

# Dynamics of charged fluids and $1/\ell$ perturbation expansions

Miloslav Znojil<sup>1</sup> and Uwe Günther<sup>2</sup>

<sup>1</sup> Nuclear Physics Institute ASCR, 250 68 Řež, Czech Republic

<sup>2</sup> Research Center Dresden-Rossendorf, POB 510119, D-01314 Dresden, Germany

E-mail: [znojil@ujf.cas.cz](mailto:znojil@ujf.cas.cz) and [u.guenther@fzd.de](mailto:u.guenther@fzd.de)

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## Abstract

Some features of the calculation of fluid dynamo systems (spherically symmetric  $\alpha^2$ -dynamos) in magnetohydrodynamics are studied, the problem connected with the presence of mixed (Robin) boundary conditions is addressed and a new treatment for it is proposed. The perturbation formalism of large- $\ell$  expansions is shown to be applicable and its main technical steps are outlined.

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## 1. Introduction

The magnetohydrodynamics of conductive fluids plays an important role in the explanation of the existence and stability of the magnetic field of the Earth as well as of the magnetic fields of stars and galaxies studied in astrophysics [1–15]. In the theoretical description of the motion of such fluids an important role is played by the induction of magnetic fields which are able to generate a genuine global dynamo effect [16]. For certain spherically symmetric field configurations after a mean field approximation a so-called  $\alpha^2$ -dynamo model can be obtained [2]. Its description may be reduced [2] to the coupled pair of ordinary differential phenomenological equations [17, 18]

$$-\partial_r^2 \phi(r) + V_u(r)\phi(r) - \alpha(r)\chi(r) = -\lambda\phi(r), \quad (1)$$

$$-\partial_r^2 \chi(r) + V_d(r)\chi(r) + \partial_r \alpha(r)\partial_r \phi(r) - V_m(r)\phi(r) = -\lambda\chi(r). \quad (2)$$

They are defined on a finite interval of a single coordinate  $r \in (0, R)$  with  $R = 1$  after a rescaling. For the purely kinematic factors one has to set  $V_u(r) = V_d(r) = V_m(r)/\alpha(r) = \ell(\ell + 1)/r^2$  where the integer parameter  $\ell = 0, 1, \dots$  coincides with the angular mode number of the field. The only input information about the flow of the charged fluid (or plasma) is carried by the shape of the so-called  $\alpha$ -profile  $\alpha(r)$ .

Let us now formulate our present task as the construction of the solutions of equations (1) and (2) specified by realistic physical boundary conditions

$$\phi(0) = 0, \quad [\partial_r \phi(r)]_{r=R} + \frac{\ell}{R} \phi(R) = 0, \quad (3)$$

$$\chi(0) = \chi(R) = 0. \quad (4)$$

The presentation of our results will start by an outline of consequences of the difference between the latter two boundary conditions in section 2. In section 3 we show how this difference diminishes in proportion to the quantity  $1/\ell$ . This, of course, indicates the possible applicability of a standard perturbation expansion in this parameter. After a few introductory technical remarks made on such an approach in the separate appendix we demonstrate, in the next section 4, that the parameter  $1/\ell$  can really play the role of a measure of perturbation of the solvable  $\ell \rightarrow \infty$  limit. In the subsequent sections 5 and 6 we then describe the explicit realization of such a programme in more detail. Finally, section 7 summarizes all the key ingredients of our present new method of solution of the magnetohydrodynamical  $\alpha^2$ -dynamo problem by the asymptotic series in terms of certain rational powers of  $1/\ell$ .

## 2. The dynamo problem in two-space formulation

There exist several formal analogies of the dynamo problem with quantum mechanics [7, 9–11]. We intend to employ here some of these analogies as a methodical guide. A deeper discussion of this aspect of the problem can be found in [18–20]. Nevertheless, one should keep in mind that these analogies are violated, first of all, by the presence of mixed-type boundary conditions.

### 2.1. The doublet of bases

The most challenging mathematical feature of the physical boundary conditions (3) and (4) is that they prescribe the use of *different* spaces, say,  $\mathcal{V}_u$  and  $\mathcal{V}_d$  for the respective channel functions  $\phi(r)$  (representing the so-called poloidal mode of the magnetic field) and  $\chi(r)$  (representing the complementary, toroidal mode). In a compactified Dirac's bra-ket notation this means that one is forced to employ different symbols for elements of each of these spaces. Let us employ the usual vector or ket-symbol  $|\chi\rangle \in \mathcal{V}_d$  in place of the function  $\chi(r)$  emphasizing that for this Hilbert-space element the Dirichlet boundary condition (4) at  $r = R = 1$  is entirely standard. A modified abbreviation  $|\phi\rangle \in \mathcal{V}_u$  will be introduced for  $\phi(r)$  which obeys the less standard (mixed, Robin) boundary condition (3) containing the derivative.

In the new notation one may introduce orthonormalized bases of eigenvectors resulting from the spectral representations of self-adjoint differential operators in any of the two vector spaces  $\mathcal{V}_u$  and  $\mathcal{V}_d$ . We shall postulate

$$-\partial_r^2 + V_u(r) = \sum_{m=0}^{\infty} |m\rangle \tau_m \langle m|, \quad (5)$$

$$-\partial_r^2 + V_d(r) = \sum_{n=0}^{\infty} |n\rangle \varrho_n \langle n|, \quad (6)$$

knowing that due to the simplicity of  $V_{u,d}(r)$  and due to the coincidence of the boundary conditions in the origin, all the basis functions  $\phi_n(r) = \langle r|n\rangle$  as well as  $\chi_n(r) = \langle r|n\rangle$  must be

proportional to Bessel special functions ([21] may be consulted for more details). The basis states have the same closed form with different scaling,

$$\varphi_m(r) = M_m r^{1/2} J_{\ell+1/2}(\sqrt{\tau_m} r), \quad \chi_n(r) = N_n r^{1/2} J_{\ell+1/2}(\sqrt{\varrho_n} r). \quad (7)$$

The kinetic operator eigenvalues  $\tau_p$  and  $\varrho_q$  are given as roots of the appropriate combinations of Bessel functions at  $r = R = 1$ . Their numerical values are available with arbitrary precision, therefore.

