Visualization of Branch Points in $\mathcal{PT}$-Symmetric Waveguides

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The visualization of an exceptional point in a $\mathcal{PT}$-symmetric directional coupler (DC) is demonstrated. In such a system the exceptional point can be probed by varying only a single parameter. Using the Rayleigh-Schrödinger perturbation theory we prove that the spectrum of a $\mathcal{PT}$-symmetric Hamiltonian is real as long as the radius of convergence has not been reached. We also show how one can use a $\mathcal{PT}$-symmetric directional coupler to measure the radius of convergence for non-$\mathcal{PT}$-symmetric structures. For such systems the physical meaning of the rather mathematical term radius of convergence is exemplified.

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In the past several years, following the seminal paper by Bender and Boettcher [1], non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians have caught a lot of attention (see [2] and references therein). Under certain conditions $\mathcal{PT}$-symmetric Hamiltonians can have a completely real spectrum and thus can serve, under the appropriate inner products, as the Hamiltonians for unitary quantum systems [3].

Recently, a few suggestions of $\mathcal{PT}$-symmetric "Hamiltonians" have been made using optical waveguides with complex refractive indices or boundary terms [4–6]. The equivalence of the Maxwell and Schrödinger equations in certain regimes provides a physical system in which the properties of $\mathcal{PT}$-symmetric operators can be studied and exemplified. Moreover, such realizations can be very useful in connecting general mathematical concepts with observable physical measurements. As will be seen below, in the realization of a $\mathcal{PT}$-symmetric system one such clarifiable concept is the radius of convergence of a perturbation expansion.

An extremely interesting property of $\mathcal{PT}$-symmetric operators is the transition from a completely real spectrum into a nonstrictly real spectrum. This property has come to be known as exact/spontaneously-broken $\mathcal{PT}$ symmetry. Exact $\mathcal{PT}$ symmetry refers to the case where the entire spectrum is real. In any other case the $\mathcal{PT}$ symmetry is said to be broken. Usually, the transition between exact and spontaneously broken $\mathcal{PT}$ symmetry can be controlled by a parameter in the Hamiltonian. This parameter serves as a measure of the non-Hermiticity.

Bender *et al.* [7] showed that the reality of the spectrum is explained by the real secular equations one can write for $\mathcal{PT}$-symmetric matrices. These secular equations will depend on the non-Hermiticity parameter and, consequently, yield either real or complex solutions. Delabaere *et al.* [8] showed for the one-parameter family of complex cubic oscillators that pairs of eigenvalues cross each other at Bender and Wu branch points. Dorey *et al.* [9], after proving the reality of the spectrum for a family of $\mathcal{PT}$-symmetric Hamiltonians, showed [10] that at the point where the energy levels cross, a supersymmetry is broken and not only the eigenvalues but also the eigenfunctions become the same.

An important class of $\mathcal{PT}$-symmetric Hamiltonians are of the form $\hat{H}(\lambda) = H_0 + i\lambda V$, where H_0 (and V) are real and symmetric (antisymmetric) with respect to parity so that $[\mathcal{PT}, \hat{H}] = 0$. When $\lambda = 0$ the Hamiltonian is Hermitian and the entire spectrum is real. The spectrum remains real even when $\lambda \neq 0$ as long as $\lambda < \lambda_c$. At this critical value and beyond, pairs of eigenvalues collide and become complex; see for example [11]. Although the above-mentioned proofs can be applied also for this special class of $\mathcal{PT}$-symmetric Hamiltonians, a much more physically oriented proof can be given. Such a proof, see below, is not only reachable by a wider audience, relying solely on a perturbational analysis of the Hamiltonian, but also provides new insights into the physical system.

