block of h_2, i.e. an eigenvalue of geometric multiplicity 1 and algebraic multiplicity 2 (for details, see [22]).

[10] The basic idea can be understood as a signature preservation of Hilbert space metrics, i.e. $\forall E$ belonging to the point spectrum of h_0 the eigenfunctions ψ_E with $||\psi_E||^2 = (\psi_E, \psi_E) > 0$ should map into corresponding eigenfunctions ϕ_E of h_N with $||\phi_E||^2 = ||L\psi_E||^2 = (\psi_E, L^+L\psi_E) = (\psi_E, P_N(h_0)\psi_E) \geqslant 0$ that via (5) implies (12). Here, the equality takes place for those ψ_E for which the point E does not belong to the spectrum of h_N, i.e. if $P_N(E) = 0$ and $\phi_E = L\psi_E \equiv 0$. A more detailed analysis of corresponding sufficient conditions for spectral problems on the whole real line in case of scattering potentials and of confining potentials is given in [21].

coincides with the h_0-eigenfunction of this level, i.e. when $u(x)$ satisfies the Dirichlet BCs.

- The spectrum of h_0 coincides with the spectrum of h_N. In this case, none of the transformation functions $u_l(x)$ or $1/u_l(x)$ should be physical, i.e. satisfy Dirichlet BCs on both ends of the interval (a, b). This property should be fulfilled for all transformation functions $u_l(x)$ from which the transformation operator is built.

In all cases, we assume that the transformation functions $\{u_l(x)\}_{l=1}^M$ are linearly independent one from the other and their Wronskian $W(u_1, \ldots, u_M)$ does not vanish $\forall x \in (a, b)$.

In the remainder of this section, we briefly recall some basic properties of quantum–mechanical propagators. Once the super-Hamiltonian (2) is diagonal, it suffices to restrict to one-component (scalar) propagators.

It is well known that the propagator $K(x, y, t)$ of a non-stationary Schrödinger equation contains the complete information about the spacetime behavior of the wavefunction $\Psi(x, t)$ evolving from an initial configuration $\Psi(x, 0) = \Psi_0(x)$

$$\Psi(x, t) = \int_a^b K(x, y, t)\Psi(y, 0)\, \mathrm{d}y$$

solving in this way the probability related Cauchy problem of QM. The propagator (integration kernel) satisfies a differential equation with the Dirac delta-function as initial condition

$$[\mathrm{i}\partial_t - h(x)]K(x, y; t) = 0, \qquad K(x, y; 0) = \delta(x - y). \tag{13}$$

For non-dissipative systems, like in our case, the propagator $K(x, y; t)$ can be interpreted as coordinate representation of the unitary evolution operator $U(t)$: $K(x, y; t) = \langle x|U(t)|y\rangle$, where unitarity implies the symmetry $K^*(x, y; -t) = K(y, x; t)$.

Subsequently, we will mainly work with a spectral decomposition of propagators in terms of complete basis sets of eigenfunctions

$$K(x, y; t) = \sum_{n=0}^{N_p} \psi_n(x, t)\psi_n^*(y) + \int_{-\infty}^{\infty} \mathrm{d}k\, \psi_k(x, t)\psi_k^*(y), \tag{14}$$

where summation over the point spectrum and integration over the continuous (essential) spectrum are understood. By SUSY transformations we will only induce changes in the point spectrum of h_0 so that, for simplicity, we will work with decompositions over discrete sets of basis functions (corresponding to point spectra) keeping in mind that extensions to the continuous spectrum are straightforward.

It is clear that the defining equation for the propagator of the non-stationary Schrödinger equation with super-Hamiltonian (2) can be trivially decomposed as

$$[\mathrm{i}I\partial_t - H]\mathbf{K}(x, y; t) = 0, \qquad \mathbf{K} = \begin{bmatrix} K_0(x, y; t) & 0 \\ 0 & K_N(x, y; t) \end{bmatrix},$$
$$\mathbf{K}(x, y; 0) = I\delta(x - y).$$

3. Propagators related by first-order intertwiners

In this section, we study the structure of propagators interrelated by first-order SUSY transformations. Such SUSY transformations are generated from a single function $u(x)$ and they act as basic building blocks in chain representations of higher-order intertwiners. Although any Nth-order intertwiner may be represented as a chain of first-order intertwiners [3], one has to distinguish between chains which are completely reducible within a given

Hilbert space $\mathcal{H}$ [23] and chains which are partially or completely irreducible in $\mathcal{H}$. Complete reducibility means that apart from h_0 and h_N also all intermediate first-order SUSY-related Hamiltonians h_k, $k \neq 0, N$ are self-adjoint or essentially self-adjoint in the same Hilbert space $\mathcal{H}$. In case of irreducible chains[11], several or all intermediate Hamiltonians are non-self-adjoint in $\mathcal{H}$. Below both chain types will play a role. We note that chain representations lead to extremely simplified transformation rules for higher-order intertwined propagators and allow for very efficient calculation techniques (see section 4.2).

According to (7), a first-order intertwiner has the form

$$L_x = -u'(x)/u(x) + \partial_x, \qquad h_0 u = \alpha u.$$

Due to $E - \alpha \geqslant 0$ (see (12)) and depending on the concrete form of the function $u(x)$, the intertwiner L may result in the following three types of relations between the spectra of the Hamiltonians h_0 and h_1 (see, e.g. [2]):

(i) for $\alpha = E_0$ and $u = \psi_0$, the ground state level E_0 of h_0 is removed from the spectrum of h_1;
(ii) h_1 has a new and deeper ground state level $E_{-1} = \alpha < E_0$ than h_0;
(iii) the spectra of h_1 and h_0 completely coincide ($\alpha < E_0$).

In order to create a potential $V_1(x)$ which is nonsingular on the whole interval $(a, b) \ni x$, the function $u(x)$ should be nodeless inside this interval. This property is evidently fulfilled for type (i) relations since the function $u(x)$ coincides in this case with the ground state eigenfunction $u(x) = \psi_0(x)$. In cases (ii) and (iii), the nodelessness should be ensured by an appropriate choice of $u(x)$, a choice which is always possible because of the 'oscillation' theorem (see e.g. [17]). In case (iii), it implies $\alpha < E_0$.

Introducing the Green function

$$G_0(x, y; E) = \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(y)}{E_m - E} \tag{15}$$

of the stationary h_0-Schrödinger equation at fixed energy E[12] (see, e.g. [24]) and its 'regularized' version

$$\widetilde{G}_0(x, y, E_0) = \sum_{m=1}^{\infty} \frac{\psi_m(x)\psi_m(y)}{E_m - E_0} = \lim_{E \to E_0} \left[G_0(x, y, E) - \frac{\psi_0(x)\psi_0(y)}{E_0 - E} \right],$$

the corresponding structural relations for the propagators can be summarized as follows.

Theorem 1. *The propagators $K_1(x, y; t)$ and $K_0(x, y; t)$ of non-stationary Schrödinger equations with SUSY-intertwined Hamiltonians h_1 and h_0 are interrelated with each other and with the Green functions $G_0(x, y; E)$ and $\widetilde{G}_0(z, y, E_0)$ in the following way.*

Type (i) relation:

$$K_1(x, y, t) = L_x L_y \int_a^b K_0(x, z, t) \widetilde{G}_0(z, y, E_0)\, \mathrm{d}z. \tag{16}$$

Type (ii) relation:

$$K_1(x, y, t) = L_x L_y \int_a^b K_0(x, z, t) G_0(z, y, \alpha)\, \mathrm{d}z + \phi_{-1}(x)\phi_{-1}(y)\, \mathrm{e}^{-\mathrm{i}\alpha t}. \tag{17}$$

[11] For a careful analysis of different kinds of irreducible transformations, we refer to [20].
[12] We assume that h_0 has a purely discrete spectrum. As it was already stated in the preliminaries, a generalization to the continuous spectrum is straightforward.

Type (iii) relation:

$$K_1(x, y, t) = L_x L_y \int_a^b K_0(x, z, t) G_0(z, y, \alpha)\, dz. \tag{18}$$

Proof. We start from the type (ii) relation and represent the propagator $K_1(x, y; t)$ in terms of the basis functions $\phi_m(x, t)$ of the Hamiltonian h_1 (cf (14)). We note that the explicit time-independence of h_1 implies a factorization $\phi_m(x, t) = \phi_m(x) \exp(-iE_m t)$, with $\phi_m(x)$ purely real-valued. Expressing ϕ_m in terms of the corresponding wavefunctions of the Hamiltonian $h_0, \phi_m = N_m L\psi_m$, with $N_m = (E - \alpha)^{-1/2}$ a normalization constant (see, e.g. [2]), we arrive at

$$\begin{aligned} K_1(x, y, t) &= \sum_{m=-1}^{\infty} \phi_m(x)\phi_m(y)\, e^{-iE_m t} \\ &= L_x L_y \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(y)}{E_m - \alpha} e^{-iE_m t} + \phi_{-1}(x)\phi_{-1}(y)\, e^{-i\alpha t}. \end{aligned}$$

The wavefunction ϕ_{-1} of the new ground state is proportional to the inverse power of the transformation function $u(x), \phi_{-1} = N/u(x)$, where N is a normalization factor. It remains to express the time-dependent phase factor in terms of the propagator. This can be easily done using the evident property of the bound state solutions of the Schrödinger equation

$$\int_a^b K_0(x, z, t)\psi_m(z)\, dz = \psi_m(x)\, e^{-iE_m t}, \tag{19}$$

so that the previous equation reads

$$K_1(x, y, t) = L_x L_y \int_a^b K_0(x, z, t) \sum_{m=0}^{\infty} \frac{\psi_m(z)\psi_m(y)}{E_m - \alpha}\, dz + \phi_{-1}(x)\phi_{-1}(y)\, e^{-i\alpha t}. \tag{20}$$

The sum in this relation can be identified as the Green function (15). Due to $E_m - \alpha > 0, \forall E_m \in \operatorname{spec}(h_0)$ this Green function is regular $\forall E_m$ and the proof for type (ii) transformations is complete.

The proof for type (i) and (iii) transformations follows the same scheme. The formally regularized Green function $\widetilde{G}_0(z, y, E_0)$ in (i) results from the fact that the ground state with energy E_0 is not present in the spectrum of h_1 so that a sum $\sum_{m>0}$ appears and the ground state contribution has to be subtracted from $G_0(x, y, \alpha = E_0)$. In case of a type (iii) transformation, a sum $\sum_{m=0}^{\infty}$ over the complete set of eigenfunctions appears in (20) and no new state occurs. □

We conclude this section by reshaping relation (16) for the propagator of a system with removed original ground state, i.e. of a type (i) transformed system. The corresponding result can be formulated as follows.

Theorem 2. *For transformations with $u(x) = \psi_0(x)$, the propagator $K_1(x, y; t)$ of the resulting system can be represented as*

$$K_1(x, y; t) = -\frac{1}{u(y)} L_x \int_a^y K_0(x, z; t) u(z)\, dz = \frac{1}{u(y)} L_x \int_y^b K_0(x, z; t) u(z)\, dz. \tag{21}$$

First of all, we recall that $\psi_0(x)$ being the ground state function of h_0 satisfies the zero-boundary conditions. To facilitate the proof of theorem 2, we need the following two lemmas.

Lemma 1.

$$L_y \lim_{E\to E_0}\left(G_0(z,y,E)-\frac{\psi_0(z)\psi_0(y)}{E_0-E}\right)=\lim_{E\to E_0} L_y G_0(z,y,E).$$

Proof. This result follows from the explicit representation of $G_0(z, y, E)$ in terms of basis functions. On the one hand, it holds

$$L_y \lim_{E\to E_0}\left(G_0(z,y,E)-\frac{\psi_0(z)\psi_0(y)}{E_0-E}\right)=\sum_{n=1}^{\infty}\frac{\psi_n(z)L_y\psi_n(y)}{E_n-E_0},$$

whereas on the other hand the kernel property (annihilation) of the ground state $L\psi_0 = 0$ gives

$$\lim_{E\to E_0}\left(L_y G_0(z,y,E)\right)=\lim_{E\to E_0}\left(\sum_{n=1}^{\infty}\frac{\psi_n(z)L_y\psi_n(y)}{E_n-E}\right)=\sum_{n=1}^{\infty}\frac{\psi_n(z)L_y\psi_n(y)}{E_n-E_0}. \qquad \square$$

Lemma 2. *Let $f_l(x, E)$ and $f_r(x, E)$ satisfy the Schrödinger equation*

$$h_0 f(x,E) := -f''(x,E)+V_0(x)f(x,E)=Ef(x,E), \qquad x\in(a,b) \tag{22}$$

and boundary conditions

$$f_l(a,E)=0, \qquad f_r(b,E)=0. \tag{23}$$

Let also $E = E_0$ be the ground state level of h_0 with $\psi_0(x)$ as the ground state function (we assume that h_0 has at least one discrete level), then

$$\lim_{E\to E_0}\frac{f_l(x,E)L_y f_r(y,E)}{W_E(f_r,f_l)}=-\frac{\psi_0(x)\int_y^b\psi_0^2(z)\,\mathrm{d}z}{\psi_0(y)\int_a^b\psi_0^2(z)\,\mathrm{d}z}, \tag{24}$$

$$\lim_{E\to E_0}\frac{f_r(x,E)L_y f_l(y,E)}{W_E(f_r,f_l)}=\frac{\psi_0(x)\int_a^y\psi_0^2(z)\,\mathrm{d}z}{\psi_0(y)\int_a^b\psi_0^2(z)\,\mathrm{d}z}, \tag{25}$$

where $L_y = -u'(y)/u(y) + \partial_y$ with $u(y) \equiv \psi_0(y)$.