More care must be paid to the operators which couple the channels. Once they represent a map between  $\mathcal{V}_u$  and  $\mathcal{V}_d$  we can only employ a non-diagonal, full-matrix formula for the operator of the multiplication by the function  $\alpha(r)$  in equation (1),

$$\alpha(r) = \sum_{k,j=0}^{\infty} |k\rangle \{k|\alpha(r)|j\rangle \langle j|. \quad (8)$$

The technique of the evaluation of the matrix elements  $\{k|\alpha(r)|j\rangle$  (by integration) is standard. *Mutatis mutandis*, the same comment applies to the other channel-coupling operator

$$\partial_r \alpha(r) \partial_r - V_m(r) = - \sum_{k,j=0}^{\infty} |k\rangle \omega_{k,j} \langle j| \quad (9)$$

in equation (2).

## 2.2. Linear algebraic form of the coupled equations

The spectral series (5) and (6) as well as the full-matrix formulae (8) and (9) should be inserted in the coupled set of differential equations (1) and (2). Using the pair of the natural ansätze

$$|\phi\rangle = \sum_{k=0}^{\infty} |k\rangle \phi_k, \quad |\chi\rangle = \sum_{j=0}^{\infty} |j\rangle \chi_j, \quad (10)$$

we get the set of relations

$$\sum_{m=0}^{\infty} |m\rangle (\tau_m + \lambda) \phi_m = \sum_{k,j=0}^{\infty} |k\rangle \{k|\alpha(r)|j\rangle \chi_j, \quad (11)$$

$$\sum_{k,j=0}^{\infty} |k\rangle \omega_{k,j} \phi_j = \sum_{n=0}^{\infty} |n\rangle (\varrho_n + \lambda) \chi_n. \quad (12)$$

Once we assume the completeness of both orthonormalized basis sets  $\{|p\rangle\}_{p=0}^{\infty}$  and  $\{|q\rangle\}_{q=0}^{\infty}$  spanning respective infinite-dimensional vector spaces  $\mathcal{V}_u$  and  $\mathcal{V}_d$  we are allowed to multiply the previous pair of equations by the respective bra-vectors  $\langle p|$  and  $\langle q|$  from the left. This leads to the final formulation of the dynamo problem in the form of the infinite set of linear algebraic equations,

$$(\tau_p + \lambda) \phi_p = \sum_{j=0}^{\infty} \langle p|\alpha(r)|j\rangle \chi_j, \quad p = 0, 1, \dots, \quad (13)$$

$$\sum_{j=0}^{\infty} \omega_{q,j} \phi_j = (\varrho_q + \lambda) \chi_q, \quad q = 0, 1, \dots \quad (14)$$

These equations may only be solved numerically (in this case it makes sense to start from the elementary elimination, say, of all the unknown quantities  $\phi_p$  using equation (13)) or perturbatively (in the latter case it seems better to preserve the linearity of the whole set in  $\lambda$ ).

### 2.3. The simplified model with constant $\alpha(r)$

Both operators (8) and (9) become perceivably simpler for constant  $\alpha$ -profiles. It allows us to simplify equation (11),

$$\sum_{k=0}^{\infty} |k\rangle (\tau_k + \lambda) \phi_k = \alpha_0 \sum_{k,j=0}^{\infty} |k\rangle \langle k|j\rangle \chi_j.$$

In parallel, a factorization of the matrix  $\omega_{kj}$  in equation (9),

$$\partial_r \alpha_0 \partial_r - \alpha_0 \ell(\ell + 1)/r^2 = -\alpha_0 \sum_{k,j=0}^{\infty} |k\rangle \langle k|j\rangle \tau_j \langle j|$$

converts equation (12) into the simpler relation

$$\alpha_0 \sum_{n,m=0}^{\infty} |n\rangle \langle n|m\rangle \tau_m \phi_m = \sum_{n=0}^{\infty} |n\rangle (\varrho_n + \lambda) \chi_n.$$

As a result one gets the pair of coupled linear algebraic equations

$$(\tau_k + \lambda) \phi_k = \alpha_0 \sum_{j=0}^{\infty} \langle k|j\rangle \chi_j, \quad (15)$$

$$\alpha_0 \sum_{m=0}^{\infty} \langle k|m\rangle \tau_m \phi_m = (\varrho_k + \lambda) \chi_k, \quad k = 0, 1, \dots \quad (16)$$

which replaces equations (13) and (14) at  $\alpha(r) = \alpha_0$ . For numerical purposes we may easily eliminate one of these relations and get the single Feshbach-type set for  $k = 0, 1, \dots$ ,

$$\sum_{m=0}^{\infty} \left[ (\tau_k + \lambda) \delta_{k,m} - \alpha_0^2 \sum_{j=0}^{\infty} \langle k|j\rangle \frac{1}{(\varrho_j + \lambda)} \langle j|m\rangle \tau_m \right] \phi_m = 0 \quad (17)$$

or, alternatively,

$$\sum_{m=0}^{\infty} \left[ (\varrho_k + \lambda) \delta_{k,m} - \alpha_0^2 \sum_{j=0}^{\infty} \langle k|j\rangle \frac{\tau_j}{(\tau_j + \lambda)} \langle j|m\rangle \right] \chi_m = 0, \quad (18)$$

i.e., after a slight modification,

$$\sum_{m=0}^{\infty} \left[ (\varrho_k + \lambda - \alpha_0^2) \delta_{k,m} + \alpha_0^2 \sum_{j=0}^{\infty} \langle k|j\rangle \frac{\lambda}{(\tau_j + \lambda)} \langle j|m\rangle \right] \chi_m = 0.$$

We see that the knowledge of the single matrix of overlaps  $\langle k|j\rangle$  is the only input needed for the standard numerical solution of these equations. Unfortunately, the latter observation is in fact of no immediate impact upon applications since the choice of the constant  $\alpha(r) = \alpha_0 \neq 0$  is not too realistic [1, 2]. Even in a more formal evaluation it seems oversimplified as it allows many specific relations between its basis states (7). One should note that the eigenvalues even become available as roots of a certain quadratic superposition of Bessel functions as a consequence [2]. In this sense our toy problem  $\alpha(r) = \alpha_0$  remains exactly solvable. Nevertheless, it deserves full attention as a very useful methodical guide to more realistic situations. Our present ambition will concentrate upon the proposal and description of its new, promising and fairly natural *perturbative* treatment.

### 3. The confluence of bases in spaces $\mathcal{V}_u$ and $\mathcal{V}_d$ at $\ell \gg 1$

There exist several important differences between the present eigenvalue problem (13) and (14) and its quantum-mechanical coupled-channel analogues [22]. First, one should note the difference in the sign convention (the energies of quantum mechanics would be  $E = -\lambda$ ). Second, the present eigenvalue problem is not self-adjoint in the form which would be usual in quantum mechanics (consult again [21] for a deeper discussion of this aspect). Third, one usually does not encounter Robin boundary conditions in quantum mechanics. In this sense our present problem is perceptibly more complicated.