Consider the family of Hamiltonians given by $\hat{H} = H_0 + \lambda V = H_0 + i|\lambda|V$. The existence of a branch point in the Hamiltonian's spectrum determines the radius of convergence of a series expansion of the energy in λ. Friedland and one of us [12] proved that for two real symmetric matrices $\mathbf{H}_0$ and $\mathbf{V}$ that do not commute there exists *at least* one branch point $\lambda_{\rm bp}$ for which $\frac{dE}{d\lambda}|_{\lambda=\lambda_{\rm bp}} = \infty$. Therefore, the expansion of the energy in powers of λ converges only as long as $|\lambda| < |\lambda_{\rm bp}|$. The most common situation is when the branch point is associated with the coalescence of two eigenfunctions and the two corresponding eigenvalues. Such a point in the spectrum is often referred to as an *exceptional point* [13]. Exceptional points in physical systems have been studied, e.g., in [14,15]. Recently, exceptional points have been observed experimentally in microwave cavities [16]. In general, as stated in the theorem above, the value of the parameter λ at which

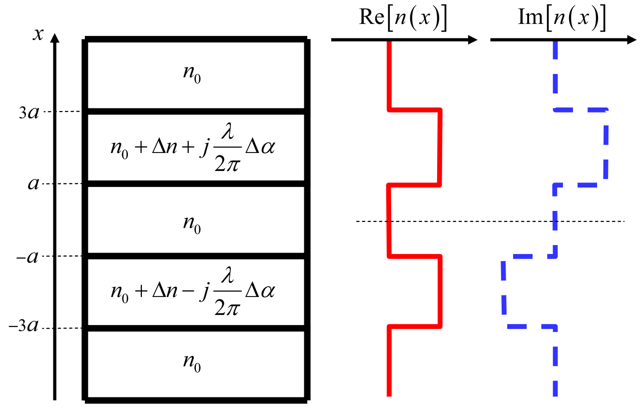

FIG. 1 (color online). A $\mathcal{PT}$-symmetric directional coupler. The structure consists of two coupled slab waveguides. The real and imaginary part of the refractive index is also portrayed. The refractive index only varies in the x direction.

the branch point occurs is a complex number. This demands the control of the Hamiltonian by at least two parameters [15]. From the evidence of branch points in previous studies of $\mathcal{PT}$-symmetric Hamiltonians, e.g., [11], it is plausible to assume that the branch point is located on the imaginary axis, i.e., $\lambda_{bp} = i|\lambda_{bp}|$. Hence, the two parameter dependence reduces to a dependence on a single real parameter. Therefore, the study of exceptional points in $\mathcal{PT}$-symmetric systems is strongly simplified.

The relation between the reality of the spectrum and the branch point at λ_{bp} can be seen from a Rayleigh-Schrödinger perturbation analysis. Consider the following time-independent Schrödinger equation:

$$(H_0 + \lambda V)\Psi_j(\lambda) = E_j(\lambda)\Psi_j(\lambda), \tag{1}$$

where again, H_0 (and V) are real and symmetric (antisymmetric) with respect to parity and are both Hermitian operators. Assuming that $|\lambda| < |\lambda_{bp}|$, one can expand the eigenvalues and eigenfunctions in a *convergent* power series in powers of λ: $E_j(|\lambda| < |\lambda_{bp}|) = \sum_{n=0}^{\infty} \lambda^n E_j^{(n)}$ and $\Psi_j(|\lambda| < |\lambda_{bp}|) = \sum_{n=0}^{\infty} \lambda^n \Psi_j^{(n)}$, where $E_j^{(n)}(\Psi_j^{(n)})$; $n = 0, 1, 2\ldots$ are the *real* energy (wavefunction) correction terms of the Rayleigh-Schrödinger perturbation expansion. The $(2n + 1)$-rule stated by Wigner [17] implies that

$$E_j = \langle \chi_j^{(n)} | H_0 + \lambda V | \chi_j^{(n)} \rangle + O(\lambda^{2n+2}) \tag{2}$$

where $\chi_j^{(n)}(x) = \sum_{k=0}^{n} \lambda^k \psi_j^{(k)}$ and $\langle \chi_j^{(n)} | \chi_j^{(n)} \rangle = 1$. Therefore, following the $(2n + 1)$-rule,