Remark 1. The Wronskian $W_E(f_r, f_l) = f_r(x, E)f_l'(x, E) - f_l(x, E)f_r'(x, E)$ is x-independent so that its dependence on the spatial variable is not indicated.

Proof. First, we note that according to (9) $L_y f_r(y, E) = W[u(y), f_r(y, E)]/u(y)$. Next, since both $u(x) = \psi_0(x)$ and $f_r(x, E)$ satisfy the same Schrödinger equation (22) (with $E = E_0$ for $u(x)$), it holds $\partial W[u(x), f_r(x, E)]/\partial_x = (E_0 - E)u(x) f_r(x, E)$ and hence

$$W[u(y),f_r(y,E)]=(E-E_0)\int_y^b u(z)f_r(z,E)\,\mathrm{d}z, \tag{26}$$

where we have used the property $W[u(y), f_r(y, E)]_{y=b} = 0$ which follows from the BCs for $u(y)$ and $f_r(y, E)$. Via (26) we find

$$L_y f_r(y,E)=\frac{E-E_0}{u(y)}\int_y^b u(z)f_r(z,E)\,\mathrm{d}z \tag{27}$$

and hence

$$\frac{L_y f_r(y,E)}{W_E(f_r,f_l)}=-\frac{E-E_0}{f_l(b,E)}\frac{\int_y^b u(z)f_r(z,E)\,\mathrm{d}z}{f_r'(b,E)u(y)}, \tag{28}$$

where it has been used that the Wronskian $W_E(f_r, f_l)$ is x-independent and can be calculated at $x = b$ where $f_r(b, E) = 0$. Since the spectrum of h_0 is non-degenerate, the ground state function is unique up to an arbitrary constant factor and, hence, $u(x) = \psi_0(x)$, $f_r(x, E_0)$ and $f_l(x, E_0)$ have to be proportional to each other

$$f_{r,l}(x, E_0) = C_{r,l} u(x) \tag{29}$$

and for $E \to E_0$ only the first fraction in the rhs of (28) remains undetermined. The l'Hospital rule gives for this limit

$$\lim_{E\to E_0} \frac{E - E_0}{f_l(b, E)} = \frac{1}{\dot f_l(b, E_0)}, \tag{30}$$

where the dot denotes the derivative with respect to E. Making use of (30) and

$$\dot f_l(b, E_0) f_l'(b, E_0) = \int_a^b f_l^2(z, E_0)\,\mathrm{d}z \tag{31}$$

(which we prove below), relation (28) yields

$$\lim_{E\to E_0} \frac{f_l(x, E) L_y f_r(y, E)}{W_E(f_r, f_l)} = -f_l(x, E_0) \frac{f_l'(b, E_0)}{f_r'(b, E_0)} \frac{\int_y^b u(z) f_r(z, E_0)\,\mathrm{d}z}{u(y) \int_a^b f_l^2(z, E_0)\,\mathrm{d}z} \tag{32}$$

and via (29) it leads to the result (24). The proof of (25) follows the same lines with evident changes.

Finally, it remains to derive equation (31). This is easily accomplished by multiplying the Schrödinger equation (22) for $f = f_l(x, E)$ by $\dot f_l(x, E)$, its derivative with respect to E by $f = f_l(x, E)$ and integrating their difference over the interval (a, b). The intermediate result

$$\begin{aligned}\int_a^b f_l^2(x, E)\,\mathrm{d}x &= \dot f_l'(a, E) f_l(a, E) - f_l'(a, E) \dot f_l(a, E) \\ &\quad - \dot f_l'(b, E) f_l(b, E) + f_l'(b, E) \dot f_l(b, E)\end{aligned} \tag{33}$$

reduces to (31) via BC (23) and its derivative with respect to E (what cancels the first two terms) and the limit $E = E_0$, its implication (29) and the BC for $u(x)$. □

Proof of theorem 2. For the Green function $G(x, y, E_0)$ in (16), we use the standard representation in terms of two linearly independent solutions $f_{l,r}$ of the h_0-Schrödinger equation introduced in lemma 2 (see, e.g. [16]):

$$G(x, y, E) = [f_l(x, E) f_r(y, E)\Theta(y - x) + f_l(y, E) f_r(x, E)\Theta(x - y)]/W_E(f_r, f_l), \tag{34}$$

where Θ denotes, as usual, the Heaviside step function. Then, relation (16) takes the form

$$\begin{aligned}K_1(x, y, t) = L_x L_y \int_a^b K_0(x, z, t) \lim_{E\to E_0} \Bigg[&\frac{f_l(z, E) f_r(y, E)}{W_E(f_r, f_l)} \Theta(y - z) \\ &+ \frac{f_l(y, E) f_r(z, E)}{W_E(f_r, f_l)} \Theta(z - y) - \frac{\psi_0(z)\psi_0(y)}{E_0 - E} \Bigg] \mathrm{d}z,\end{aligned}$$

where the step functions can be resolved to give

$$\begin{aligned}K_1(x, y, t) &= L_x L_y \int_a^y K_0(x, z, t) \lim_{E\to E_0} \left[\frac{f_l(z, E) f_r(y, E)}{W_E(f_r, f_l)} - \frac{\psi_0(z)\psi_0(y)}{E_0 - E} \right] \mathrm{d}z \\ &\quad + L_x L_y \int_y^b K_0(x, z, t) \lim_{E\to E_0} \left[\frac{f_l(y, E) f_r(z, E)}{W_E(f_r, f_l)} - \frac{\psi_0(z)\psi_0(y)}{E_0 - E} \right] \mathrm{d}z.\end{aligned}$$

Explicitly acting with the differential operator L_y on the integrals with the variable y-boundary yields

$$K_1(x,y,t) = L_x \int_a^y K_0(x,z,t) L_y \lim_{E\to E_0} \left[\frac{f_l(z,E) f_r(y,E)}{W_E(f_r,f_l)} - \frac{\psi_0(z)\psi_0(y)}{E_0 - E} \right] dz$$
$$+ L_x \int_y^b K_0(x,z,t) L_y \lim_{E\to E_0} \left[\frac{f_l(y,E) f_r(z,E)}{W_E(f_r,f_l)} - \frac{\psi_0(z)\psi_0(y)}{E_0 - E} \right] dz \qquad (35)$$

whereas via lemma 1 the intertwiner L_y and the limit $\lim_{E\to E_0}$ can be interchanged to give

$$K_1(x,y,t) = L_x \left\{ \int_a^y K_0(x,z,t) \lim_{E\to E_0} \frac{f_l(z,E) L_y f_r(y,E)}{W_E(f_r,f_l)} dz \right.$$
$$\left. + \int_y^b K_0(x,z,t) \lim_{E\to E_0} \frac{f_r(z,E) L_y f_l(y,E)}{W_E(f_r,f_l)} dz \right\}. \qquad (36)$$

Application of lemma 2 leads to

$$K_1(x,y,t) = \frac{L_x}{\psi_0(y) \int_a^b \psi_0^2(q)\, dq} \left\{ - \int_y^b \psi_0^2(q)\, dq \int_a^y K_0(x,z,t)\psi_0(z)\, dz \right.$$
$$\left. + \int_a^y \psi_0^2(q)\, dq \int_y^b K_0(x,z,t)\psi_0(z)\, dz \right\}, \qquad (37)$$

which we further reshape by expressing the integral with respect to q over the interval (y,b) by the difference of two integrals over the intervals (a,b) and (a,y). Substitution of $\psi_0 = u$ in the first term results in

$$K_1(x,y,t) = -\frac{1}{u(y)} L_x \int_a^y K_0(x,z,t) u(z)\, dz$$
$$+ \frac{1}{\psi_0(y) \int_a^b \psi_0^2(q)\, dq} \int_a^y \psi_0^2(q)\, dq\, L_x \int_a^b K_0(x,z,t)\psi_0(z)\, dz. \qquad (38)$$

The very last integral is nothing but the ground state stationary wavefunction $\psi_0(x,t) = u(x)\exp(-iE_0 t)$. Therefore, since $L_x u(x) = 0$, we obtain the first equality in (21). The second equality results from applying a similar transformation to the second term in (37).

□

The following remarks are in order. First, we have to note that the integral representation (21) is only valid in case of first-order SUSY transformations which remove the ground state level. If one wants to create a level in a problem on the whole real line, one has to use a transformation function $u(x)$ which diverges for $x \to \pm\infty$ ensuring in this way the normalizability and Dirichlet BCs of the new ground state wavefunction $\phi_{-1}(x) \propto 1/u(x)$. An attempt to calculate the propagator K_1 via (21) would usually lead to a divergent integral. The correct approach is to use (17) in this case. Jauslin [7] using a different procedure obtained the same result (21) both for removing and creating a level, but he completely ignored questions of convergence or divergence of the corresponding integrals. In concrete calculations, he avoided divergent integrals by considering the heat equation only.

4. Higher-order transformations

4.1. Addition of new levels

Let us consider an Nth-order ($N = 2, 3, \ldots$) polynomial supersymmetry corresponding to the appearance of N additional levels in the spectrum of h_N compared to the spectrum of h_0. In this case, new levels may appear both below the ground state energy of h_0 (reducible supersymmetry) and between any two neighbor levels of h_0 (irreducible supersymmetry, see, e.g. [21]). The propagator for the transformed equation can be found in the following way. We develop $K_N(x, y; t)$ over the complete orthonormal set $\{\phi_m(x, t)\}$ of eigenfunctions of h_N and express all ϕ_m with eigenvalues already contained in the spectrum of h_0 in terms of ψ_m, i.e. $\phi_m = N_m L\psi_m$. The normalization constants N_m for transformations fulfilling condition (12) have the form [3]

$$N_m = [(E-\alpha_0)(E-\alpha_1)\cdots(E-\alpha_{N-1})]^{-1/2}.$$

All other eigenfunctions of h_N which correspond to new levels and which are not contained in spec(h_0) we keep untouched. This yields

$$K_N(x, y, t) = L_x L_y \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(y)}{(E_m-\alpha_0)\cdots(E_m-\alpha_{N-1})}\,\mathrm{e}^{-\mathrm{i}E_m t} + \sum_{n=0}^{N-1} \phi_n(x)\phi_n(y)\,\mathrm{e}^{-\mathrm{i}\alpha_n t}.$$

Here we interchanged the derivative operators present in $L_{x,y}$ and the summation. This interchange is justified because the propagators are understood not as the usual functions but as the generalized functions [25] (which, in particular, may be regular, i.e. defined with the help of locally integrable functions). It remains to express $\psi_m(x)\exp(-\mathrm{i}E_m t)$ with the help of (19) in terms of $K_0(x, z, t)$, to make use of the identity

$$\prod_{n=0}^{N-1} \frac{1}{E-\alpha_n} = \sum_{n=0}^{N-1} \left(\prod_{j=0, j\neq n}^{N-1} \frac{1}{\alpha_j - \alpha_n} \right) \frac{1}{E-\alpha_n}$$

and to represent the sum over m in terms of the Green function $G_0(z, y, \alpha_n)$. As a result, one arrives at

$$K_N(x, y, t) = L_x L_y \sum_{n=0}^{N-1} \left(\prod_{j=0, j\neq n}^{N-1} \frac{1}{\alpha_j - \alpha_n} \right) \int_a^b K_0(x, z, t) G_0(z, y, \alpha_n)\,\mathrm{d}z + \sum_{n=0}^{N-1} \phi_n(x)\phi_n(y)\,\mathrm{e}^{-\mathrm{i}\alpha_n t}. \quad (39)$$

4.2. *Removal of levels*

In this section, we need information on the intermediate transformation steps of Nth-order SUSY transformations which goes beyond that presented in section 2. Therefore, we start with a more detailed description of the transformation operators and solutions of the Schrödinger equation at each transformation step.