#### 3.1. Boundary conditions

A formal key to the simplification of equations (13) and (14) can be sought in a *decrease* of the difference between the Robin and Dirichlet boundary conditions (3) and (4) at  $r = R = 1$  with the growth of  $\ell$ . In a way which significantly weakens the above-mentioned constant-profile assumption  $\alpha(r) = \alpha_0$  let us merely assume now that  $\alpha(r)$  remains more or less constant in a small vicinity of  $R = 1$ . In this case with  $r \approx R$ , equations (1) and (2) acquire the following approximate form,

$$\begin{aligned} -\partial_r^2 \phi(r) + (\kappa^2 + \lambda)\phi(r) - \alpha(R)\chi(r) &= 0, & \kappa^2 &= \frac{\ell(\ell + 1)}{R^2}, \\ -\partial_r^2 \chi(r) + (\kappa^2 + \lambda)\chi(r) + \alpha(R)\partial_r^2 \phi(r) - \alpha(R)\kappa^2 \phi(r) &= 0, & r &\approx R. \end{aligned} \tag{19}$$

This system (with constant coefficients) possesses eigenfrequencies  $\mu = \mu[\lambda, \kappa, \alpha(R)]$  obtainable directly from the corresponding characteristic equation

$$(\mu^2 + \kappa^2 + \lambda)^2 = \alpha^2(R)(\mu^2 + \kappa^2).$$

In the light of boundary conditions at  $r = R$  we may expect that  $\chi(r) = D \sin \mu(r - R)$  and  $\phi(r) = C \sin \mu(r - S)$  where  $S \approx R$  is unknown and where  $R = 1$ . An estimate of the value of  $S$  may be deduced via insertion from equation (3),

$$\mu \cos \mu(R - S) + \frac{\ell}{R} \sin \mu(R - S) = 0. \tag{20}$$

For all  $\ell > 0$  this enables us to conclude that  $S > R = 1$  so that  $\tau_k < \varrho_k$  at  $k = 0, 1, \dots$

At very large angular excitations the following explicit frequency-independent estimate results from equation (20),

$$S = 1 + \frac{1}{\ell} + \mathcal{O}\left(\frac{1}{\ell}\right)^2. \tag{21}$$

We see that the two boundary conditions (3) and (4) and, hence, also the two bases spanning the spaces  $\mathcal{V}_u$  and  $\mathcal{V}_d$  will coincide in the limit  $\ell \rightarrow \infty$ ,

$$\lim_{\ell \rightarrow \infty} \tau_n = \varrho_n, \quad \lim_{\ell \rightarrow \infty} |n\rangle \rightarrow |n\rangle, \quad n = 0, 1, \dots \tag{22}$$

This is an important observation. From the practical point of view it means that we shall have

$$\lim_{\ell \rightarrow \infty} \langle q|n\rangle = \delta_{q,n} \quad \lim_{\ell \rightarrow \infty} \langle q|\alpha(r)|n\rangle = \langle q|\alpha(r)|n\rangle \tag{23}$$

for all the indices  $q \geq 0$  and  $n \geq 0$  and for all reasonable functions  $\alpha(r)$ .

### 3.2. Illustration: exact solvability of the $\ell \gg 1$ model at a constant $\alpha$

As the only dynamical input in equations (1) and (2) the  $\alpha$ -profile function is a key to all the practical and phenomenological considerations. At the same time, all its not too large deviations from a constant mean  $\alpha$ -profile  $\alpha_0$  may be treated perturbatively in the way outlined in [21].

Let us now turn attention to the unperturbed problem with  $\alpha(r) = \alpha_0$  where we add as another assumption that  $\ell \gg 1$ . Due to the limit of the orthogonality rule (23) the resulting doubly simplified algebraic equations (16) become completely decoupled. At every index  $k$  the condition of vanishing secular determinant remains trivial,

$$\det \begin{pmatrix} \tau_k + \lambda & -\alpha_0 \\ -\alpha_0 \tau_k & \varrho_k + \lambda \end{pmatrix} \approx 0, \quad \tau_k \approx \varrho_k, \quad \ell \gg 1$$

and its closed solution exists,

$$\lambda_{1,2} = \frac{1}{2}[-\tau_k - \varrho_k \pm \sqrt{(\tau_k - \varrho_k)^2 + 4\varrho_k \alpha_0^2}]. \quad (24)$$

In the normalization  $\phi_k = \alpha_0$  the closed form of the coefficients is equally elementary,

$$(\chi_k)_{1,2} = \tau_k + \lambda_{1,2} = \frac{1}{2}[\tau_k - \varrho_k \pm \sqrt{(\tau_k - \varrho_k)^2 + 4\varrho_k \alpha_0^2}]. \quad (25)$$

As long as we have  $\tau_k \approx \varrho_k$ , the latter recipe reproduces exactly the definition  $\lambda = \lambda_k^\pm = -\varrho_k \pm \alpha_0 \sqrt{\varrho_k}$  of the spectrum as obtained earlier under an alternative assumption of the high conductivity [6, 12] of the dynamo's fluid (cf equation (13) in [21]).

## 4. Effective simplifications of boundary conditions

Several complications outlined in section 2 disappear when one returns to the zeroth-order  $\ell \rightarrow \infty$  approximation in equation (21) [21]. For this reason we intend to pay attention to the next, first-order level of approximation in  $1/\ell$ .

### 4.1. Approximation using a pair of Dirichlet boundary conditions

We feel strongly motivated by the observation that equation (21) in fact *prescribes* an 'effective' replacement of the Robin (i.e., 'difficult') boundary condition (3) by the Dirichlet (i.e., 'easy') boundary condition, say,

$$\phi(0) = 0, \quad \phi(S) = 0, \quad S = S(\ell) = 1 + \frac{1}{\ell}. \quad (26)$$

One should note that the new condition is specified just by a rightward shift of the end of the interval. An important 'user-friendly' feature of this shift should be seen in the fact that on the first-order level of precision the value of  $S(\ell)$  remains *independent* of all the other parameters. We may expect that the approximation (3)  $\rightarrow$  (26) will represent a fairly reliable and nontrivial approximation at all angular mode numbers which are not too small. The idea might even remain applicable in the very 'realistic' domain of the smallest angular mode number  $\ell$  where a further improvement of the choice of  $S(\ell)$  could be sought via equation (20) whenever necessary.

