Homotopic Arnold tongues deformation of the MHD α^2-dynamo

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We consider a mean-field α^2-dynamo with helical turbulence parameter $\alpha(r) = \alpha_0 + \gamma\Delta\alpha(r)$ and a boundary homotopy with parameter $\beta \in [0,1]$ interpolating between Dirichlet (idealized, $\beta = 0$) and Robin (physically realistic, $\beta = 1$) boundary conditions. It is shown that the zones of oscillatory solutions at $\beta = 1$ end up at the diabolical points for $\beta = 0$ under the homotopic deformation. The underlying network of the diabolical points for $\beta = 0$ substantially determines the choreography of eigenvalues and thus the character of the dynamo instability for $\beta = 1$. Using perturbation theory we derive the first-order approximations to the resonance (Arnold's) tongues in the $(\alpha_0, \beta, \gamma)$-space, which turn out to be cones in the vicinity of the diabolical points, selected by the Fourier coefficients of $\Delta\alpha(r)$. The space orientation of the 3D tongues is determined by the Krein signature of the modes involved in the diabolical crossings at the apexes of the cones. The Krein space induced geometry of the resonance zones explains the subtleties in finding α-profiles leading to oscillatory dynamos, and it explicitly predicts the locations of the spectral exceptional points, which are important ingredients in the recent theories of polarity reversals of the geomagnetic field.

The mean field MHD α^2-dynamo [1–3] in its kinematic regime is described by a linear induction equation for the magnetic field. For spherically symmetric $\alpha-$profiles $\alpha(r)$ the vector of the magnetic field is decomposed into poloidal and toroidal components and expanded in spherical harmonics with degree l and order m. After additional time separation, the induction equation reduces to a set of $l-$decoupled boundary eigenvalue problems [3], which we write in matrix form, convenient for the implementation of perturbation theory [4–6]

$$\mathbf{L}\mathbf{u} := \begin{pmatrix} 1 & 0 \\ -\alpha(r) & 1 \end{pmatrix} \partial_r^2 \mathbf{u} + \begin{pmatrix} 0 & 0 \\ -\partial_r\alpha(r) & 0 \end{pmatrix} \partial_r \mathbf{u} + \begin{pmatrix} -\frac{l(l+1)}{r^2} - \lambda & \alpha(r) \\ \alpha(r)\frac{l(l+1)}{r^2} & -\frac{l(l+1)}{r^2} - \lambda \end{pmatrix} \mathbf{u} = 0, \quad \mathfrak{U}\mathfrak{u} = 0. \quad (1)$$

The matrix $\mathfrak{U} := [\mathfrak{A}, \mathfrak{B}] \in \mathbb{C}^{4\times 8}$ in the boundary conditions consists of the blocks

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta l + 1 - \beta & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (2)$$

The vector-function $\mathbf{u} \in \tilde{\mathcal{H}} = L_2(0,1) \oplus L_2(0,1)$ lives in the Hilbert space $(\tilde{\mathcal{H}}, (.,.))$ with inner product $(\mathbf{u}, \mathbf{v}) = \int_0^1 \overline{\mathbf{v}}^T \mathbf{u}\, dr$, where the overbar denotes complex conjugation, and the vector $\mathfrak{u}$ is defined as

$$\mathfrak{u}^T := \left(\mathbf{u}^T(0), \partial_r \mathbf{u}^T(0), \mathbf{u}^T(1), \partial_r \mathbf{u}^T(1)\right) \in \mathbb{C}^8. \quad (3)$$

We assume that $\alpha(r) := \alpha_0 + \gamma\Delta\alpha(r)$, where $\Delta\alpha(r)$ is a smooth real function $C^2(0,1) \ni \Delta\alpha(r) : (0,1) \to \mathbb{R}$ with $\int_0^1 \Delta\alpha(r) dr = 0$. For a fixed $\Delta\alpha(r)$ the differential expression $\mathbf{L}$ depends on the parameters α_0 and γ, while β interpolates between idealized ($\beta = 0$) boundary conditions, corresponding to an infinitely conducting exterior, and physically realistic ones ($\beta = 1$) corresponding to a non-conducting exterior of the dynamo region. For constant $\alpha-$profiles $\alpha(r) \equiv \alpha_0 =$const and $\beta = 0$ the spectrum and eigenvectors of the operator matrix pencil $\mathbf{L}(\lambda)$ are [3]

$$\lambda_n^\varepsilon = \lambda_n^\varepsilon(\alpha_0) = -\rho_n + \varepsilon\alpha_0\sqrt{\rho_n} \in \mathbb{R}, \quad \varepsilon = \pm, \quad \mathbf{u}_n^\varepsilon = \begin{pmatrix} 1 \\ \varepsilon\sqrt{\rho_n} \end{pmatrix} u_n \in \mathbb{R}^2 \otimes L_2(0,1), \quad n \in \mathbb{Z}^+, \quad (4)$$

where $u_n(r)$ are Riccati-Bessel functions and $\rho_n > 0$ are the squares of Bessel function $J_{l+\frac{1}{2}}$ zeros [3]. The eigenvectors $\mathbf{u}_n^+, \mathbf{u}_n^- \in \mathcal{K}_\pm \subset \mathcal{K}$ correspond to Krein space $(\mathcal{K}, [.,.])$ states of positive and negative signature

$$[\mathbf{u}_m^\pm, \mathbf{u}_n^\pm] = \pm 2\sqrt{\rho_n}\delta_{mn}, \; [\mathbf{u}_m^\pm, \mathbf{u}_n^\mp] = 0, \quad [.,.] = (\mathbf{J}.,.), \quad \mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (5)$$