Let us consider a chain of N first-order transformations

$$h_N \overset{L_{N,N-1}}{\longleftarrow} h_{N-1} \overset{L_{N-1,N-2}}{\longleftarrow} \cdots \overset{L_{2,1}}{\longleftarrow} h_1 \overset{L_{1,0}}{\longleftarrow} h_0$$

built from operators $L_{k+1,k}$ which intertwine neighbor Hamiltonians h_k and h_{k+1} as $L_{k+1,k}h_k = h_{k+1}L_{k+1,k}$. We assume all Hamiltonians h_k, $k = 0, \ldots, N$ self-adjoint or essentially self-adjoint in the same Hilbert space $\mathcal{H}$ so that the SUSY-transformation chain itself is completely reducible. Furthermore, we assume that at each transformation step the ground state of the corresponding Hamiltonian is removed. This means that after N linear SUSY transformations the first N states of the h_0-system are removed and the $N + 1$ st state of h_0 maps into the ground state of h_N. In our analysis, these first $N + 1$ states of h_0 will play a crucial role and we denote them by $u_{0,n}$, $n = 0, \ldots, N$. Furthermore, we use a numbering for the solutions

$u_{k,n}$ of the Schrödinger equations of the SUSY-chain Hamiltonians h_k, $k = 0, \ldots, N$, which is 'synchronized' with the level numbering of h_0, meaning that a function $u_{k,n}$ is related to the spectral parameter E_n. We have to distinguish between physical solutions, which correspond to the existing bound states of h_k and which have indices $n = k, \ldots, N$, and unphysical auxiliary solutions $u_{k,n}$ with $n = 0, \ldots, k-1$ which we construct below. The ground state eigenfunction of a Hamiltonian h_k is given by $u_{k,k}$ and for $k < N$ it is annihilated by the SUSY-intertwiner $L_{k+1,k}$

$$L_{k+1,k}u_{k,k} = 0.$$

The bound state functions $u_{k+1,n}$, $n = k+1, \ldots, N$ of h_{k+1} may be obtained by acting with the SUSY-intertwiner

$$L_{k+1,k} = -u_{k,k,x}/u_{k,k} + \partial_x, \qquad L_{k+1,k}f = \frac{W(u_{k,k}, f)}{u_{k,k}} \tag{40}$$

on the corresponding eigenfunctions of h_k

$$L_{k+1,k}u_{k,n} = u_{k+1,n}, \qquad n = k+1, \ldots, N.$$

Next, we note that the chain of k ($k = 2, \ldots, N$) first-order transformations is equivalent to a single kth-order transformation (7) generated by the transformation functions $u_{0,0}, u_{0,1}, \ldots, u_{0,k-1}$. Furthermore, the transformation operators obey the composition rules

$$L_{k+1,k}L_{k,l} = L_{k+1,l}, \qquad l = 0, \ldots, k-1, \qquad k = 1, \ldots, N-1, \tag{41}$$

so that, e.g., the second-order transformation operator $L_{k+2,k}$ intertwines the Hamiltonians h_k and h_{k+2}

$$L_{k+2,k}h_k = h_{k+2}L_{k+2,k}, \qquad k = 0, \ldots, N-2.$$

The Nth-order transformation operator $L_{N,0}$ is then inductively defined as $L_{N,0} = L_{N,N-1}L_{N-1,0}$. Obviously, it annihilates the N lowest states of the original Hamiltonian h_0, i.e. $u_{0,0}, \ldots, u_{0,N-1} \in \operatorname{Ker} L_{N,0}$.

As further ingredient for the derivation of the propagator-mapping, we need the set of unphysical auxiliary functions $u_{N,n}$, $n = 0, \ldots, N-1$. We construct them as $u_{N,n} = L_{N,0}\widetilde{u}_{0,n}$, where the functions $\widetilde{u}_{0,n}$ are the unphysical solutions of the h_0-Schrödinger equation at energies E_n which are linearly independent from the eigenfunctions $u_{0,n}$. Normalizing $\widetilde{u}_{0,n}$ by the condition $W(u_{0,n}, \tilde{u}_{0,n}) = 1$ and integrating this Wronskian gives

$$\widetilde{u}_{0,n}(x) = u_{0,n}(x)\int_{x_0}^{x}\frac{\mathrm{d}y}{u_{0,n}^2(y)}$$

and finally

$$u_{N,n}(x) = L_{N,0}u_{0,n}(x)\int_{x_0}^{x}\frac{\mathrm{d}y}{u_{0,n}^2(y)}, \qquad n = 0, \ldots, N-1. \tag{42}$$

In the appendix, we show that

$$\begin{aligned} u_{N,n} &= C_{N,n}\frac{W_n(u_{0,0}, \ldots, u_{0,N-1})}{W(u_{0,0}, \ldots, u_{0,N-1})}, \\ C_{N,n} &= (E_{N-1} - E_n)(E_{N-2} - E_n)\cdots(E_{n+1} - E_n), \qquad n = 0, \ldots, N-2, \end{aligned} \tag{43}$$

$$u_{N,N-1} = \frac{W_{N-1}(u_{0,0}, \ldots, u_{0,N-1})}{W(u_{0,0}, \ldots, u_{0,N-1})} = \frac{W(u_{0,0}, \ldots, u_{0,N-2})}{W(u_{0,0}, \ldots, u_{0,N-1})}. \tag{44}$$

The ground state function of h_N is obtained by acting with $L_{N,0}$ on the Nth excited state of h_0:

$$u_{N,N} = L_{N,0}u_{0,N} = \frac{W(u_{0,0}, \ldots, u_{0,N})}{W(u_{0,0}, \ldots, u_{0,N-1})}. \tag{45}$$

As explained in the preliminaries (section 2) and starting from the present section, we will often use the abbreviations $W_N(x)$, $W_{N,n}(x)$ for Wronskians built from transformation functions $u_{0,0}(x), \ldots, u_{0,N-1}(x)$—indicating explicitly the dependence on the spatial variable and the number of transformation functions entering the Wronskians.

Within the above framework, the N first discrete levels $E_0, E_1, \ldots, E_{N-1}$ have been removed from the spectrum of h_0 by choosing the ground state functions $u_{k,k}$ of the Hamiltonians h_k as intermediate transformation functions. For such a construction, the transformed propagator may be calculated according to the following theorem.

Theorem 3. *Let the N first eigenfunctions $u_{0,n} \equiv u_n = \psi_n, n = 0, \ldots, N-1$ of h_0 be the SUSY-transformation functions. Then, the propagators $K_N(x, y; t)$ and $K_0(x, y; t)$ of the Schrödinger equations with Hamiltonians h_N and h_0 are interrelated as*

$$K_N(x, y; t) = (-1)^N L_{N,0,x} \sum_{n=0}^{N-1} (-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_a^y K_0(x, z; t) u_{0,n}(z)\, \mathrm{d}z \tag{46}$$

$$= (-1)^{N-1} L_{N,0,x} \sum_{n=0}^{N-1} (-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_y^b K_0(x, z; t) u_{0,n}(z)\, \mathrm{d}z. \tag{47}$$

Proof. The proof of these relations can be given by induction. We start with (47). For $K_1(x, y; t)$ the statement is proven in (21). Assuming that (47) holds for $K_N(x, y; t)$ we verify its validity for $K_{N+1}(x, y; t)$. The corresponding Hamiltonians h_{N+1} and h_N are intertwined by the *linear* transformation $L_{N+1,N}$ so that (21) is applicable and $K_{N+1}(x, y; t)$ can be represented as

$$K_{N+1}(x, y; t) = \frac{1}{u_{N,N}(y)} L_{N+1,N,x} \int_y^b K_N(x, z; t) u_{N,N}(z)\, \mathrm{d}z.$$

Replacing K_N by (47) and making use of relations (43), (44) and the composition rule (41) gives

$$(-1)^{N-1} K_{N+1}(x, y; t) = \frac{1}{u_{N,N}(y)} L_{N+1,0,x}$$
$$\times \sum_{n=0}^{N-1} (-1)^n C_{Nn}^{-1} \int_y^b \mathrm{d}z \int_z^b \mathrm{d}q\, u_{N,n}(z) u_{N,N}(z) K_0(x, q; t) u_{0,n}(q). \tag{48}$$

The integration region of the double integral is the upper triangle of the rectangle $y < z, q < b$ in the (z, q)-plane. We replace this double integral by the difference of two double integrals over the whole rectangle and the lower triangle, respectively,

$$(-1)^{N-1} K_{N+1}(x, y; t) = \frac{1}{u_{N,N}(y)} L_{N+1,0,x} \sum_{n=0}^{N-1} (-1)^n C_{Nn}^{-1}$$
$$\times \left[\int_y^b \mathrm{d}z\, K_0(x, z; t) u_{0,n}(z) \int_y^b \mathrm{d}q\, u_{N,n}(q) u_{N,N}(q) \right.$$
$$\left. - \int_y^b \mathrm{d}z\, K_0(x, z; t) u_{0,n}(z) \int_z^b \mathrm{d}q\, u_{N,n}(q) u_{N,N}(q) \right]. \tag{49}$$

Here, we reshape the two integrals of the type $\int_\xi^b \mathrm{d}q\, u_{N,n}(q) u_{N,N}(q)$ as follows. First, we note that $u_{N,n}$ and $u_{N,N}$ are solutions of the same Schrödinger equation with Hamiltonian h_N

and therefore

$$\int_{\xi}^{b} \mathrm{d}q\, u_{N,n}(q)u_{N,N}(q) = \frac{W\left[u_{N,N}(\xi), u_{N,n}(\xi)\right]}{E_n - E_N} - w_{b,n}$$
$$= \frac{u_{N,N}L_{N+1,N}u_{N,n}}{E_n - E_N} - w_{b,n}, \tag{50}$$

where $w_{b,n} := W[u_{N,N}(b), u_{N,n}(b)]/(E_n - E_N)$ and where the second equality was obtained via (40). Applying the general relation (43) to $L_{N+1,N}u_{N,n} = u_{N+1,n}$ leads finally to

$$\int_{\xi}^{b} \mathrm{d}q\, u_{N,n}(q)u_{N,N}(q) = -C_{N,n}u_{N,N}(\xi)\frac{W_{N+1,n}(\xi)}{W_{N+1}(\xi)} - w_{b,n}. \tag{51}$$

With (51) as substitution rule, the propagator (49) takes the form

$$(-1)^{N-1}K_{N+1}(x, y; t) = -L_{N+1,0,x}\sum_{n=0}^{N-1}(-1)^n\frac{W_{N+1,n}(y)}{W_{N+1}(y)}\int_{y}^{b} K_0(x, z; t)u_{0,n}(z)\,\mathrm{d}z$$
$$+ \frac{1}{u_{N,N}(y)}L_{N+1,0,x}\int_{y}^{b}\mathrm{d}z K_0(x, z; t)\frac{u_{N,N}(z)}{W_{N+1}(z)}\sum_{n=0}^{N-1}(-1)^n u_{0,n}(z)W_{N+1,n}(z). \tag{52}$$

(The terms containing $w_{b,n}$ exactly canceled.) The sum

$$S_N := \sum_{n=0}^{N-1}(-1)^n u_{0,n}W_{N+1,n}(u_{0,0}, \ldots, u_{0,N}) \tag{53}$$

in the second term can be calculated explicitly. Comparison with the evident determinant identity[13]

$$0 = \begin{vmatrix} u_{0,0} & \cdots & u_{0,N-1} & u_{0,N} \\ u_{0,0} & \cdots & u_{0,N-1} & u_{0,N} \\ u'_{0,0} & \cdots & u'_{0,N-1} & u'_{0,N} \\ \vdots & \ddots & \vdots & \vdots \\ u^{(N-1)}_{0,0} & \cdots & u^{(N-1)}_{0,N-1} & u^{(N-1)}_{0,N} \end{vmatrix} = \begin{vmatrix} u_{0,0} & \cdots & u_{0,N-1} & 0 \\ u_{0,0} & \cdots & u_{0,N-1} & u_{0,N} \\ u'_{0,0} & \cdots & u'_{0,N-1} & u'_{0,N} \\ \vdots & \ddots & \vdots & \vdots \\ u^{(N-1)}_{0,0} & \cdots & u^{(N-1)}_{0,N-1} & u^{(N-1)}_{0,N} \end{vmatrix}$$
$$+ \begin{vmatrix} 0 & \cdots & 0 & u_{0,N} \\ u_{0,0} & \cdots & u_{0,N-1} & u_{0,N} \\ u'_{0,0} & \cdots & u'_{0,N-1} & u'_{0,N} \\ \vdots & \ddots & \vdots & \vdots \\ u^{(N-1)}_{0,0} & \cdots & u^{(N-1)}_{0,N-1} & u^{(N-1)}_{0,N} \end{vmatrix} \tag{54}$$

shows that (53) coincides with the decomposition of the first determinant in the second line of (54) over the elements of its first row. Hence, it holds

$$S_N = -(-1)^N u_{0,N}W(u_{0,0}, \ldots, u_{0,N-1}).$$

Representing the ground state eigenfunctions $u_{N,N}$ in (52) via (45) in terms of Wronskian fractions, we find that

$$\frac{u_{N,N}(z)}{W_{N+1}(z)}\sum_{n=0}^{N-1}(-1)^n u_{0,n}(z)W_{N+1,n}(z) = -(-1)^N u_{0,N}(z)$$

[13] In the determinant of the first line in (54), the first row is equal to the second row so that the determinant vanishes identically.

and, hence, that the second term in (52) is nothing but the absent $n = N$ summand of the sum in the first term. As a result, we arrive at

$$K_{N+1}(x, y; t) = (-1)^N L_{N+1,0,x} \sum_{n=0}^{N} (-1)^n \frac{W_{N+1,n}(y)}{W_{N+1}(y)} \int_y^b K_0(x, z; t) u_{0,n}(z)\, \mathrm{d}z$$

what completes the proof of (47). The representation (46) follows from (47) and the relations

$$\int_a^b K_0(x, z, t) u_{0,n}(z)\, \mathrm{d}z = u_{0,n}(x) \exp(-\mathrm{i}E_n t), \qquad L_{N,0,x} u_{0,n}(x) = 0. \qquad \square$$

We note that in formulae (46) and (47) only one-dimensional integrals are present. In this way, they may turn out more convenient for concrete calculations than similar equations derived in [7].