As next step a (purely formal) transition to all the real axis of  $r \in \mathbb{R}$  will be mediated by the immersion of both coupled (i.e., poloidal and toroidal) subsystems of the whole system in

infinitely deep square wells  $V_Y^{(\text{SQW})}(r)$ . They are defined as very large (or infinite) everywhere except the interval  $(0, Y)$  where they should vanish,

$$V_Y^{(\text{SQW})}(r) = \begin{cases} +\infty, & r \in (-\infty, 0), \\ 0, & r \in (0, Y), \\ +\infty, & r \in (Y, \infty). \end{cases} \tag{27}$$

This means that we change the definition of the kinematic factors in equations (1) and (2) where we set  $Y = S = 1 + 1/\ell > 1$  and  $Y = R = 1$ , respectively,

$$\begin{aligned} V_u(r) &= V_u^{(\text{SQW})}(r) = \frac{\ell(\ell + 1)}{r^2} + V_S^{(\text{SQW})}(r), \\ V_d(r) &= V_d^{(\text{SQW})}(r) = \frac{\ell(\ell + 1)}{r^2} + V_R^{(\text{SQW})}(r). \end{aligned} \tag{28}$$

The latter convention is purely formal and its use merely emphasizes the essence of the introduction of the approximation (26).

#### 4.2. Models with smeared boundaries

In the context of section 4.1 the use of the spectral series (6) remains based on closed solutions of the Sturm–Liouville problem,

$$-\partial_r^2 \chi_n(r) + \frac{\ell(\ell + 1)}{r^2} \chi_n(r) + V_R^{(\text{SQW})}(r) \chi_n(r) = \varrho_n \chi(r). \tag{29}$$

The trick extends to the Robin boundary condition immediately. The parallel construction of the second auxiliary spectral series (5) is based on the solution of the similar equation

$$-\partial_r^2 \phi_n(r) + \frac{\ell(\ell + 1)}{r^2} \phi_n(r) + V_S^{(\text{SQW})}(r) \phi_n(r) = \tau_n \phi(r). \tag{30}$$

As long as the effect of the Dirichlet boundary conditions is mimicked by the action of an infinitely deep square-well potential, a perceivable reduction of the complexity of the basis  $|n\rangle$  as well as of the modified space  $\mathcal{V}_u$  is achieved. It is also more easy to evaluate quantities  $\tau_n$ .

In summary, the effective simplification of boundary conditions of section 4.1 looks satisfactory. It is possible to argue that although the numerical *performance* of the resulting simplified bases could be enhanced significantly by the modification of boundary conditions, the *precision* of results may still be kept under control by making the modification ‘infinitesimal’ (i.e., controllably small). One can only feel dissatisfied by the non-analyticity of square wells (27) which does not seem to open any easy way towards understanding of the observable simplifications at  $\ell \rightarrow \infty$ .

For this reason it would make sense to replace the non-analytic auxiliary square wells in equations (29) and (30) by some ‘infinitesimally’ modified *analytic* approximants. One should add that the latter idea need not necessarily be in any conflict even with the underlying experimental setup. Moreover, a transition from the *discontinuous* functions  $V_{S,R}^{(\text{SQW})}(r)$  to many of their available respective analytic alternatives  $U_{u,d}^{(AA)}(r)$  may be expected better tuned to the *analytic* essence of perturbation methods.

In this spirit let us introduce models with smeared boundaries characterized by a final redefinition of the corresponding non-analytic functions (28),

$$\begin{aligned} V_u(r) &= V_u^{(AA)}(r) = \frac{\ell(\ell + 1)}{r^2} + U_u^{(AA)}(r), \\ V_d(r) &= V_d^{(AA)}(r) = \frac{\ell(\ell + 1)}{r^2} + U_d^{(AA)}(r). \end{aligned} \tag{31}$$

The two requirements of a sufficiently precise fit to the square wells,

$$U_d^{(AA)}(r) \approx V_R^{(\text{SQW})}(r), \quad U_u^{(AA)}(r) \approx V_S^{(\text{SQW})}(r) \quad (32)$$

are the only limitations of our free choice of the new *analytic* auxiliary potentials  $U$ .

### 5. The idea of $1/\ell$ perturbation expansions

The effective shift of one of the boundary conditions in equation (26) obviously *introduces* a ‘hidden’ *kinematical* parameter  $1/\ell$  in the *dynamics* of our present  $\alpha^2$ -dynamo models. The numerical smallness of this parameter is one of our present most important observations. We believe that the existence of such a small parameter offers an entirely new hint and encouragement for a new development and/or (typically, perturbative) simplifications of many existing practical calculations.

In principle, a prospective way of doing so might consist, e.g., in a systematic improvement of the linear-algebraic and weighted-residual approximation techniques based on the truncation of the infinite matrices to their finite-size approximants. A complementary picture might be provided by the Fourier-series-based Galerkin techniques of [21] etc. Still, we believe that the best use of the smallness of  $1/\ell$  in our present  $\alpha^2$ -dynamo models can be achieved via a direct use of the various large- $\ell$  expansions as tested, in the various contexts, e.g., in [23–32].

Unfortunately, the actual potential of the large- $\ell$  expansions has only rarely been tested out of its natural quantum-mechanical domain. Thus, only this experience as outlined briefly in the appendix is, at present, available as our preliminary methodical guide. Still, we believe that it offers a sufficiently strong encouragement of the transfer of the  $1/\ell$  asymptotic-series techniques to the present MHD equations (1) and (2) accompanied by the boundary conditions (3) and (4).

Let us now complement the outline of the method as presented in the appendix by a more detailed account of its features which transcend the routine applications, say, of [33, 34].

#### 5.1. Coupled channels and the smeared boundary conditions

In the generic context of  $1/\ell$ -expansions we have to clarify, first of all, the explicit perturbation account of the coupling of the poloidal and toroidal modes. In the sequel we shall pick up again equations (13) and (14) with  $\alpha(r) = \alpha_0$ . Formally, this enables us to rearrange the original differential equations,

$$-\partial_r^2 \phi(r) + V_u(r)\phi(r) - \alpha_0 \chi(r) = -\lambda \phi(r), \quad (33)$$

$$-\partial_r^2 \chi(r) + V_d(r)\chi(r) - \alpha_0[\alpha_0 \chi(r) - \lambda \phi(r)] = -\lambda \chi(r). \quad (34)$$

This simplification will also enhance the transparency of our considerations, paving the way towards their extension to more realistic nonconstant  $\alpha$ -profiles.