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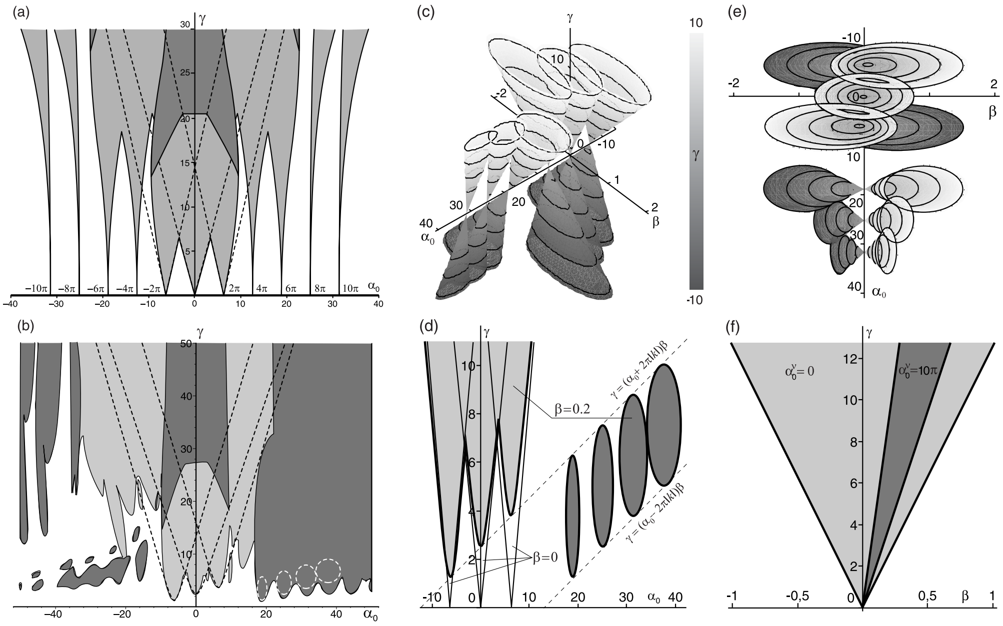

Fig. 1 $l = 0$, $\Delta\alpha(r) = \cos(4\pi r)$: (a) Arnold's tongues corresponding to the decaying (light gray) and growing (dark gray) oscillations of the magnetic field with their linear approximation (dashed lines) for (a) $\beta = 0$ and (b) $\beta = 0.2$; 3D cones of the principal resonances with inclination determined by the Krein signature of the double eigenvalues (c) and their projections and cross-sections (d)-(f).

Two branches $\lambda_m^\delta, \lambda_n^\varepsilon$ with $n \neq m$ intersect at a point $(\alpha_0^\nu, \lambda_0^\nu)$ with $\alpha_0^\nu := \varepsilon\sqrt{\rho_n} + \delta\sqrt{\rho_m}$, $\quad \lambda_0^\nu := \varepsilon\delta\sqrt{\rho_n\rho_m}$, corresponding to a double eigenvalue $\lambda_0^\nu = \lambda_0^\varepsilon = \lambda_0^\delta$ with two linearly independent eigenvectors $\mathbf{u}_n^\varepsilon$ and $\mathbf{u}_m^\delta$, i.e. at a semi-simple eigenvalue (diabolical point) of algebraic and geometric multiplicity two [3, 4].

The magnetic field of a dynamo is maintained by its non-decaying modes ($\Re\lambda > 0$). When additionally one of the dominant modes is oscillating ($\Im\lambda \neq 0$) then via nonlinear back-reactions ($\alpha-$quenching) the dynamo becomes prone to polarity reversals. The relevant eigenvalue branches with $\Re\lambda > 0, \Im\lambda \neq 0$ can be induced by inhomogeneously deforming (perturbing) the constant $\alpha-$profile, $\alpha(r) = \alpha_0 + \gamma\Delta\alpha(r)$, and simultaneously varying the boundary conditions. With the help of perturbation theory [4–6] we derive a first-order approximation of the eigenvalues, originating from the splitting of the double semi-simple eigenvalue λ_0^ν at the node $(\alpha_0^\nu, \lambda_0^\nu)$ of the spectral mesh induced by a variation of the parameters α_0, β, and γ. In the simplest case $l = 0$ and $\Delta\alpha(r) = \cos(2\pi kr)$, $k \in \mathbb{Z}$, we find that close to a diabolical crossing of $(\varepsilon n)-$ and $(\varepsilon n + 2k)-$mode branches the complex eigenvalues are located in the $(\alpha_0, \beta, \gamma)-$parameter space within the conus interiors

$$-4k^2(\alpha_0 \pm 2\pi(n-|k|))^2 + n(2|k|-n)(\gamma \pm 2\pi(n-|k|)\beta)^2 > n(2|k|-n)4\pi^2\beta^2k^2, \quad n = 1, 2, \ldots, |k|, \tag{6}$$

when $\lambda_0^\nu < 0$, i.e. for decaying field modes, and

$$4k^2\left(\alpha_0 \pm 2\pi(n+|k|)\right)^2 + n(2|k|+n)\left(\gamma \pm 2\pi(n+|k|)\beta\right)^2 < n(2|k|+n)4\pi^2\beta^2k^2, \quad n = 1, 2, \ldots. \tag{7}$$

when $\lambda_0^\nu > 0$, i.e. for overcritical (growing) field modes (see Fig. 1(c)-(f)). The Krein signature determines the inclination of the cones with respect to the $(\beta = 0)-$plane as well as the sign of the real parts of the oscillatory modes. In this way it separates the two eigenvalue groups with qualitatively different oscillatory dynamo behavior. A similarly defining effect of the Krein signature has been observed in stability problems for rotating elastic bodies of revolution having frictional contact [6, 7].

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