Furthermore, we note the following. Theorem 3 is proven for the case when the N lowest discrete levels are removed from the spectrum of h_0 starting from the ground state level. This scenario corresponds to reducible supersymmetry. In order to see which of the conditions on the transformation functions $u_{0,n}$ used for the construction of the propagator representations (46) and (47) are indeed necessary conditions, one may simply insert $K_N(x, y; t)$ directly into the Schrödinger equation (13). It turns out that neither the condition of level deletion starting from the ground state nor a deletion of a level block without surviving levels inside is used. This means that equations (46) and (47) hold for any choice of transformation functions provided their Wronskian does not vanish inside the interval (a, b), i.e. it holds for reducible as well as for irreducible SUSY-transformation chains. A necessary but in general not sufficient condition for the nodelessness of the Wronskian is inequality (12) (for further details, see [21]).

4.3. Strictly isospectral transformations

Strictly isospectral transformations can be generated with the help of unphysical solutions of the Schrödinger equation as transformation functions. In this section, we extend theorem 3 to models defined over the whole real line $(a, b) = (-\infty, \infty)$ by assuming transformation functions which vanish at one of the infinities $x \to -\infty$ or $x \to \infty$ and violate the Dirichlet BCs at the opposite infinities ($x \to \infty$ or $x \to -\infty$).

In accordance with (46) and (47), we formulate the corresponding relaxed version of theorem 3 as follows.

Theorem 4. *Let the transformation functions $u_n(x)$ vanish at only one of the infinities $x \to -\infty$ or $x \to \infty$ of the real axis $\mathbb{R}$. Then, the propagators $K_N(x, y; t)$ and $K_0(x, y; t)$ of the Schrödinger equations with h_N and h_0 as Hamiltonians are related as follows:*

for $u_n(x \to -\infty) \to 0$:

$$K_N(x, y; t) = (-1)^N L_x \sum_{n=0}^{N-1} (-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_{-\infty}^{y} K_0(x, z; t) u_n(z)\, \mathrm{d}z; \qquad (55)$$

for $u_n(x \to \infty) \to 0$:

$$K_N(x, y; t) = (-1)^{N-1} L_x \sum_{n=0}^{N-1} (-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_{y}^{\infty} K_0(x, z; t) u_n(z)\, \mathrm{d}z; \qquad (56)$$

for $u_k(x \to -\infty) \to 0, k = 0, \dots, M$ *and* $u_m(x \to \infty) \to 0, m = M+1, \dots, N-1$:

$$K_N(x, y; t) = (-1)^N L_x \sum_{k=0}^{M} (-1)^n \frac{W_{N,k}(y)}{W_N(y)} \int_{-\infty}^{y} K_0(x, z; t) u_k(z)\, dz$$
$$+ (-1)^{N-1} L_x \sum_{m=M+1}^{N-1} (-1)^m \frac{W_{N,m}(y)}{W_N(y)} \int_{y}^{\infty} K_0(x, z; t) u_m(z)\, dz. \tag{57}$$

Proof. We have to verify that the initial condition $K_0(x, y, 0) = \delta(x-y)$ and the Schrödinger equations $(\mathrm{i}\partial_t - h_{0x})K_0(x, y, t) = 0$ and $(\mathrm{i}\partial_t - h_{0y})K_0(x, y, t) = 0$ fulfilled by the original propagator $K_0(x, y, t)$ map into corresponding relations for the final propagator $K_N(x, y, t)$, i.e. that $K_N(x, y, 0) = \delta(x-y)$, $(\mathrm{i}\partial_t - h_{Nx})K_N(x, y, t) = 0$ and $(\mathrm{i}\partial_t - h_{Ny})K_N(x, y, t) = 0$ are satisfied. We demonstrate the explicit proof for the setup with $u_k(x \to -\infty) \to 0$ omitting the technically identical considerations for the other cases.

We start by noticing that the intertwiner L_x maps solutions of the Schrödinger equation for h_0 into solutions of the Schrödinger equation for h_N and, hence, $(\mathrm{i}\partial_t - h_{Nx})K_N(x, y, t) = 0$ is automatically satisfied.

Next, we consider the initial condition $K_N(x, y; 0) = \delta(x-y)$ which should be fulfilled by the rhs of (57). With $K_0(x, y, 0) = \delta(x-y)$ and $\int_{-\infty}^{y} \delta(x-z) u_n(z)\, dz = \theta(y-x) u_n(x)$, we have from (57)

$$K_N(x, y; 0) = (-1)^N \sum_{n=0}^{N-1} (-1)^n \frac{W_{N,n}(y)}{W_N(y)} L_{N,0,x}[\theta(y-x) u_n(x)]. \tag{58}$$

In the Crum–Krein formula (see (7))

$$L_{N,0,x}[\theta(x-y) u_n(x)] = \frac{W[u_0(x), \dots, u_{N-1}(x), \theta(x-y) u_n(x)]}{W(u_0(x), \dots, u_{N-1}(x))}, \tag{59}$$

we represent the derivatives $\partial_x^m[\theta(y-x) u_n(x)]$, $m = 0, \dots, N$ as

$$\partial_x^m[\theta(y-x) u_n(x)] = \sum_{k=0}^{m-1} C_k^m \theta_x^{(m-k)}(y-x) u_n^{(k)}(x) + \theta(y-x) u_n^{(m)}(x), \tag{60}$$

$\theta_x^{(m-k)}(y-x) := \partial_x^{m-k}\theta(y-x)$. Taking into account that

$$\begin{vmatrix} u_0(x) & \cdots & u_{N-1}(x) & \theta(y-x) u_n(x) \\ u_0'(x) & \cdots & u_{N-1}'(x) & \theta(y-x) u_n'(x)) \\ \vdots & \ddots & \vdots & \vdots \\ u_0^{(N)}(x) & \cdots & u_{N-1}^{(N)}(x) & \theta(y-x) u_n^{(N)} \end{vmatrix} = 0, \qquad n = 0, \dots, N-1$$

and making use of the linearity properties of determinants, we reshape (59) as

$$L_{N,0,x}[\theta(x-y) u_n(x)] = \frac{1}{W_N(x)} \begin{vmatrix} u_0(x) & \cdots & u_{N-1}(x) & 0 \\ u_0'(x) & \cdots & u_{N-1}'(x) & -\delta(x-y) u_n(x) \\ \vdots & \ddots & \vdots & \vdots \\ u_0^{(N)}(x) & \cdots & u_{N-1}^{(N)}(x) & \sum_{k=0}^{N-1} C_k^N \theta_x^{(N-k)}(y-x) u_n^{(k)}(x) \end{vmatrix}. \tag{61}$$

Expanding this determinant with regard to the elements of the last column, we find

$$L_{N,0,x}[\theta(x-y) u_n(x)] = \frac{(-1)^N}{W_N(x)} \sum_{m=1}^{N} (-1)^m M_{Nm}(x) \sum_{k=0}^{m-1} C_k^m \theta_x^{(m-k)}(y-x) u_n^{(k)}(x), \tag{62}$$

where

$$M_{Nm}(x) = \begin{vmatrix} u_0 & \cdots & u_{N-1} \\ u_0' & \cdots & u_{N-1}' \\ \vdots & \ddots & \vdots \\ u_0^{(m-1)} & \cdots & u_{N-1}^{(m-1)} \\ u_0^{(m+1)} & \cdots & u_{N-1}^{(m+1)} \\ \vdots & \ddots & \vdots \\ u_0^{(N)} & \cdots & u_{N-1}^{(N)} \end{vmatrix}, \qquad u_j = u_j(x), \quad m = 1, 2, \ldots, N$$

are the corresponding minors.

For the verification of the relation $K_N(x, y; 0) = \delta(x - y)$, we use the definition of the Dirac δ-function

$$\int_{-\infty}^{\infty} K_N(x, y; 0) f(x)\, dx = f(y), \tag{63}$$

where $f(x)$ is a sufficiently smooth test function with compact support. With (58) and (62), the lhs of (63) reads

$$\sum_{n=0}^{N-1}\sum_{m=1}^{N}\sum_{k=0}^{m-1}(-1)^{n+m} C_k^m \frac{W_{N,n}(y)}{W_N(y)} \int_{-\infty}^{\infty} dx\, \theta_x^{(m-k)}(y-x) \frac{M_{Nm}(x) f(x) u_n^{(k)}(x)}{W_N(x)}. \tag{64}$$

As the next step, we use $\theta_x^{(m-k)}(y-x) = -\delta^{(m-k-1)}(x-y)$ and multiple integration by parts[14] to remove the derivatives from the θ-functions:

$$\text{lhs of (63)} = \sum_{n=0}^{N-1}\sum_{m=1}^{N}\sum_{k=0}^{m-1}(-1)^{n-k} C_k^m \frac{W_{N,n}(y)}{W_N(y)} \partial_y^{m-k-1}\left[\frac{M_{Nm}(y) f(y) u_n^{(k)}(y)}{W_N(y)}\right]. \tag{65}$$

The relation

$$\frac{1}{W_N(y)}\sum_{n=0}^{N-1}(-1)^n W_{N,n}(y) u_n^{(j)}(y) = (-1)^N \delta_{j,N-1}, \qquad j = 0, \ldots, N-1 \tag{66}$$

reduces this multiple sum to

$$\text{lhs of (63)} = (-1)^N \frac{M_{NN}(y) f(y)}{W_N(y)} \sum_{k=0}^{N-1}(-1)^k C_k^N \tag{67}$$

and because of $M_{NN}(y) = W_N(y)$ and $\sum_{k=0}^{N-1}(-1)^k C_k^N = (-1)^N$ (cf 4.2.1.3 in [29]) the condition (63) is satisfied.

It remains to prove that the Schrödinger equation $(\mathrm{i}\partial_t - h_{Ny})K_N(x, y, t) = 0$ is fulfilled too. By explicit substitution of equation (55), we have

$$\begin{aligned}(\mathrm{i}\partial_t - h_{Ny})K_N(x, y, t) &= (-1)^N L_x \sum_{n=0}^{N-1}(-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_{-\infty}^{y} \mathrm{i}\partial_t K_0(x, z; t) u_n(z)\, dz \\ &\quad + L_x \sum_{n=0}^{N-1}(-1)^n \left[\left(\frac{W_{N,n}(y)}{W_N(y)}\right)'' - V_N(y)\frac{W_{N,n}(y)}{W_N(y)}\right] \int_{-\infty}^{y} K_0(x, z; t) u_n(z)\, dz \\ &\quad + L_x \sum_{n=0}^{N-1}(-1)^n \left[2\left(\frac{W_{N,n}(y)}{W_N(y)}\right)' u_n(y) + \frac{W_{N,n}(y)}{W_N(y)}(u_n(y)\partial_y + u_n'(y))\right] K_0(x, y; t).\end{aligned} \tag{68}$$

[14] For the theory of distributions (generalized functions), see, e.g. [25] (in particular, vol 1, p 26).

First, we note that due to relation (66) and its derivative the last sum vanishes. Taking further into account that $W_{N,n}(y)/W_N(y)$ is a solution of the Schrödinger equation for h_N at energy E_n (cf (10)) and replacing $\mathrm{i}\partial_t K_0(x,z;t) \to h_{0z}K_0(x,z;t)$, one reduces equation (68) to

$$(\mathrm{i}\partial_t - h_{Ny})K_N(x,y,t) = (-1)^N L_x \sum_{n=0}^{N-1}(-1)^n \frac{W_{N,n}(y)}{W_N(y)} \int_{-\infty}^{y} [(h_{0z} - E_n)K_0(x,z;t)]u_n(z)\,\mathrm{d}z. \tag{69}$$

Integrating by parts and making use of $(h_{0z} - E_n)u_n(z) = 0$, the asymptotical behavior $u(z \to -\infty) \to 0, u'(z \to -\infty) \to 0$ and relation (66), one finds that the rhs in (69) vanishes and, hence, the Schrödinger equation $(\mathrm{i}\partial_t - h_{Ny})K_N(x,y,t) = 0$ is fulfilled. □

4.4. General polynomial supersymmetry

The three different types of transformations considered above may be combined in various ways to produce a supersymmetry of more general type. In general, from the spectrum of the original Hamiltonian h_0, q levels may be removed and p additional levels may be added, $p+q \leqslant N$ producing in this way the spectral set of h_N. The inequality would correspond to SUSY-transformation chains between h_0 and h_N which contain isospectral transformations. For further convenience, we split the spectra of h_0 and h_N according to their transformation-related contents as

$$\begin{aligned} &\mathrm{spec}(h_0) = \{\varepsilon_i, \beta_j, E_k\} + \mathrm{spec}_c(h_0), \qquad i = 1,\ldots,q; \quad j = 1,\ldots,N-(p+q+r) \\ &\mathrm{spec}(h_N) = \{\lambda_l, \beta_j, E_k\} + \mathrm{spec}_c(h_N), \qquad l = 1,\ldots,p; \quad j = 1,\ldots,N-(p+q+r), \end{aligned} \tag{70}$$

where the discrete levels E_k and the continuous spectrum $\mathrm{spec}_c(h_0) = \mathrm{spec}_c(h_N)$ are not affected by the SUSY transformations. The set of transformation constants $\{\alpha_n\}_{n=0}^{N-1} = \{\varepsilon_i, \lambda_l, \beta_j, \gamma_k\}$ corresponds to p new discrete levels $\lambda_n \in \mathrm{spec}(h_N)$ not present in $\mathrm{spec}(h_0)$, q levels $\varepsilon_i \in \mathrm{spec}(h_0)$ not present in $\mathrm{spec}(h_N)$, $N-(p+q+r)$ levels β_j present in both spectra and r constants γ_k not coinciding with any energy level of both Hamiltonians, $\gamma_k \notin \mathrm{spec}(h_0) \cup \mathrm{spec}(h_N)$. Transformations induced at constants $\alpha_n = \beta_n, \gamma_n$ are strictly isospectral.