$$E_j^{(2n+1)} = \langle \psi_j^{(n)} | V | \psi_j^{(n)} \rangle \tag{3}$$

where the nth order correction to the exact eigenfunction Ψ_j is the solution of the following equation:

$$\psi_j^{(n)}(x) = G_0'(E_j^{(0)}) V(x) |\psi_j^{(n-1)}\rangle - \sum_{k=1}^{n-1} E_j^{(k)} G_0'(E_j^{(0)}) |\psi_j^{(n-k)}\rangle, \tag{4}$$

and $G_0'(E_j^{(0)}) = \sum_{q \neq j} \langle x | \psi_q^{(0)} \rangle \langle \psi_q^{(0)} | (E_j^{(0)} - E_q^{(0)})^{-1}$. The parity symmetry of the zeroth-order Hamiltonian ensures that its eigenfunction $\psi_q^{(0)}(x)$ has either *even* or *odd* parity. That is, $\psi_j^{(n)}(x) = (-1)^{j+n-1} \psi_j^{(n)}(-x)$ for $j = 1, 2, \ldots$ and $n = 0, 1, \ldots$. One can now immediately conclude that

$$E_j^{(2n+1)} = 0. \tag{5}$$

Therefore, $E_j(|\lambda| < |\lambda_{bp}|) = \sum_{n=0}^{\infty} \lambda^{2n} E_j^{(2n)}$, and, consequently, for $\lambda = i|\lambda|$ the series $E_j(|\lambda| < |\lambda_{bp}|) = \sum_{n=0}^{\infty} (-1)^n |\lambda|^{2n} E_j^{(2n)}$ converges to real values.

The above analysis shows that the transition between exact and spontaneously broken $\mathcal{PT}$ symmetry occurs at a branch point where the non-Hermiticity parameter ($|\lambda|$ in our case) reaches the radius of convergence of the perturbation expansion. As we will show below this phenomenon can be observed experimentally.

The measurement of the radius of convergence of a Rayleigh-Schrödinger series expansion will be demonstrated in a $\mathcal{PT}$-symmetric waveguide configuration. A $\mathcal{PT}$-symmetric waveguide can be easily realized with a symmetric index guiding profile and an antisymmetric gain-loss profile, i.e., $n(x) = n^*(-x)$ [5]. We consider two coupled planar waveguides depicted in Fig. 1 for which the refractive index varies only in the x direction. The direction of propagation in the waveguides is taken to be the z axis. The wave equation for the transverse-electric modes then reads

$$\left(\frac{\partial^2}{\partial x^2} + k^2 n(x)^2\right)\mathcal{E}_y(x) = \beta^2 \mathcal{E}_y(x), \tag{6}$$

where the y component of the electric field is given by $E_y(x, z, t) = \mathcal{E}_y(x) e^{i(\omega t - \beta z)}$, $k = 2\pi/\lambda$, and λ is the vacuum wavelength. Clearly, the wave equation for the y component of the electric field, i.e., Eq. (6), is analogous to the one-dimensional Schrödinger equation: $[-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V(x)]\Psi(x) = E\Psi(x)$, identifying the potential as $V(x) = -\frac{1}{2}k^2 n^2$ and the energy as: $E = -\frac{1}{2}\beta^2$. As shown in Fig. 1 we couple between one gain-guiding waveguide (positive imaginary part of the refractive index) and one loss-guiding waveguide (negative imaginary part of the refractive index) [18] in order to create the $\mathcal{PT}$-symmetric structure. For simplicity we take the separation between the two coupled waveguides to be the same as the waveguides' width, i.e., $2a$. Note that unlike in many $\mathcal{PT}$-symmetric systems, in our case the imaginary part of the refractive index, i.e., the complex part of the potential, vanishes at $x \to \pm\infty$. Therefore, one can impose the boundary conditions on the real axis, i.e., $\mathcal{E}_y(x) \to 0$ as