In the  $1/\ell$ -series perspective of the appendix, we have to add now a discussion of a few specific features of  $\alpha^2$  dynamos. In a preparatory step let us emphasize that the necessary underlying transition to the smeared boundary conditions will imply the necessity of the use of the coordinates along the whole real axis,  $r \in \mathbb{R}$ . For any smeared-boundary analytic potential (31) we will then be allowed to write the corresponding coupled equations (33) and (34) in the partitioned operator form

$$\begin{pmatrix} -\partial_r^2 + V_u(r) & -\alpha_0 \\ \alpha_0 \lambda & -\partial_r^2 + V_d(r) - \alpha_0^2 \end{pmatrix} \begin{pmatrix} \phi(r) \\ \chi(r) \end{pmatrix} = -\lambda \begin{pmatrix} \phi(r) \\ \chi(r) \end{pmatrix}. \quad (35)$$



As a consequence, we need not distinguish between the two distinct linear spaces  $\mathcal{V}_u$  and  $\mathcal{V}_d$ . We may search for the smeared-boundary solutions corresponding to the low-lying eigenvalues  $-\lambda$  in the single and standard Hilbert space  $L_2(\mathbb{R})$ . Nevertheless, in the perturbation context we shall still be forced to employ the two different bases.

5.2. *Non-coincidence of the two local minima*

As a first step, paralleling the procedure of the appendix, we make an appropriate choice of the two boundary-simulating analytic functions  $U_{u,d}^{(AA)}(r) = U_{u,d}(r)$  of  $r$  and expand

$$V_{u,d}(r) = \frac{\ell(\ell + 1)}{r^2} + U_{u,d}(r) = V_{u,d}(T_{u,d}) + V'_{u,d}(T_{u,d})(r - T_{u,d}) + \dots \quad (36)$$

Then the two parallel generalizations of equation (A.3), viz.,

$$V'_u(T_u) = 0, \quad V'_d(T_d) = 0,$$

i.e., the two elementary quadratic equations

$$2\ell(\ell + 1) = T_u^3 U'_u(T_u) = T_d^3 U'_d(T_d) \quad (37)$$

define in principle the pair of inverse functions  $T_{u,d} = T_{u,d}(\ell)$ . By assumption they should grow with  $\ell$ , say, in such a way that the two new independent measures of smallness  $1/T_{u,d}(\ell)$  converge to zero in the limit  $\ell \rightarrow \infty$ .

The pair of the coordinates  $T_{u,d}$  of the respective minima of  $V_{u,d}(r)$  will be different in general. In a way paralleling the guidance by the appendix the respective shapes of  $V_{u,d}(r)$  near their minima will be specified by the next Taylor-series term with the coefficient proportional to the minus-fourth power of the scaling factor,

$$V''_{u,d}(T_{u,d}) = \frac{6\ell(\ell + 1)}{T_{u,d}^4} + U''_{u,d}(T_{u,d}) = \frac{3U'_{u,d}(T_{u,d})}{T_{u,d}} + U''_{u,d}(T_{u,d}) \equiv 2\sigma_{u,d}^{-4}. \quad (38)$$

In this way we arrive at the zeroth-order descendant of equation (35),

$$\begin{pmatrix} -\partial_\xi^2 + \xi^2 + v_u & -\alpha_0 \sigma_u^2 \\ \alpha_0 \lambda^{[0]} \sigma_d^2 & -\partial_\zeta^2 + \zeta^2 + v_d - \alpha_0^2 \sigma_d^2 \end{pmatrix} \begin{pmatrix} \phi^{[0]}(r) \\ \chi^{[0]}(r) \end{pmatrix} = -\lambda^{[0]} \begin{pmatrix} \sigma_u^2 \phi^{[0]}(r) \\ \sigma_d^2 \chi^{[0]}(r) \end{pmatrix}, \quad (39)$$

where we have introduced the additional abbreviation  $\sigma_{u,d}^2 V_{u,d}(T_{u,d}) \equiv v_{u,d}$  and where we have to keep in mind that the symbol  $r$  is just an abbreviation for  $r = T_u + \sigma_u \xi$  or  $r = T_d + \sigma_d \zeta$ .

6. Feasibility of perturbation constructions

6.1. Algebraic form of equations in zeroth-order limit  $\ell \rightarrow \infty$

As a next step, guided by the notation used in subsection 2.1, we may innovate the spectral series (5) and (6) writing

$$-\partial_\xi^2 + \xi^2 = \sum_{m=0}^{\infty} |m\rangle \hat{\tau}_m \langle m|, \quad \hat{\tau}_m = 2m + 1, \quad (40)$$

$$-\partial_\zeta^2 + \zeta^2 = \sum_{n=0}^{\infty} |n\rangle \hat{\varrho}_n \langle n|, \quad \hat{\varrho}_n = 2n + 1. \quad (41)$$

The specific doubling of the bra-ket symbols has been chosen here to underline the specific feature of both sets of eigenvectors: in the zeroth-order perturbation limit  $\ell \rightarrow \infty$  the

eigenvalues become well known at all subscripts,  $\hat{\tau}_k = \hat{\varrho}_k = 2k + 1$ , and also the eigenvectors coincide with the two distinct special cases of the harmonic-oscillator bases defined in terms of Hermite polynomials [33].

It is necessary to add that even in the limit  $\ell \rightarrow \infty$  the latter two harmonic-oscillator basis sets remain distinct in general. A return to the appendix reveals that due to the difference between the Robin and Dirichlet boundary conditions at  $\ell < \infty$  the two choices of  $V_{u,d}^{(AA)}$  (say, in the anharmonic form sampled by equation (A.1)) must be necessarily non-identical. Hence, also the resulting harmonic-oscillator basis functions will differ in both of their  $\ell$ -dependent shifts  $T = T_{u,d}(\ell)$  and scaling factors  $\sigma_{u,d}$ . This means that we have to rewrite our unperturbed equation (39) in the spectral-series-like form paralleling equations (8) and (9),

$$\begin{pmatrix} \sum_{m=0}^{\infty} |m\rangle\langle m| (\hat{\tau}_m + v_u) \{ \{m| & -\sigma_u^2 \alpha_0 \sum_{m,j=0}^{\infty} |m\rangle\langle m| \{ \{m|j\rangle\langle j| \} \\ \sigma_d^2 \alpha_0 \lambda \sum_{n,p=0}^{\infty} |n\rangle\langle n| \{ \{p| & \sum_{n=0}^{\infty} |n\rangle\langle n| (\hat{\varrho}_n + v_d - \sigma_d^2 \alpha_0^2) \{ \{n| \} \end{pmatrix} \begin{pmatrix} \phi(r) \\ \chi(r) \end{pmatrix} \\ = -\lambda \begin{pmatrix} \sigma_u^2 \phi(r) \\ \sigma_d^2 \chi(r) \end{pmatrix}, \quad \ell \leq \infty. \quad (42)$$