Summarizing the previous results, the following expression for the propagator $K_N(x,y,t)$ can be given

$$\begin{aligned} K_N(x,y,t) = L_x L_y \sum_{n=0}^{N-1} \left(\prod_{j=0, j\neq n}^{N-1} \frac{1}{\alpha_n - \alpha_j} \right) \int_a^b K_0(x,z,t)\widetilde{G}_0(z,y,\alpha_n)\,\mathrm{d}z \\ + \sum_{\lambda_n} \phi_{\lambda_n}(x)\phi_{\lambda_n}(y)\,\mathrm{e}^{-\mathrm{i}\lambda_n t} + \sum_{\beta_n} \phi_{\beta_n}(x)\phi_{\beta_n}(y)\,\mathrm{e}^{-\mathrm{i}\beta_n t}, \end{aligned} \tag{71}$$

where for $\alpha_n = \varepsilon_n, \beta_n$

$$\widetilde{G}_0(z,y,\alpha_n) = \lim_{E\to\alpha_n} \left[G_0(z,y,E) - \frac{\psi_n(z)\psi_n(y)}{\alpha_n - E} \right]$$

and $\widetilde{G}_0(z,y,\alpha_n) = G_0(z,y,\alpha_n)$ otherwise.

5. Applications

5.1. Particle in a box

As first application we consider a free particle in a box, i.e. the Schrödinger equation with $V_0(x) \equiv 0$ and Dirichlet BCs at the ends of the finite interval $(0, 1)$. The eigenfunctions of this problem are the well known $\psi_{n-1}(x) = \sqrt{2}\sin(n\pi x)$, $n = 1, 2, \ldots$, with energies $E_{n-1} = n^2\pi^2$. The corresponding propagator reads [26]

$$K_{\text{box}0}(x, y, t) = \tfrac{1}{2}\left[\vartheta_3^-(x, y; t) - \vartheta_3^+(x, y; t)\right],$$

with

$$\vartheta_3^-(x, y; t) := \vartheta_3\left(\frac{x-y}{2}\bigg| -\frac{\pi t}{2}\right), \qquad \vartheta_3^+(x, y; t) := \vartheta_3\left(\frac{x+y}{2}\bigg| -\frac{\pi t}{2}\right)$$

and $\vartheta_3(q|\tau)$ denoting the third theta function [27].

As SUSY partner problem, we choose a model which we derive by removing the ground state level E_0 by a linear (one-step) SUSY-mapping with $u = \psi_0 = \sin \pi x$ as a transformation function[15]. According to (11) with $N = 1$ this leads to the Schrödinger equation with potential $V_1(x) = 2\pi^2/\sin^2(\pi x)$, i.e. a singular Sturm–Liouville problem. The propagator of this problem can be represented via (21) as

$$K_{\text{box}1}(x, y, t) = -\frac{1}{2\sin\pi y} L_x \int_0^y \left[\vartheta_3^-(x, z; t) - \vartheta_3^+(x, z; t)\right] \sin(\pi z)\,\mathrm{d}z \tag{72}$$

or after explicit substitution of $L_x = -\pi\cot(\pi x) + \partial_x$ as

$$\begin{aligned} K_{\text{box}1}(x, y, t) = {}& \frac{\pi\cot(\pi x)}{2\sin(\pi y)} \int_0^y \left[\vartheta_3^-(x, z; t) - \vartheta_3^+(x, z; t)\right] \sin(\pi z)\,\mathrm{d}z \\ & - \frac{\pi}{2\sin(\pi y)} \int_0^y \left[\vartheta_3^-(x, z; t) + \vartheta_3^+(x, z; t)\right] \cos(\pi z)\,\mathrm{d}z \\ & + \frac{1}{2}\left[\vartheta_3^-(x, y; t) + \vartheta_3^+(x, y; t)\right]. \end{aligned} \tag{73}$$

Here after using the property $\partial_x\vartheta_3^{\pm}(x, z, t) = \pm\partial_z\vartheta_3^{\pm}(x, z, t)$ we integrated in (72) by parts.

5.2. Harmonic oscillator

Here we consider the Hamiltonian

$$h_0 = -\partial_{xx}^2 + \frac{x^2}{4}.$$

Using a two-fold transformation with transformation functions

$$u_1(x) = \psi_2(x) = (x^2 - 1)\,\mathrm{e}^{-x^2/4}, \qquad u_2(x) = \psi_3(x) = x(x^2 - 3)\,\mathrm{e}^{-x^2/4}$$

corresponding to the second and third excited state eigenfunctions of h_0, we obtain with the help of (11) a perturbed harmonic oscillator potential [3]

$$V^{(2,3)}(x) = \frac{8x^2}{x^4 + 3} - \frac{96x^2}{(x^4 + 3)^2} + \frac{x^2}{4} + 2 \tag{74}$$

which for large $|x|$ behaves like the original harmonic oscillator potential, but for small $|x|$ shows two shallow minima. For completeness, we note that the transformation functions $u_2(x)$ and $u_3(x)$ have nodes, whereas their Wronskian $W(u_2, u_3) = (x^4 + 3)\,\mathrm{e}^{-x^2/2}$ is nodeless so that the corresponding second-order SUSY-transformation itself is well defined, but irreducible.

The propagator for the Schrödinger equation with Hamiltonian $h^{(2,3)} = -\partial_x^2 + V^{(2,3)}(x)$ can be constructed from the well-known propagator

$$K_{\text{osc}}(x, y, t) = \frac{1}{\sqrt{4\pi \mathrm{i} \sin t}}\,\mathrm{e}^{\frac{\mathrm{i}[(x^2+y^2)\cos t - 2xy]}{4\sin t}}$$

[15] There exist other types of transformations leading to regular transformed Sturm–Liouville problems. But the solutions of the resulting Schrödinger equations will violate the Dirichlet BCs [9] so that a special analysis is needed.

for the h_0-Schrödinger equation (see, e.g. [28]) via relation (56). The occurring integrals $\int_y^\infty K_{\mathrm{osc}}(x,z,t)u_n(z)\,\mathrm{d}z$ can be represented as derivatives with respect to the auxiliary current J of the generating function

$$S(J) = \frac{1}{\sqrt{4\pi\mathrm{i}\sin t}}\int_y^\infty \exp\left[\frac{\mathrm{i}[(x^2+z^2)\cos t - 2xz]}{4\sin t} - \frac{z^2}{4} + Jz\right]\mathrm{d}z$$
$$= \frac{1}{2}\exp\left(\frac{-2\mathrm{i}t - x^2}{4} + (\mathrm{i}J^2\sin t + Jx)\exp(-\mathrm{i}t)\right)$$
$$\times\left(1 + \mathrm{erf}\left[\frac{-\sqrt{\mathrm{i}}\exp\left(-\frac{\mathrm{i}t}{2}\right)(2J\sin t + \mathrm{i}(y\exp(\mathrm{i}t) - x))}{2\sqrt{\sin t}}\right]\right),$$

where in S we indicated only the essential variable J and omitted the other ones (x, y and t) on which S also depends. As a result, we obtain

$$K^{(2,3)}(x,y,t) = \mathrm{e}^{y^2/2}L_x\left(\frac{y(y^2-3)}{y^4+3}\left[\frac{\partial^2 S(J)}{\partial J^2} - S(J)\right] - \frac{y^2-1}{y^4+3}\left[\frac{\partial^3 S(J)}{\partial J^3} - 3\frac{\partial S(J)}{\partial J}\right]\right)_{J=0}.$$

The technique can be straightforwardly generalized to second-order SUSY transformations built on any eigenfunction pair

$$u_1(x) = \psi_k(x) = p_k(x)\,\mathrm{e}^{-x^2/4}, \qquad u_2(x) = \psi_{k+1}(x) = p_{k+1}(x)\,\mathrm{e}^{-x^2/4}$$

of the Hamiltonian h_0. Equation (11) yields the corresponding generalized potentials $V^{(k,k+1)}(x)$ (see [3])

$$V^{(k,k+1)}(x) = -2\frac{Q_k''(x)}{Q_k(x)} + 2\left[\frac{Q_k'(x)}{Q_k(x)}\right]^2 + \frac{x^2}{4} + 4, \tag{75}$$

where

$$Q_k(x) := p_{k+1}^2(x) - p_k(x)p_{k+2}(x), \qquad W(u_k,u_{k+1}) = -Q_k(x)\exp(-x^2/2)$$

are built from the re-scaled Hermite polynomials $p_k(x) = 2^{-k/2}H_k(x/\sqrt{2})$.

With the help of (46), the propagator is obtained as

$$K^{(k,k+1)}(x,y,t) = \mathrm{e}^{y^2/2}L_x\left(\frac{p_{k+1}(y)}{Q_k(y)}[p_k(\partial_J)S(J)] - \frac{p_k(y)}{Q_k(y)}[p_{k+1}(\partial_J)S(J)]\right)_{J=0}, \tag{76}$$

with $p_k(\partial_J)$ denoting the kth-order differential operator obtained from $p_k(x)$ by replacing $x^n \to \frac{\partial^n}{\partial J^n}$.

Finally, we note that the potentials (75) behave asymptotically like $x^2/4$ for $|x| \to \infty$ and have k shallow minima at their bottom.

5.3. Transparent potentials

Here we apply our propagator calculation method to transparent potentials which are SUSY partners of the zero potential in models defined over the whole real axis. We note that the propagator for a one-level transparent potential was previously calculated by Jauslin [7] for the Fokker–Planck equation. The propagator for a two-level potential can be found in [9] where the general form of the propagator for an N-level transparent potential has been given as a conjecture. Here we will prove this conjecture.

We construct the N-level transparent potential from the zero potential with the help of the following N solutions of the Schrödinger equation with $V_0(x) \equiv 0$ (see, e.g. [3] for further details and a closed expression of the potential):

$$\begin{aligned} u_{2j}(x) &= \cosh(a_{2j}x + b_{2j}), \\ u_{2j+1}(x) &= \sinh(a_{2j+1}x + b_{2j+1}), \qquad j = 0, 1, \ldots, [(N-1)/2] \end{aligned} \tag{77}$$

as transformation functions. Here, $a_i, b_i \in \mathbb{R}, a_i > a_{i+1} > 0$ is assumed and $[(N-1)/2]$ denotes the integer part of $(N-1)/2$.

The factorization constants $\alpha_j = -a_j^2$ define the positions of the discrete levels (point spectrum) $E_j = \alpha_j < 0$ of $h_N = -\partial_x^2 + V_N(x)$, whereas the continuous part of the spectrum of h_N fills the positive real half-line. The eigenfunctions of the discrete levels normalized to unity have the form [2]

$$\varphi_n = \left[\frac{a_n}{2}\prod_{j=0, j\neq n}^{N-1}\left|a_n^2 - a_j^2\right|\right]^{1/2}\frac{W(u_0, u_1, \ldots, u_{n-1}, u_{n+1}, \ldots, u_{N-1})}{W(u_0, u_1, \ldots, u_{N-1})}. \tag{78}$$

The propagator K_0 and the Green function G_0 for the free particle are well known [1]

$$\begin{aligned} K_0(x, y; t) &= \frac{1}{\sqrt{4\pi \mathrm{i}t}}\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}}, \\ G_0(x, y, E) &= \frac{\mathrm{i}}{2\kappa}\,\mathrm{e}^{\mathrm{i}\kappa|x-y|}, \qquad \mathrm{Im}\,\kappa > 0, \qquad E = \kappa^2, \end{aligned} \tag{79}$$

so that according to (39) the propagator of the transformed system can be calculated as

$$\begin{aligned} K_N(x, y, t) &= \frac{L_x L_y}{\sqrt{4\pi \mathrm{i}t}}\sum_{n=0}^{N-1}\left(\prod_{j=0, j\neq n}^{N-1}\frac{1}{\alpha_n - \alpha_j}\right)\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}(x-z)^2}{4t} - a_n|z-y|\right)\frac{\mathrm{d}z}{2a_n} \\ &\quad + \sum_{n=0}^{N-1}\varphi_n(x)\varphi_n(y)\,\mathrm{e}^{-\mathrm{i}\alpha_n t} =: K_{cN}(x, y, t) + K_{dN}(x, y, t). \end{aligned} \tag{80}$$

In the last line we separated contributions from the continuous spectrum, $K_{cN}(x, y, t)$, from contributions from the discrete spectrum, $K_{dN}(x, y, t)$.