$x \to \pm\infty$. The effect of changing the separation of two coupled $\mathcal{PT}$-symmetric waveguides was studied in [5], and showed that as in regular (non-$\mathcal{PT}$-symmetric) directional couplers the coupling length is an increasing monotonic function of the separation. The beat time period usually used to describe quantum beating systems, e.g., [19], is exchanged in the study of waveguides by a beat length $L = 2\pi/\Delta\beta$, where $\Delta\beta$ is the difference between the propagation constants of the two modes. As described below a different method of controlling the beat length is by changing the non-Hermiticity of the potential which can be controlled by the gain (loss) coefficient $\Delta\alpha$. In order to illustrate the control of the beat length using the non-Hermiticity parameter we choose the following parameters for the waveguide structure shown in Fig. 1: The background index is taken to be $n_0 = 3.3$, the vacuum wavelength $\lambda = 1.55$ μm, the real index difference between the waveguides and the background material $\Delta n = 10^{-3}$, and the separation between the waveguides which equals the waveguides' width $2a = 5$ μm. The parameters are chosen such that each waveguide contains only a single guided mode before we couple them. The coupled guided modes are calculated by diagonalizing the matrix representation of Eq. (6) in a sine basis. The "Hamiltonian" matrix is non-Hermitian and one needs to take care when normalizing the eigenvectors. We choose to normalize our eigenvectors according to the so-called *c product* [20], i.e., $(\mathcal{E}_n|\mathcal{E}_m) = \langle \mathcal{E}_n^*|\mathcal{E}_m\rangle = \delta_{n,m}$. The coupled waveguides support two guided modes. The propagation constants of the two modes are plotted in Fig. 2 as a function of the non-Hermiticity parameter. Increasing $\Delta\alpha$ causes the propagation constants of the two modes to move towards each other up to a critical point, i.e., the branch point. Beyond the branch point the propagation constants become complex conjugates of one another. As long as $\mathcal{PT}$ symmetry remains exact, i.e., $\Delta\alpha < \Delta\alpha_c \simeq 8.4$, the two guided modes can be classified according to the parity of the real part of the transverse electric field just as in the non-$\mathcal{PT}$-symmetric case. The critical value of $\Delta\alpha$ corresponds to a branch point, i.e., an exceptional point, where the two modes coalesce. At the branch point both the propagation constants and the corresponding electric field become equal. One can, therefore, study the exceptional point in a $\mathcal{PT}$-symmetric wave-guide by varying only a *single parameter*. Past the critical value of $\Delta\alpha$, the waveguides support one gain-guiding mode and one loss-guiding mode. The transverse field past the branch point no longer retains the symmetry properties of the $\mathcal{PT}$ operator, but rather each of the two modes becomes localized in one of the waveguides. The progression of the propagation constants on the real axis towards the branch point can be visualized experimentally by observing the beat length (time, in the corresponding quantum mechanical problem) of the sum field for the $\mathcal{PT}$-symmetric waveguide. Figure 3 displays the power distribution,

$$|E_y(x,z)|^2 = |\tfrac{1}{\sqrt{2}}[\mathcal{E}_1(x)e^{-i\beta_1 z} + \mathcal{E}_2(x)e^{-i\beta_2 z}]|^2, \quad (7)$$

for three values of $\Delta\alpha$. As the value of $\Delta\alpha$ approaches the critical value the beat length increases. This is a direct observation of the movement of the propagation constants towards each other on the real axis. Near the critical value of $\Delta\alpha$, i.e., the exceptional point, the sum field no longer oscillates between the waveguides but rather travels in both waveguides simultaneously. Therefore, by tuning the value of $\Delta\alpha$ one can visualize the movement of the eigenmodes towards the branch point or away from it, as seen in Fig. 3. The effect of the non-Hermiticity on the beat length is clearly reflected in Fig. 3 where the beat length goes to infinity as the branch point is approached. However, this visualization of the branch point is not the only information one can get from such an experiment. By finding what is the critical value of $\Delta\alpha$ which corresponds to the branch point one can infer the radius of convergence for any symmetric waveguide structure with an antisymmetric perturbation. Therefore, while the visualization is done on a $\mathcal{PT}$-symmetric waveguide, the critical value of $\Delta\alpha$ corresponds to the maximum value of an added antisymmetric index profile which can still be treated within perturbation theory.