Differently said, the resulting matrix problem remains purely numerical even after the insertion of the present counterpart

$$|\phi\rangle\rangle = \sum_{k=0}^{\infty} |k\rangle\rangle \phi_k, \quad |\chi\rangle\rangle = \sum_{j=0}^{\infty} |j\rangle\rangle \chi_j \quad (43)$$

of equations (10), giving the final infinite coupled set of unperturbed equations

$$\left( \frac{\hat{\tau}_k + v_u}{\sigma_u^2} + \lambda \right) \phi_k - \alpha_0 \sum_{j=0}^{\infty} \{ \{k|j\rangle\rangle \chi_j = 0, \quad (44)$$

$$\left( \frac{\hat{\varrho}_k + v_d}{\sigma_d^2} + \lambda - \alpha_0^2 \right) \chi_k + \alpha_0 \lambda \sum_{m=0}^{\infty} \{ \{k|m\rangle\rangle \phi_m = 0, \quad k = 0, 1, \dots \quad (45)$$

which is just a smoothed-boundary parallel to equations (15) and (45).

## 6.2. Conditions of solvability of the zeroth-order equations

For a given pair of smoothed square wells  $U_{u,d}^{(AA)}(r)$  which mimic boundary conditions the core of the applicability of the large- $\ell$  perturbation theory may now be identified with a guarantee of coincidence of the two bases in the limit  $\ell \rightarrow \infty$ . Indeed, only in such a case one can find the application of the standard textbook algorithms of perturbation expansions [34] sufficiently economical and well motivated, especially in comparison with the fairly efficient purely numerical methods of linear algebra.

More explicitly, we require the exact solvability of the zeroth-order version of the set of equations (44) and (45). It may only be achieved when  $|j\rangle\rangle \rightarrow |j\rangle$  for all  $j = 0, 1, \dots$  in the limit  $\ell \rightarrow \infty$ . In such a case we shall get the orthogonality rule  $\{ \{k|j\rangle\rangle = \delta_{k,j}$  so that our set of equations (44) and (45) decouples in infinitely many pairs of equations

$$\left( \frac{\hat{\tau}_k + v_u}{\sigma_u^2} + \lambda \right) \phi_k - \alpha_0 \chi_k = 0, \quad \alpha_0 \lambda \phi_k + \left( \frac{\hat{\varrho}_k + v_d}{\sigma_d^2} + \lambda - \alpha_0^2 \right) \chi_k = 0,$$

numbered by index  $k = 0, 1, \dots$ . With their secular determinant vanishing,

$$\det \begin{pmatrix} \hat{\tau}_k + v_u + \lambda \sigma_u^2 & -\alpha_0 \sigma_u^2 \\ \alpha_0 \lambda \sigma_d^2 & \hat{\varrho}_k + v_d + (\lambda - \alpha_0^2) \sigma_d^2 \end{pmatrix} = 0, \quad \ell \gg 1$$

we arrive at the exact solution

$$\lambda_{1,2} = \frac{1}{2} \left[ -\frac{\hat{\tau}_k + v_u}{\sigma_u^2} - \frac{\hat{\varrho}_k + v_d}{\sigma_d^2} \pm \sqrt{\left(\frac{\hat{\tau}_k + v_u}{\sigma_u^2} - \frac{\hat{\varrho}_k + v_d}{\sigma_d^2}\right)^2 + 4 \frac{\hat{\tau}_k + v_u}{\sigma_u^2} \alpha_0^2} \right], \tag{46}$$

where we have to insert  $\hat{\tau}_k = 2k + 1 = \hat{\varrho}_k$  and  $\sigma_u^2 = \sigma_d^2$ .

Although formulae (46) look formally very similar to the examples we studied in subsection 3.2, the key difference lies in the fact that the coincidence of the bases may now be achieved easily. In the light of equation (A.4) the necessary and sufficient conditions of this coincidence  $|j\rangle = |j\rangle\rangle$  reads

$$T_u = T_d = T, \quad V_u''(T_u) = V_d''(T_d). \tag{47}$$

Due to equation (37) the first rule means that

$$U_u'(T) = U_d'(T) \tag{48}$$

while the validity of equation (38) then implies that the rest of equation (47) is equivalent to the condition

$$U_u''(T) = U_d''(T). \tag{49}$$

We may conclude that the pair of equations (48) and (49) (i.e., in a geometric language the so-called osculation of the two curves) represents the necessary and sufficient condition of the required coincidence  $|j\rangle = |j\rangle\rangle$  of the zeroth-order bases.

In practice we may expect that once we fix the value of  $T \gg 1$ , we may guarantee the solvability of our present illustrative zeroth-order  $\alpha^2$ -dynamo eigenvalue problem simply by the choice of the pair of the smooth-boundary-simulating potentials which obey the formula

$$U_{u,d}(r) = U_{u,d}(T) + A(r - T) + B(r - T)^2 + f_{u,d}(r - T)(r - T)^3. \tag{50}$$

This formula contains the *same* pair of parameters  $A$  and  $B$  and *two different* functions  $f_{u,v}(r - T)$  which remain regular at  $r = T$  as well as *two different 'offsets'*  $U_{u,d}(T)$ . Obviously, the latter formula offers enough freedom for an arbitrarily precise and explicit necessary fit (32) of the boundary conditions.

### 7. Summary

Despite our explicit knowledge of the exact basis states (7), the immediate linear-algebraic recipe (13) and (14) for the construction of the solutions of differential equations (1) and (2) is a numerical task. The necessary matrix elements must be computed as integrals of the products of the pairs of Bessel functions with given and, in principle, arbitrary phenomenological input function  $\alpha(r)$ . The matrix elements rarely remain available in closed form and sophisticated numerical methods are necessary for their evaluation. Moreover, even if the precision of the matrix elements themselves proves satisfactory for a given  $\alpha(r)$  in (8) and (9), the final solution of the eigenvalue problem requires an infinite-dimensional matrix diagonalization.

In this context we revealed that the quantity  $1/\ell$  represents a 'hidden' natural small parameter in the problem. This encouraged us to search for an efficient non-numerical construction of the solutions. An ambitious candidate has been sought and found in the perturbation method of the so-called large- $\ell$  expansions. A detailed adaptation of the key ingredients of this technique to the specific needs of the coupled differential  $\alpha^2$ -dynamo equations has been performed here in some detail.