The integral in $K_{cN}(x, y, t)$ can easily be calculated since the primitive of the integrand is well known (see, e.g. integral 1.3.3.17 of [29]). Using the well studied convergency conditions of the error-function erfc from [30], we find

$$\begin{aligned} I(a, x, y, t) &:= \frac{1}{\sqrt{4\pi \mathrm{i}t}}\int_{-\infty}^{\infty}\exp\left(\frac{\mathrm{i}(x-z)^2}{4t} - a|z-y|\right)\frac{\mathrm{d}z}{2a} \\ &= \frac{\mathrm{e}^{\mathrm{i}a^2 t}}{4a}\left[\mathrm{e}^{a(x-y)}\,\mathrm{erfc}\left(a\sqrt{\mathrm{i}t} + \frac{x-y}{2\sqrt{\mathrm{i}t}}\right) + \mathrm{e}^{a(y-x)}\,\mathrm{erfc}\left(a\sqrt{\mathrm{i}t} - \frac{x-y}{2\sqrt{\mathrm{i}t}}\right)\right], \\ &\quad a > 0, \quad t > 0 \end{aligned} \tag{81}$$

and with it

$$K_{cN}(x, y; t) = L_x L_y \sum_{n=0}^{N-1}\left(\prod_{m=0, m\neq n}^{N-1}\frac{1}{\alpha_n - \alpha_m}\right) I(a_n, x, y, t). \tag{82}$$

Introducing the notation

$$\mathrm{erf}_{\pm}(a) := \mathrm{erf}\left(a\sqrt{\mathrm{i}t} \mp \frac{x-y}{2\sqrt{\mathrm{i}t}}\right),$$

where implicit dependence on x, y and t is understood, we formulate the final expression for the propagator as follows.

Theorem 5. *The propagator for an N-level transparent potential has the form*

$$K_N(x,y;t) = \frac{1}{\sqrt{4\pi \mathrm{i}t}}\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + \sum_{n=0}^{N-1}\left(\frac{a_n}{4}\prod_{j=0,j\neq n}^{N-1}\left|a_n^2-a_j^2\right|\right) \times \frac{W_{N,n}(x)W_{N,n}(y)}{W_N(x)W_N(y)}\,\mathrm{e}^{\mathrm{i}a_n^2 t}[\mathrm{erf}_+(a_n)+\mathrm{erf}_-(a_n)]. \tag{83}$$

For the proof of this theorem we need the additional lemma as follows.

Lemma 3. *Let $\{\alpha_i\}_{i=0}^{N-1}$ be a set of N non-coinciding complex numbers $a_i \neq a_{j\neq i} \in \mathbb{C}$ and*[16] *$n \in \mathbb{Z}^+$. Then the following identity holds:*

$$\sum_{i=0}^{N-1}\alpha_i^n\left(\prod_{j=0,j\neq i}^{N-1}\frac{1}{\alpha_i-\alpha_j}\right) = \delta_{n,N-1}. \tag{84}$$

Proof. Consider an auxiliary function

$$f(z) = \frac{z^n}{(z-\alpha_0)(z-\alpha_1)\cdots(z-\alpha_{N-1})}$$

which is analytic in any finite part of the complex z-plane except for N simple poles $\alpha_0,\ldots,\alpha_{N-1}$. From the residue theorem follows

$$\sum_{i=0}^{N-1}\mathrm{res}\, f(\alpha_i) = \sum_{i=0}^{N-1}\alpha_i^n\left(\prod_{j=0,j\neq i}^{N-1}\frac{1}{\alpha_i-\alpha_j}\right) = -\mathrm{res}\, f(\infty)$$

what with the residue at infinity yields (84). □

As next step we prove theorem 5.

Proof. Without loss of generality we may set $b_j = 0$. We start with the propagator component $K_{cN}(x,y,t)$ in (82) related to the continuous spectrum. First we note that the function I in (82) depends only via the difference $x-y$ on the spatial coordinates so that the action of ∂_y in L_y can be replaced by $\partial_y^n \to (-1)^n\partial_x^n$. Hence, the composition of the two Nth-order transformation operators L_xL_y acts as an effective $2N$ th-order differential operator in x

$$L_xL_y = R_0 + R_1\partial_x + \cdots + R_{2N}\partial_x^{2N} \tag{85}$$

with coefficient functions $R_n = R_n(x,y)$. Accordingly, (82) takes the form

$$K_{cN}(x,y;t) = \sum_{n=0}^{N-1}\left(\prod_{m=0,m\neq n}^{N-1}\frac{1}{\alpha_n-\alpha_m}\right)\left[R_0 + R_1\partial_x + \cdots + R_{2N}\partial_x^{2N}\right]I(a_n,x,y,t), \tag{86}$$

where (a_n,x,y,t) is given in (81). From the explicit structure of the first derivatives of $I(a_n,x,y,t)$,

$$\begin{aligned}
\frac{\partial I(a_n,x,y,t)}{\partial x} &= \frac{1}{4}[-\mathrm{e}^{-a_n(x-y)}\,\mathrm{erfc}_+(a_n) + \mathrm{e}^{a_n(x-y)}\,\mathrm{erfc}_-(a_n)]\,\mathrm{e}^{\mathrm{i}a_n^2 t},\\
\frac{\partial^2 I(a_n,x,y,t)}{\partial x^2} &= \frac{-1}{\sqrt{4\pi \mathrm{i}t}}\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + a_n^2 I(a_n),\\
\frac{\partial^3 I(a_n,x,y,t)}{\partial x^3} &= \frac{-1}{2t\sqrt{4\pi \mathrm{i}t}}(x-y)\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}}\\
&\quad + \frac{a_n^2}{4}[-\mathrm{e}^{-a_n(x-y)}\,\mathrm{erfc}_+(a_n) + \mathrm{e}^{a_n(x-y)}\,\mathrm{erfc}_-(a_n)]\,\mathrm{e}^{\mathrm{i}a_n^2 t},
\end{aligned} \tag{87}$$

[16] We use the standard notation $\mathbb{Z}^+ = \{0,1,2,\ldots\}$ for the natural numbers with zero included.

we find (by induction) the general expression of an even-order derivative

$$\frac{\partial^{2m} I(a_n, x, y, t)}{\partial x^{2m}} = \sum_{k=0}^{m-1} a_n^{2k} A_{km}(x, y, t)\, \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + \frac{a_n^{m-1}}{4}\, \mathrm{e}^{\mathrm{i}a_n^2 t} \left[\mathrm{e}^{-a_n(x-y)} \operatorname{erfc}_+(a_n) + \mathrm{e}^{a_n(x-y)} \operatorname{erfc}_-(a_n)\right] \tag{88}$$

and of an odd-order derivative

$$\frac{\partial^{2m-1} I(a_n, x, y, t)}{\partial x^{2m-1}} = \sum_{k=0}^{m-2} a_n^{2k} A_{km}(x, y, t)\, \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} - \frac{a_n^{m-1}}{4}\, \mathrm{e}^{\mathrm{i}a_n^2 t} \left[\mathrm{e}^{-a_n(x-y)} \operatorname{erfc}_+(a_n) - \mathrm{e}^{a_n(x-y)} \operatorname{erfc}_-(a_n)\right]. \tag{89}$$

Subsequently, it will turn out that from all the non-vanishing coefficient functions $A_{km}(x, y, t)$ only the explicit structure of a single one will be needed

$$A_{N-1,2N}(x, y, t) = -(4\pi \mathrm{i}t)^{-1/2} \tag{90}$$

as well as the vanishing of a large number of the other coefficients

$$A_{km}(x, y, t) \equiv 0 \qquad \Longleftrightarrow \qquad \begin{cases} m = 2l & \cap \quad k > l-1 \\ m = 2l+1 & \cap \quad k > l-1. \end{cases}$$

Furthermore, we will use the abbreviation

$$I_m(a_n) = \frac{1}{4a_n}\, \mathrm{e}^{\mathrm{i}a_n^2 t}[(-1)^m\, \mathrm{e}^{-a_n(x-y)} \operatorname{erfc}_+(a_n) + \mathrm{e}^{a_n(x-y)} \operatorname{erfc}_-(a_n)] \tag{91}$$

and we will need the explicit form of $a_n^{2(N-1)}$ and a_n^{2N} terms in the highest-order derivative

$$\frac{\partial^{2N} I(a_n, x, y, t)}{\partial x^{2N}} = \cdots - a_n^{2(N-1)} \frac{1}{\sqrt{4\pi \mathrm{i}t}}\, \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + a_n^{2N} I_{2N}. \tag{92}$$

The complete propagator component $K_{cN}(x, y; t)$ in (86) can now be rewritten as

$$\begin{aligned} K_{cN}(x, y; t) = {} & \sum_{n=0}^{N-1} \left(\prod_{m=0, m\neq n}^{N-1} \frac{1}{\alpha_n - \alpha_m} \right) R_0 I_0(a_n) \\ & + \sum_{n=0}^{N-1} \left(\prod_{m=0, m\neq n}^{N-1} \frac{1}{\alpha_n - \alpha_m} \right) a_n R_1 I_1(a_n) \\ & + \sum_{n=0}^{N-1} \left(\prod_{m=0, m\neq n}^{N-1} \frac{1}{\alpha_n - \alpha_m} \right) R_2 \left[-\frac{1}{\sqrt{4\pi \mathrm{i}t}}\, \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + a_n^2 I_2(a_n) \right] + \cdots \\ & + \sum_{n=0}^{N-1} \left(\prod_{m=0, m\neq n}^{N-1} \frac{1}{\alpha_n - \alpha_m} \right) R_{2N} \left[\sum_{k=0}^{N-1} a_n^{2k} A_{k,2N}(x, y, t)\, \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + a_n^{2N} I_{2N}(a_n) \right]. \end{aligned}$$

Comparison with lemma 3 shows that due to $a_n^{2k} = (-\alpha_n)^k$ only a single term containing $\exp(\mathrm{i}(x-y)^2/4t)$ does not vanish. It is the $k = N-1$ term in the very last sum which, with

$R_{2N} = (-1)^N$ and $A_{N-1,2N}(x, y, t)$ as given in (90), yields exactly the propagator (79) of the free particle. All other terms containing $\exp(\mathrm{i}(x-y)^2/4t)$, after interchanging the sums $\sum_{n=0}^{N-1}\cdots$ and $\sum_{k=0}^{N-1}\cdots$, cancel out due to lemma 3. Thus, the above formula reduces to

$$K_{cN}(x, y; t) = \frac{1}{\sqrt{4\pi\mathrm{i}t}}\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + \sum_{n=0}^{N-1}\left(\prod_{m=0,m\neq n}^{N-1}\frac{1}{\alpha_n-\alpha_m}\right)$$
$$\times\left[R_0I_0(a_n) + R_1a_nI_1(a_n) + R_2a_n^2I_2(a_n) + R_3a_n^3I_3(a_n) + \cdots + R_{2N}a_n^{2N}I_{2N}(a_n)\right]. \tag{93}$$

Expressing $I_m(a_n)$ via (91) in terms of erfc_+ and erfc_-, one finds contributions

$$\frac{\mathrm{erfc}_+(a_n)}{4a_n}\,\mathrm{e}^{\mathrm{i}a_n^2t}\left(R_0 - a_nR_1 + a_n^2R_2 - a_n^3R_3 + \cdots + a_n^{2N}R_{2N}\right)\mathrm{e}^{-a_n(x-y)},$$
$$\frac{\mathrm{erfc}_-(a_n)}{4a_n}\,\mathrm{e}^{\mathrm{i}a_n^2t}\left(R_0 + a_nR_1 + a_n^2R_2 + a_n^3R_3 + \cdots + a_n^{2N}R_{2N}\right)\mathrm{e}^{a_n(x-y)},$$

which can be represented as

$$\frac{\mathrm{erfc}_+(a_n)}{4a_n}\,\mathrm{e}^{\mathrm{i}a_n^2t}\left(R_0 + R_1\partial_x + R_2\partial_x^2 + R_3\partial_x^3 + \cdots + R_{2N}\partial_x^{2N}\right)\mathrm{e}^{-a_n(x-y)},$$
$$\frac{\mathrm{erfc}_-(a_n)}{4a_n}\,\mathrm{e}^{\mathrm{i}a_n^2t}\left(R_0 + R_1\partial_x + R_2\partial_x^2 + R_3\partial_x^3 + \cdots + R_{2N}\partial_x^{2N}\right)\mathrm{e}^{a_n(x-y)}.$$