The $\mathcal{PT}$-symmetric system proposed above also allows a unique perspective on exceptional points. Usually, see, for example, [16], in order to detect an exceptional point one has to go around it in some parameter space. Such a circulation around an exceptional point produces a geometrical phase that can then be measured. In our case, however, we are able to go directly through the exceptional point by varying only a single parameter. This is not merely

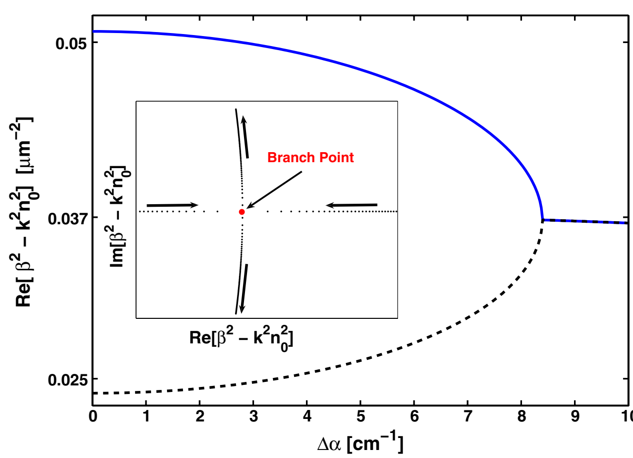

FIG. 2 (color online). The two trapped modes of the waveguide depicted in Fig. 1 as a function of the non-Hermiticity parameter strength. The eigenmodes approach each other on the real axis as $\Delta\alpha$ increases until a critical value of $\Delta\alpha_c \sim 8.4$ is reached. At the critical value one finds a branch (exceptional) point where the two modes coalesce. Beyond the branch point the directional coupler sustains one gain-guiding mode and one loss-guiding mode.

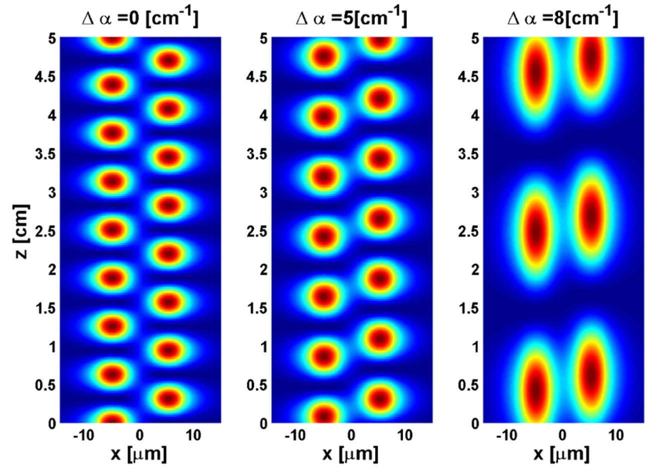

FIG. 3 (color online). The power distribution for a propagating sum field consisting of the two guided modes, see Eq. (7), for three values of $\Delta\alpha$. As can be readily observed, the beat length (analogous to the beat time period in quantum mechanics) increases as the value of $\Delta\alpha$ approaches the critical value.

a more direct method of detecting the exceptional point but also opens the door to previously unobservable quantities. One such quantity is the visualization of the self-orthogonal state that in our suggested experiment can be observed directly.

Although the studied waveguide is $\mathcal{PT}$ symmetric our measuring devices still follow the regular properties of the Hermitian "world." Consequentially, although the propagation constants of the modes of the waveguide are strictly real the integrated intensity is not conserved, i.e., $\frac{d}{dz}\times\int_{-\infty}^{\infty}|E_y(x,z)|^2dx \neq 0$. This can be easily seen from Fig. 3, where for $\Delta\alpha = 8$ cm^{-1} the intensity drops almost to zero between oscillations. In terms of the new inner product one can find a different conserved property instead of the integrated intensity. In the case of $\mathcal{PT}$-symmetric waveguides one can find that the conserved property is $\frac{d}{dz}\int_{-\infty}^{\infty}E_y^*(-x,z)E_y(x,z)dx = 0$. Yet another effect that can be observed in the suggested experiment is that the maximum intensity reached by the initially normalized sum field increases as the branch point is approached. This can be understood by observing that as one approaches the self-orthogonal state the overlap between the two functions comprising the sum field increases.