As a first step of the realization of such a project the purely Dirichlet specification of the toroidal mode  $\chi(r)$  has been selected and interpreted as if resulting from the action of an

infinitely deep square well  $V^{(\text{SQW})}(r)$  which vanishes precisely inside the corresponding finite interval of  $r$ . This trick gives an equivalent picture and at the same time it offers a guide to the simplification of the treatment of the more complicated poloidal mode  $\phi(r)$  constrained by the Robin boundary condition. In this sense the Robin constraint has been reinterpreted as the same Dirichlet boundary condition modified by its shift to some slightly higher value of an ‘effective’ boundary point.

In the second step towards the realization of perturbation solutions a ‘softening’ of the ‘rigid’ boundary conditions has been introduced via a replacement of the ‘deep well’  $V^{(\text{SQW})}(r)$  by an element of a large family of its suitable analytic analogues and descendants  $V^{(\text{AA})}(r)$ . For the sake of definiteness the effects and consequences of this ‘softening’ have been illustrated on the power-law well  $V^{(\text{AA})}(r) \sim \sum a_K x^K$  with a suitable, not necessarily integer exponent  $K \gg 1$ . We reminded the readers how the  $1/\ell$  technique works in this older and, by the way, numerically extremely successful [26] application.

In the last third of our paper we demonstrated that and how the ‘softening’ of the boundary conditions opens a mathematically consistent path towards the use of the inverse mode number  $\ell$  as *the* effective small parameter of the theory of spherically symmetric  $\alpha^2$ -dynamoes. In particular we showed that the asymmetric character of the coupling of the toroidal and poloidal channels leads to certain unexpected and nontrivial formal challenges and we were successful in finding their resolution in the use of a certain flexibility in the choice of the softened square wells.

Let us add in conclusion that the details and technical aspects of our present method of construction will strongly depend on assumptions concerning the structure of the input  $\alpha$ -profile  $\alpha(r)$ . In fact, a strong sensitivity of the results on such details has been observed and studied using a more standard perturbation theory in [21]. Our present text skipped the details of all the wealth of phenomena related to the variations of  $\alpha(r)$ . In the light of our present emphasis on the description and clarification of new methods we often restricted our attention to the mere constant choice of  $\alpha(r) = \alpha_0$ . This limitation seems to have left a lot of space for its phenomenologically motivated weakening or even complete removal.

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## Appendix. $1/\ell$ expansions in quantum mechanics

Quantum anharmonic oscillators and their Schrödinger equations

$$[-\partial_r^2 + V(r)]\psi(r) = \varrho\psi(r), \quad V(r) = \frac{\ell(\ell+1)}{r^2} + \omega^2 r^{2K} \quad (\text{A.1})$$

are often studied in the domain of  $\ell \gg 1$ . Reviews [26, 27] of the corresponding techniques may be consulted for many technical details. In a sketchy outline of these techniques let us emphasize that equation (A.1) might be also reinterpreted, in our present  $\alpha^2$ -dynamo context, as one of the eligible smooth-boundary alternatives to equation (29) or (30). The single-channel Sturm–Liouville problem (A.1) defines then the basis states  $|m\rangle$  or  $|n\rangle$  entering the spectral series (6) or (5), respectively.

In a preparatory step of the current  $1/\ell$  recipe let us recollect that our function  $V(r)$  may be rewritten as the Taylor series

$$V(r) = \frac{\ell(\ell + 1)}{r^2} + \omega^2 r^K = V(T) + V'(T)(r - T) + \frac{1}{2}V''(T)(r - T)^2 + \dots \tag{A.2}$$

At a point  $T$  defined as the unique global minimum of  $V(r)$  we have, in particular,

$$V'(T) = 0 \implies \frac{2\ell(\ell + 1)}{T^3} = K\omega^2 T^{K-1} \implies T^{k+2} = \frac{2\ell(\ell + 1)}{K\omega^2}. \tag{A.3}$$

This means that for large  $\ell$  the coordinate  $T = T(\ell)$  is also large. We may use the new measure of smallness  $1/T$  in place of the original small parameter  $1/\ell$  in the formulae

$$\begin{aligned} V(T) &= \frac{K\omega^2 T^{K+2}}{2T^2} + \omega^2 T^K = \omega^2 T^K (1 + K/2), \quad V'(T) = 0, \\ V''(T) &= K\omega^2 T^{K-2}(K + 2), \quad V'''(T) = K\omega^2 T^{K-3}(K^2 - 3K - 10), \dots \end{aligned}$$

We insert them in equations (A.2) and (A.1) and change variables in such a way that

$$r - T = \sigma\xi, \quad \sigma^4 = \frac{2}{K(K + 2)\omega^2 T^{K-2}}. \tag{A.4}$$

This reduces equation (A.1) to the formally equivalent equation

$$\begin{aligned} -\partial_r^2 \psi(r) + \frac{K(K + 2)}{2}\omega^2 T^{K-2}(r - T)^2 \psi(r) \\ + \frac{K(K^2 - 3K - 10)}{6}\omega^2 T^{K-3}(r - T)^3 \psi(r) + \dots = [\varrho - V(T)]\psi(r) \end{aligned} \tag{A.5}$$

and, after rescaling, to the perturbed harmonic oscillator,

$$\left[-\partial_\xi^2 + \xi^2 + \kappa_3 \xi^3 + \kappa_4 \xi^4 + \dots\right] \psi(T + \sigma\xi) = \varepsilon \psi(T + \sigma\xi). \tag{A.6}$$

In the zeroth-order approximation its low-lying unperturbed spectrum is well known and equidistant,

$$\varepsilon = \sigma^2[\varrho - V(T)] \approx \varepsilon_0 = 1, 3, 5, \dots$$

and all the higher order perturbation corrections may be evaluated easily [33]. Moreover, their size may be made arbitrarily small by the choice of a sufficiently large  $\ell$  since it is easy to show that we have  $\kappa_3 = \mathcal{O}(T^{-1/2-K/4})$ ,  $\kappa_4 = \mathcal{O}(T^{-1-K/2})$  etc. This means that we get the pure and exactly solvable harmonic oscillator in the limit  $\ell \rightarrow \infty$ . Perturbation theory helps us then to evaluate the corrections at any finite  $\ell < \infty$  and for all the low-lying levels with  $n = 0, 1, \dots$ ,

$$\varrho_n = V(T) + \sigma^{-2}(\varepsilon_0 + \varepsilon_1 + \dots) \tag{A.7}$$

with  $\varepsilon_j = \mathcal{O}(\kappa_{j+2}) = \mathcal{O}(T^{-j/2-jK/4})$  for all perturbation orders  $j = 1, 2, \dots$ , i.e., with

$$\varrho_n = \omega^2(1 + K/2)T^K + \omega(2n + 1)\sqrt{K(1 + K/2)}T^{K/2-1} + \dots$$

in our illustrative example.