Comparison with (85) shows that the sums yield simply $L_xL_y\exp[\pm a_n(x-y)]$. Recalling furthermore that the transformation operators L_x and L_y are given by (7) and that the transformation functions have the form (77) we arrive after some algebra at[17]

$$L_x\,\mathrm{e}^{\pm a_nx} = (\mp 1)^{n+1}(-1)^{N+n}a_n\prod_{j=0,j\neq n}^{N-1}\left(a_j^2-a_n^2\right)\frac{W_{N,n}(x)}{W_N(x)} \tag{94}$$

and, hence, at

$$L_xL_y\,\mathrm{e}^{\pm a_n(x-y)} = (-1)^{n+1}a_n^2\prod_{j=0,j\neq n}^{N-1}\left(a_n^2-a_j^2\right)^2\frac{W_{N,n}(x)W_{N,n}(y)}{W_N(x)W_N(y)}. \tag{95}$$

Substituting (95) into (93), we obtain

$$K_{cN}(x, y; t) = \frac{1}{\sqrt{4\pi\mathrm{i}t}}\,\mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} + \frac{1}{4}\sum_{n=0}^{N-1}\mathrm{e}^{\mathrm{i}a_n^2t}a_n\left(\prod_{m=0,m\neq n}^{N-1}\frac{(-1)^{n+1}\left(a_n^2-a_m^2\right)^2}{\alpha_n-\alpha_m}\right)$$
$$\times\frac{W_{N,n}(x)W_{N,n}(y)}{W_N(x)W_N(y)}[\mathrm{erfc}_+(a_n) + \mathrm{erfc}_-(a_n)]. \tag{96}$$

A further simplification can be achieved by recalling that $\alpha_n = -a_n^2$ and that the parameters a_i are ordered as $a_0 < a_1 < \cdots < a_{N-1}$. Setting $\left(a_m^2-a_n^2\right) = -\left(a_n^2-a_m^2\right)$ in the denominator for $m = 0, \ldots, n-1$ gives an additional sign factor $(-1)^n$. The propagator sum $K_N = K_{cN}+K_{dN}$

[17] We note that equation (94) is compatible with (78). The function $\exp(-a_nx)$ is a solution of the initial Schrödinger equation related to one of the factorization constants and it is linearly independent of the corresponding factorization solution. Therefore, $L_x\exp(-a_nx)$ up to a constant factor is one of the bound state functions of h_N given in (78).

resulting from continuous and discrete spectral components reads then

$$K_N(x, y; t) = \frac{1}{\sqrt{4\pi \mathrm{i}t}} \mathrm{e}^{\frac{\mathrm{i}(x-y)^2}{4t}} - \frac{1}{4} \sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}a_n^2 t} a_n \left(\prod_{m=0,m\neq n}^{N-1} |a_n^2 - a_m^2| \right)$$
$$\times \frac{W_{N,n}(x) W_{N,n}(y)}{W_N(x) W_N(y)} [\mathrm{erfc}_+(a_n) + \mathrm{erfc}_-(a_n)]$$
$$+ \frac{1}{2} \sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{i}a_n^2 t} a_n \left(\prod_{m=0,m\neq n}^{N-1} |a_n^2 - a_m^2| \right) \frac{W_{N,n}(x) W_{N,n}(y)}{W_N(x) W_N(y)}, \tag{97}$$

and via the relation $\mathrm{erfc}(z) = 1 - \mathrm{erf}(z)$ it leads to the expression in the theorem. □

6. Conclusion

A careful study of propagators for SUSY partner Hamiltonians is given. Assuming the partner Hamiltonians linked by a polynomial supersymmetry of the general type, we derived user-friendly expressions interrelating the corresponding associated propagators. We applied our general technique to derive propagators for SUSY partner Hamiltonian for 'a particle in a box', for transparent potentials and for a family of SUSY partner potentials of the harmonic oscillator.

As one of the next steps of this study, we see an application of our general theorem 2 to shape invariant potentials (see, e.g. [2]). The main feature of such potentials is that the removal of the ground state level by a one-step SUSY transformation leads to a new potential which differs from the initial potential only by reshaped parameter values. Hence, on the one hand, its propagator is just the same initial propagator taken at the reshaped parameter values and, on the other hand, it can be obtained by applying theorem 2. This will result in some integro-differential identities for propagators of shape invariant potentials and one can expect a deeper insight in the properties of these propagators.

Since propagators may also be defined in terms of Feynman path integrals, the results obtained in the present paper will be useful in exploring new classes of path integrals for exactly solvable models derived with the help of supersymmetric quantum mechanics. Due to the very wide sphere of applications of path integrals which comprises mathematical physics [32, 33], quantum field theory [33], quantum mechanics, polymer physics and statistics and reaches as far as financial markets [33] as well as to numerous SUSY QM-related applications [2], which unexpectedly may be seen in, e.g., quantum dot phenomena [6] and dynamo processes of magnetohydrodynamics [31], we hope that our general method may become a useful tool in intersections of these fields. In particular, we think that transparent potentials, with their propagators carefully studied in section 5.3, may be useful in modelling of N-level atoms placed in a strong laser fields. Such a possibility is based on the recently discovered (pseudo-)supersymmetry of two level systems [35] and the recent work [36] where propagators have been used to study atom–laser interactions.

Acknowledgments

The work has been partially supported by the grants RFBR-06-02-16719 (BFS and AMP), SS-5103.2006.2 (BFS), by the Russian Dynasty Foundation (AMP) and by the German Research Foundation DFG, grant GE 682/12-3 (UG).

Appendix

Here we derive a representation of the unphysical auxiliary solutions $u_{N,n}$, $n = 0, \ldots, N-1$, of the h_N-Schrödinger equation in terms of Wronskian fractions. We start by recalling that the functions $u_{N,n}$ are defined by acting with an Nth-order (polynomial) SUSY-transformation operator $L_{N,0}$ on the unphysical solutions $\widetilde{u}_{0,n}$ of the Schrödinger equation with Hamiltonian h_0 at energies $E_n = \alpha_n$. The functions are linearly independent from their physical counterparts $u_{0,n}$ and normalized as $W(u_{0,n}, \widetilde{u}_{0,n}) = 1$. The operator $L_{N,0}$ itself is constructed from $u_{0,n}$, $n = 0, \ldots, N-1$, as transformation functions.

Below we show by induction that the unphysical solutions $u_{N,0}$ defined in (42) as

$$u_{N,n} = L_{N,0}\widetilde{u}_{0,n} = L_{N,0}u_{0,n}\int_{x_0}^{x}\frac{\mathrm{d}y}{u_{0,n}^2(y)}$$

have the following representation in terms of Wronskian fractions

$$u_{N,n} = C_{N,n}\frac{W_n(u_{0,0}, \ldots, u_{0,N-1})}{W(u_{0,0}, \ldots, u_{0,N-1})},$$
$$C_{N,n} := (E_{N-1} - E_n)(E_{N-2} - E_n)\cdots(E_{n+1} - E_n), \qquad n = 0, \ldots, N-2, \tag{A.1}$$

$$u_{N,N-1} = \frac{W_{N-1}(u_{0,0}, \ldots, u_{0,N-1})}{W(u_{0,0}, \ldots, u_{0,N-1})} = \frac{W(u_{0,0}, \ldots, u_{0,N-2})}{W(u_{0,0}, \ldots, u_{0,N-1})}. \tag{A.2}$$

For $N = 1$, the first-order transformation operator $L_{1,0}$ is constructed with the help of $u_{0,0}$. Therefore, applying (9) with $f = \widetilde{u}_{0,0}$ yields

$$u_{1,0} = L_{1,0}\widetilde{u}_{0,0} = \frac{1}{u_{0,0}}$$

which obviously agrees with the statement. Applying (9) with $f = u_{0,1}$, we obtain

$$u_{1,1} = L_{1,0}u_{0,1} = \frac{W(u_{0,0}, u_{0,1})}{u_{0,0}}. \tag{A.3}$$

In order to prove the statement for $N = 2$, we build the linear SUSY-operator $L_{2,1}$ from $u_{1,1}$ and act with it on the unphysical solutions $\widetilde{u}_{1,1}$ and $u_{1,0}$. As a first result, we obtain

$$u_{2,1} = L_{2,1}\widetilde{u}_{1,1} = \frac{1}{u_{1,1}} = \frac{u_{0,0}}{W(u_{0,0}, u_{0,1})},$$

where (9) and (A.3) have been used. For the second function, we find

$$\begin{aligned} u_{2,0} &= L_{2,1}u_{1,0} = L_{2,1}\frac{1}{u_{0,0}} = \frac{1}{u_{0,0}u_{1,1}}L_{1,0}^{+}u_{1,1} = \frac{1}{u_{0,0}u_{1,1}}L_{1,0}^{+}L_{1,0}u_{0,1} \\ &= (E_1 - E_0)\frac{1}{u_{0,0}u_{1,1}}u_{0,1} = (E_1 - E_0)\frac{u_{0,1}}{W(u_{0,0}, u_{0,1})}. \end{aligned} \tag{A.4}$$

Here, the operator $L_{2,1} = -u_{1,1x}/u_{1,1} + \partial_x$ has been replaced by $L_{1,0}^{+} = -u_{0,0x}/u_{0,0} - \partial_x$ with the help of $(1/u_{0,0})' = -u_{0,0}'/u_{0,0}^2$. Moreover, we have used (A.3), the factorization rule $L_{1,0}^{+}L_{1,0} = h_0 - E_0$ and the Schrödinger equation $h_0u_{0,1} = E_1u_{0,1}$.

Assuming finally that the representations (A.1) and (A.2) are valid for $u_{N,n}$ and $u_{N,N-1}$ we prove them for $u_{N+1,n}$ and $u_{N+1,N}$.

We start with $u_{N,N-1} \mapsto u_{N+1,N}$. To go from h_N to h_{N+1} only the linear (one-step) SUSY transformation $L_{N+1,N}$ is required. Applying (9) and combining it with the normalization condition $W(u_{N,N}, \widetilde{u}_{N,N}) = 1$ and the Crum–Krein formula (7) [or (45)], we find the equivalence chain

$$u_{N+1,N} = L_{N+1,N}\widetilde{u}_{N,N} = \frac{W(u_{N,N}, \widetilde{u}_{N,N})}{u_{N,N}} = \frac{1}{u_{N,N}} = \frac{W(u_{0,0}, \ldots, u_{0,N-1})}{W(u_{0,0}, \ldots, u_{0,N})}.$$

Comparison with (A.2) shows that the proof is done.

The proof of the induction $u_{N,n} \mapsto u_{N+1,n}$ is less obvious. In section 4.2 the linear intertwiners $L_{m+1,m}$ have been build strictly incrementally from the corresponding ground state eigenfunctions $u_{m,m}$ (of the Hamiltonians h_m) as transformation functions. Here, we need a more general non-incremental construction scheme. In order to facilitate it, we first introduce a very detailed notation for general polynomial intertwiners $L_{k,0}$ indicating explicitly the energy levels of the transformation functions from which they are built. Based on the Crum–Krein formula (7), we set

$$L_{k,0}^{(a_1,a_2,\ldots,a_k)} f := \frac{W(u_{0,a_1}, u_{0,a_2}, \ldots, u_{0,a_k}, f)}{W(u_{0,a_1}, u_{0,a_2}, \ldots, u_{0,a_k})}, \qquad a_i \neq a_{j\neq i}, \tag{A.5}$$

with $a_i \in \mathbb{Z}^+$ being any energy level numbers of the Hamiltonian h_0. The determinant structure of (A.5) immediately implies the generalized kernel property

$$L_{k,0}^{(a_1,\ldots,n,\ldots,a_k)} u_{0,n} = 0 \tag{A.6}$$

and the invariance of the operator $L_{k,0}^{(a_1,a_2,\ldots,a_k)}$ with regard to permutations $(a_1, a_2, \ldots, a_k) \mapsto \sigma(a_1, a_2, \ldots, a_k)$

$$L_{k,0}^{\sigma(a_1,a_2,\ldots,a_k)} = L_{k,0}^{(a_1,a_2,\ldots,a_k)}. \tag{A.7}$$

Recalling that (A.5) can be built from a chain $L_{k,k-1}^{(a_k)} L_{k-1,k-2}^{(a_{k-1})} \cdots L_{1,0}^{(a_1)}$ of linear intertwiners $L_{m+1,m}^{(a_m)}$, we conclude that the ordering with regard to energy levels is inessential in the transformation chain and that we can split it into sub-chains with any permuted combination of levels

$$\begin{aligned} &\sigma(a_1, \ldots, a_k) = (b_1, \ldots, b_B, c_1, \ldots, c_C), \qquad B + C = k, \\ &L_{k,0}^{(a_1,\ldots,a_k)} = L_{B+C,B}^{(c_1,\ldots,c_C)} L_{B,0}^{(b_1,\ldots,b_B)} = L_{C+B,C}^{(b_1,\ldots,b_B)} L_{C,0}^{(c_1,\ldots,c_C)}. \end{aligned} \tag{A.8}$$

Apart from this full commutativity of the transformations we note their associativity

$$L_{m+3,m+2}^{(a_3)}\big(L_{m+2,m+1}^{(a_2)} L_{m+1,m}^{(a_1)}\big) = \big(L_{m+3,m+2}^{(a_3)} L_{m+2,m+1}^{(a_2)}\big) L_{m+1,m}^{(a_1)}. \tag{A.9}$$

Commutativity and associativity can be used to re-arrange a sequence of transformations in any required order.