We also note that the manifestation of $\mathcal{PT}$ symmetry and its resulting properties is not (theoretically) restricted to optical systems. To date, however, optical systems seem to be the most readily applicable. One could easily envision a setting using matter waves in which a condensate is placed in a double well potential where in one well particles are injected into the condensate whereas in the second well particles are removed from the condensate. Here attention should be given to the nonlinearity of the Gross-Pitaevskii equation. In order to keep the dynamics similar to that described in the optics experiment the nonlinearity should be made small. This can be achieved either by tuning the interaction between the atoms to zero or by using a very dilute sample. The experiment would also require accurate and independent control over the rates of particles injected or removed from the system. Hopefully, experimental methods will improve to allow such experiments to be done.

To conclude we wish to share the following analogy: the experiment we propose enables the measurement of the transition, through a branch point, from a real to a complex spectrum of a quantum system which is reminiscent of a transition through a bifurcation point from a stable to an unstable periodic orbit in nonlinear classical systems.

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[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[3] A. Mostafazadeh, Phys. Rev. Lett. **99**, 130502 (2007).
[4] A. Ruschhaupt, F. Delgado, and J. G. Muga, J. Phys. A **38**, L171 (2005).
[5] R. El-Ganainy, K. G. Makris, D. N. Christodoulides, and Z. H. Musslimani, Opt. Lett. **32**, 2632 (2007); K. G. Makris *et al.*, Phys. Rev. Lett. **100**, 103904 (2008); Z. H. Musslimani *et al.*, Phys. Rev. Lett. **100**, 030402 (2008).
[6] D. Krejcirik and M. Tater, J. Phys. A **41**, 244013 (2008); D. Borisov and D. Krejcirik, arXiv:0707.3039v1.
[7] C. M. Bender, M. V. Berry, and A. Mandilara, J. Phys. A **35**, L467 (2002).
[8] E. Delabaere and D. T. Trinh, J. Phys. A **33**, 8771 (2000).
[9] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A **34**, 5679 (2001).
[10] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A **34**, L391 (2001).
[11] M. Znojil, Phys. Lett. A **285**, 7 (2001); H. Langer and C. Tretter, Czech. J. Phys. **54**, 1113 (2004).
[12] N. Moiseyev and S. Friedland, Phys. Rev. A **22**, 618 (1980).
[13] T. Kato, *Perturbation Theroy of Linear Operators* (Springer, Berlin, 1966).
[14] M. V. Berry, Czech. J. Phys. **54**, 1039 (2004); J. Rubinstein, P. Sternberg, and Q. Ma, Phys. Rev. Lett. **99**, 167003 (2007); U. Günther, I. Rotter, and B. Samsonov, J. Phys. A **40**, 8815 (2007); P. Cejnar, S. Heinze, and M. Macek, Phys. Rev. Lett. **99**, 100601 (2007).
[15] H. Cartarius, J. Main, and G. Wunner, Phys. Rev. Lett. **99**, 173003 (2007).
[16] C. Dembowski *et al.*, Phys. Rev. Lett. **86**, 787 (2001); **90**, 034101 (2003).
[17] E. P. Wigner, Math. u. Naturwiss. Anzeig. d. Ungar. Akad. Wiss. **LIII**, 475 (1935).
[18] A. Siegman, J. Opt. Soc. Am. A **20**, 1617 (2003).
[19] D. J. Lovering *et al.*, Phys. Rev. Lett. **68**, 1880 (1992).
[20] N. Moiseyev, Phys. Rep. **302**, 212 (1998).