Marginally, let us note that the fact of the large size of the shift of the position of the global minimum  $T \gg 1$  of the complete, the so-called ‘effective’ potential term  $V_{\text{eff}}(r) = \ell(\ell + 1)/r^2 + V(r)$  in the Schrödinger equation resolves, as a byproduct, also the well-known puzzle that our *approximative* wavefunctions are, in general, allowed to become singular near the pole of the centrifugal component of  $V_{\text{eff}}(r)$ , i.e., near  $r = 0$ . In the present approximation-theory context, the *practical* irrelevance of the corresponding error terms is an interesting consequence of the fact that *all* our approximate wavefunctions are exponentially decaying outside the effective-potential valley at the sufficiently large  $\ell$ . This makes the

irregularity of the wavefunctions at  $r = 0 \ll T$  (which is definitely there of course, together with the equally puzzling ‘allowed tunnelling’ to  $r < 0$  [30]) entirely irrelevant. Still, it is worth noting that all these ‘omissions’ definitely cause the ultimate divergence of the *infinite*  $1/\ell$  expansions, in the manner discussed thoroughly in the specialized literature (of which we may recommend the most succinct study [26] to the interested reader’s attention).

## References

- [1] Moffatt H K 1978 *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge: Cambridge University Press)
- [2] Krause F and Rädler K-H 1980 *Mean-Field Magnetohydrodynamics and Dynamo Theory* (Berlin/Oxford: Pergamon/Akademie-Verlag)
- [3] Zeldovich Ya B, Ruzmaikin A A and Sokoloff D D 1983 *Magnetic Fields in Astrophysics* (New York: Gordon and Breach Science Publishers)
- [4] Rüdiger G and Hollerbach R 2004 *The Magnetic Universe* (Weinheim: Wiley-VCH)
- [5] Proctor M R E 1977 *Astron. Nachr.* **298** 19  
Proctor M R E 1977 *Geophys. Astrophys. Fluid Dyn.* **8** 311
- [6] Rädler K-H 1982 *Geophys. Astrophys. Fluid Dyn.* **20** 191
- [7] Sokoloff D D, Ruzmaikin A A and Shukurov A M 1983 *Geophys. Astrophys. Fluid Dyn.* **25** 293
- [8] Krause F and Meinel R 1988 *Geophys. Astrophys. Fluid Dyn.* **43** 95
- [9] Baryshnikova Y and Shukurov A 1987 *Astron. Nachr.* **308** 89
- [10] Meinel R 1989 *Astron. Nachr.* **310** 1
- [11] Meinel R and Brandenburg A 1990 *Astron. Astrophys.* **238** 369
- [12] Rädler K-H and Geppert U 1999 *ASP Conf. Ser.* **178** 151
- [13] Stefani F and Gerbeth G 2005 *Phys. Rev. Lett.* **94** 184506 (Preprint physics/0411050)
- [14] Giesecke A, Rüdiger G and Elstner D 2005 *Astron. Nachr.* **326** 693 (Preprint astro-ph/0509286)
- [15] Stefani F, Gerbeth G, Günther U and Xu M 2006 *Earth Planet. Sci. Lett.* **243** 828–40 (Preprint physics/0509118)
- [16] Gailitis A, Lielausis O, Platacis E, Gerbeth G and Stefani F 2002 *Rev. Mod. Phys.* **74** 973
- [17] Günther U and Stefani F 2003 *J. Math. Phys.* **44** 3097 (Preprint math-ph/0208012)
- [18] Günther U, Stefani F and Znojil M 2005 *J. Math. Phys.* **46** 063504 (Preprint math-ph/0501069)
- [19] Günther U, Stefani F and Gerbeth G 2004 *Czech. J. Phys.* **54** 1075 (Preprint math-ph/0407015)
- [20] Günther U and Stefani F 2005 *Czech. J. Phys.* **55** 1099 (Preprint math-ph/0506021)
- [21] Günther U and Kirillov O N 2006 *J. Phys. A: Math. Gen.* **39** 10057 (Preprint math-ph/0602013)
- [22] Znojil M 2006 *J. Phys. A: Math. Gen.* **39** 441 (Preprint quant-ph/0511085)  
Znojil M 2006 *J. Phys. A: Math. Gen.* **39** 4047 (Preprint quant-ph/0511194)
- [23] Mlodinov L and Papanicolau N 1980 *Ann. Phys., NY* **128** 314  
Mlodinov L and Papanicolau N 1981 *Ann. Phys., NY* **131** 1
- [24] Sukhatme U and Imbo T 1983 *Phys. Rev. D* **28** 418
- [25] Mlodinov L and Shatz M P 1984 *J. Math. Phys.* **25** 943
- [26] Bjerrum-Bohr N E J 2000 *J. Math. Phys.* **41** 2515 (Preprint quant-ph/0302107)
- [27] Fernandez F M 2001 *Introduction to Perturbation Theory in Quantum Mechanics* (Boca Raton: CRC Press)
- [28] Znojil M, Gemperle F and Mustafa O 2002 *J. Phys. A: Math. Gen.* **35** 5781 (Preprint hep-th/0205181)
- [29] Mustafa O and Znojil M 2002 *J. Phys. A: Math. Gen.* **35** 8929 (Preprint math-ph/0206042)
- [30] Znojil M 2003 *J. Phys. A: Math. Gen.* **36** 9929 (Preprint quant-ph/0307239)
- [31] Bíla H 2004 *Czech. J. Phys.* **54** 1049
- [32] Znojil M 2004 *Int. J. Pure Appl. Math.* **12** 79 (Preprint physics/0404123)
- [33] Landau L D and Lifshitz E M 1965 *Quantum Mechanics: Non-Relativistic Theory* (Oxford: Pergamon)
- [34] Baumgärtel H 1984 *Analytic Perturbation Theory for Matrices and Operators* (Berlin: Akademie-Verlag)  
Baumgärtel H 1985 *Operator Theory: Adv. Appl.* vol 15 (Basel: Birkhäuser)