In accordance with the intertwiners L, we denote eigenfunctions as

$$u_{k,n}^{(a_1,a_2,\ldots,a_k)} := L_{k,0}^{(a_1,a_2,\ldots,a_k)} u_{0,n}, \qquad a_i \neq n, \quad i = 1, \ldots, k. \tag{A.10}$$

Another ingredient that we need is the representation

$$\begin{aligned} \frac{W(u_{0,0}, \ldots, u_{0,N-1})}{W_n(u_{0,0}, \ldots, u_{0,N-1})} &= (-1)^{N-1-n} \frac{W(u_{0,0}, \ldots, u_{0,n-1}, u_{0,n+1}, \ldots, u_{0,N-1}, u_{0,n})}{W_n(u_{0,0}, \ldots, u_{0,N-1})} \\ &= (-1)^{N-1-n} L_{N-1,0}^{(0,\ldots,n-1,n+1,\ldots,N-1)} u_{0,n} \\ &= (-1)^{N-1-n} u_{N-1,n}^{(0,\ldots,n-1,n+1,\ldots,N-1)} =: v_{N-1,n} \end{aligned} \tag{A.11}$$

which immediately follows from the Crum–Krein formula (A.5) and definition (A.10). From the physical solution $v_{N-1,n}$ we build the operators[18]

$$L^{(n)}_{N,N-1} := L^{(n)}_{N,N-1}[v_{N-1,n}] = -\frac{v_{N-1,n,x}}{v_{N-1,n}} + \partial_x$$
$$L^{(n)+}_{N,N-1} := L^{(n)+}_{N,N-1}[v_{N-1,n}] = -\frac{v_{N-1,n,x}}{v_{N-1,n}} - \partial_x \tag{A.12}$$

and the corresponding Hamiltonian $h^{(0,\ldots,n-1,n+1,\ldots,N-1)}_{N-1}$ with

$$L^{(n)+}_{N,N-1}L^{(n)}_{N,N-1} = h^{(0,\ldots,n-1,n+1,\ldots,N-1)}_{N-1} - E_n. \tag{A.13}$$

We start the proof of the induction $u_{N,n} \mapsto u_{N+1,n}$ with the following transformations

$$\begin{aligned} u_{N+1,n} &= L^{(N)}_{N+1,N}u_{N,n} = C_{N,n}L^{(N)}_{N+1,N}\frac{W_n(u_{0,0},\ldots,u_{0,N-1})}{W(u_{0,0},\ldots,u_{0,N-1})} \\ &= C_{N,n}L^{(N)}_{N+1,N}v^{-1}_{N-1,n} \\ &= C_{N,n}\left[-\frac{u_{N,N,x}}{u_{N,N}v_{N-1,n}} - \frac{v_{N-1,n,x}}{v^2_{N-1,n}}\right] \\ &= C_{N,n}\frac{1}{u_{N,N}v_{N-1,n}}L^{(n)+}_{N,N-1}u_{N,N}. \end{aligned} \tag{A.14}$$

In order to obtain the operator product $L^{(n)+}_{N,N-1}L^{(n)}_{N,N-1}$ we use (45), the composition rule (A.8) and the Crum–Krein formula (A.5)

$$\begin{aligned} u_{N,N} &= L^{(0,1,\ldots,N-1)}_{N,0}u_{0,N} = \frac{W(u_{0,0},\ldots,u_{0,N})}{W(u_{0,0},\ldots,u_{0,N-1})} \\ &= L^{(n)}_{N,N-1}L^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,0}u_{0,N} \\ &= L^{(n)}_{N,N-1}u^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,N}, \end{aligned} \tag{A.15}$$

where

$$\begin{aligned} u^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,N} &= L^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,0}u_{0,N} \\ &= \frac{W_n(u_{0,0},\ldots,u_{0,N})}{W_n(u_{0,0},\ldots,u_{0,N-1})}. \end{aligned} \tag{A.16}$$

This gives

$$\begin{aligned} u_{N+1,n} &= C_{N,n}\frac{1}{u_{N,N}v_{N-1,n}}L^{(n)+}_{N,N-1}u_{N,N} \\ &= C_{N,n}\frac{1}{u_{N,N}v_{N-1,n}}L^{(n)+}_{N,N-1}L^{(n)}_{N,N-1}u^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,N} \\ &= C_{N,n}\frac{1}{u_{N,N}v_{N-1,n}}(E_N - E_n)u^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,N} \\ &= C_{N+1,n}\frac{W_n(u_{0,0},\ldots,u_{0,N})}{W(u_{0,0},\ldots,u_{0,N})}, \end{aligned} \tag{A.17}$$

where the last line has been obtained by expressing $u^{(0,1,\ldots,n-1,n+1,\ldots,N-1)}_{N-1,N}$, $u_{N,N}$ and $v_{N-1,n}$ via (A.16), (45) and (A.11), respectively, in terms of their Wronskian fractions. With (A.17) the proof is complete.

[18] For $C \neq 0$ holds $L^{(n)}_{N,N-1}[Cv_{N,n}] = L^{(n)}_{N,N-1}[v_{N,n}]$ so that the sign factor $(-1)^{N-1-n}$ plays no role in the operator $L^{(n)}_{N,N-1}$ itself.

References

[1] Grosche C and Steiner F 1998 *Handbook of Feynman Path Integrals* (Berlin: Springer)

[2] Sukumar C V 1985 Supersymmetric quantum mechanics of one-dimensional systems *J. Phys. A: Math. Gen.* **18** 2917–36

Cooper F, Khare A A and Sukhatme U 1995 Supersymmetry and quantum mechanics *Phys. Rep.* **251** 267–85

Junker G 1996 *Supersymmetry Method in Quantum and Statistical Method* (Berlin: Springer)

Bagchi B 2000 *Supersymmetry in Quantum and Classical Mechanics* (New York: Chapman and Hall)

Mielnik B and Rosas-Ortiz O 2004 Factorization: little or great algorithm? *J. Phys. A: Math. Gen.* **37** 10007–35

[3] Bagrov V G and Samsonov B F 1995 Darboux transformation, factorization and supersymmetry in one-dimensional quantum mechanics *Theor. Math. Phys.* **104** 1051–60

Bagrov V G and Samsonov B F 1997 Darboux transformation of the Schrödinger equation *Phys. Part. Nucl.* **28** 374–97

[4] Samsonov B F and Ovcharov I N 1995 Darboux transformations and non-classical orthogonal polynomials *Russ. Phys. J.* **38/4** 58–65

[5] Aref'eva I, Fernández D J, Hussin V, Negro J, Nieto L M and Samsonov B F 2004 Progress in supersymmetric quantum mechanics *J. Phys. A: Math. Gen.* **37** 10007–458 (special issue)

[6] Pershin Yu V and Samsonov B F 2005 Quantum dots created through spherically polarized nuclear spins *Phys. E: Low-Dimens. Syst. Nanostruct.* **28** 134–40 (*Preprint* cond-mat/0401373)

[7] Jauslin H R 1988 Exact propagator and eigenfunctions for multistable models with arbitrarily prescribed *N* lowest eigenvalues *J. Phys. A: Math. Gen.* **21** 2337–50

[8] Samsonov B F, Sukumar C V and Pupasov A M 2005 SUSY transformation of the Green function and a trace formula *J. Phys. A: Math. Gen.* **38** 7557–65 (*Preprint* quant-ph/0507160)

[9] Samsonov B F and Pupasov A M 2005 Darboux transformation of the Green's function of a regular Sturm–Liouville problem *Russ. Phys. J.* **48/10** 1020–8

[10] Samsonov B F and Pupasov A M 2006 Exact propagators for complex SUSY partners of real potentials *Phys. Lett.* A **356** 210–4 (*Preprint* quant-ph/0602218)

Pupasov A M and Samsonov B F 2005 Exact propagators for soliton potentials *Symmetry Integr. Geom.: Methods Appl. (SIGMA)* **1** 20–7 (*Preprint* quant-ph/0511238)

[11] Fernández D J C and Fernández-García N 2005 Higher order supersymmetric quantum mechanics *AIP Conf. Proc.* **744** 236–73 (*Preprint* quant-ph/0502098)

[12] Andrianov A A and Cannata F 2004 Nonlinear supersymmetry for spectral design in quantum mechanics *J. Phys. A: Math. Gen.* **37** 10297–321 (*Preprint* hep-th/0407077)

[13] Witten E 1981 Dynamical breaking of supersymmetry *Nucl. Phys.* B **185** 513–54

Witten E 1982 Constraints on supersymmetry breaking *Nucl. Phys.* B **202** 253–316

[14] Andrianov A A, Ioffe M V and Spiridonov V P 1993 Higher-derivative supersymmetry and Witten index *Phys. Lett.* A **174** 273–9 (*Preprint* hep-th/9303005)

[15] Kostyuchenko A G and Sargsyan I S 1979 *Distribution of Eigenvalues. Selfadjoint Ordinary Differential Operators* (Moscow: Nauka)

[16] Levitan B M and Sargsjan I S 1975 *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators (Translations of Mathematical Monographs vol 39)* (Providence, RI: American Mathematical Society)

Levitan B M 1984 *Inverse Sturm–Liouville Problems* (Moscow: Nauka)

[17] Berezin F A and Shubin M A 1991 *The Schrödinger Equation* (Dordrecht: Kluwer)

[18] Crum M 1955 Associated Sturm–Liouville systems *Quart. J. Math., Ser. 2* **6** 121–7

[19] Krein M G 1957 On a continual analogue of the Christoffel formula from the theory of orthogonal polynomials *Dokl. Akad. Nauk SSSR* **113** 970–3

[20] Andrianov A A and Sokolov A V 2006 Factorization of nonlinear supersymmetry in one-dimensional Quantum Mechanics I: general classification of reducibility and analysis of third order algebra *Zapiski Nauchnyh Seminarov POMI* **335** 22–49

[21] Samsonov B F 1999 New possibilities for supersymmetry breakdown in quantum mechanics and second-order irreducible Darboux transformations *Phys. Lett.* A **263** 274–80 (*Preprint* quant-ph/9904009)

Bagrov V G and Samsonov B F 2002 On irreducible second-order Darboux transformations *Russ. Phys. J.* **45** 27–33

[22] Samsonov B F 2005 SUSY transformations between digonalizable and non-diagonalizable Hamiltonians *J. Phys. A: Math. Gen.* **38** L397–403 (*Preprint* quant-ph/0503075)

[23] Samsonov B F 1996 New features in supersymmetry breakdown in quantum mechanics *Mod. Phys. Lett.* A **11** 1563–7 (*Preprint* quant-ph/9611012)

[24] Morse P and Feshbach H 1953 *Methods of Theoretical Physics* vol 1 (New York: McGraw-Hill)

[25] Gelfand I M and Shilov G E 1964 *Generalized Functions* vol 1–3 (New York: Academic)
Gelfand I M and Vilenkin N Ya 1964 *Generalized Functions* vol 4 (New York: Academic)
[26] Dodonov V V and Man'ko V I 1987 Evolution of multidimensional systems. Magnetic properties of ideal gases of charged particles *Proc. Phys. Lebedev Inst. (Trudy FIAN)* **183** 182–286
[27] Erdélyi A 1953 *Higher Trascendental Functions* (New York: McGraw-Hill)
[28] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)
[29] Prudnikov A P, Brychkov Yu A and Marichev O I 1981 *Integrals and Series* vol 1 (Moscow: Nauka)
[30] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards)
[31] Günther U, Samsonov B F and Stefani F 2007 A globally diagonalizable α^2-dynamo operator, SUSY QM and the Dirac equation *J. Phys. A: Math. Theor.* **40** F169–76 (*Preprint* math-ph/0611036)
[32] Simon B 1979 *Functional Integration and Quantum Physics* (New York: Academic)
[33] Masujima M 2000 *Path Integral Quantization and Stochastic Quantization (Springer Tracts in Modern Physics vol 165)* (Berlin: Springer)
[34] Kleinert H 1990 *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets* (Singapore: World Scientific)
[35] Samsonov B F and Shamshutdinova V V 2005 Quadratic pseudosupersymmetry in two-level systems *J. Phys. A: Math. Gen.* **38** 4715–25 (*Preprint* quant-ph/0504065)
Shamshutdinova V V, Samsonov B F and Gitman D M 2007 Two-level systems: exact solutions and underlying pseudosupersymmetry *Ann. Phys. (NY)* **322** 1043–61 (*Preprint* quant-ph/0606195)
[36] del Campo A and Muga J G 2006 Exact propagators for atom–laser interactions *J. Phys. A: Math. Gen.* **45** 14079–88 (*Preprint* quant-ph/0607079)