Naimark-Dilated $\mathcal{PT}$-Symmetric Brachistochrone

Uwe Günther[1,*] and Boris F. Samsonov[2,†]
[1]Research Center Dresden-Rossendorf, POB 510119, D-01314 Dresden, Germany
[2]Tomsk State University, 36 Lenin Avenue, 634050 Tomsk, Russia
(Received 23 July 2008; published 5 December 2008)

The quantum mechanical brachistochrone system with a $\mathcal{PT}$-symmetric Hamiltonian is Naimark-dilated and reinterpreted as a subsystem of a Hermitian system in a higher-dimensional Hilbert space. This opens a way to a direct experimental implementation of the recently hypothesized $\mathcal{PT}$-symmetric ultrafast brachistochrone regime of Bender *et al.* [Phys. Rev. Lett. **98**, 040403 (2007)] in an entangled two-spin system.

DOI: 10.1103/PhysRevLett.101.230404 PACS numbers: 03.65.Xp, 03.65.Ca, 03.65.Ud, 03.67.Lx

Introduction.—The quantum brachistochrone problem consists in finding a Hamiltonian H which evolves a given initial state $|\psi_I\rangle$ into a given final state $|\psi_F\rangle$ in a minimal time τ. Considering this problem for quantum mechanics with $\mathcal{PT}$-symmetric Hamiltonians (PTQM), Bender, Brody, Jones, and Meister (BBJM) found the surprising result [1] that the minimal evolution (passage) time $\tau_{\mathcal{PT}}$ was less than the minimal time τ_h required for the evolution induced by a Hermitian Hamiltonian [2,3]. It could be made even arbitrarily small $\tau_{\mathcal{PT}} \to 0$ in a strongly non-Hermitian regime [1,4]. If this effect of a "faster than Hermitian" evolution [1] were experimentally realizable, it would open a way to ultrafast quantum computing processes [5]. A problem still unsolved in Ref. [1] concerned the switching mechanism between the $\mathcal{PT}$-symmetric brachistochrone system and a conventional (von Neumann) quantum system necessary for an experimental implementation of the suggested ultrafast quantum process.

As shown by Mostafazadeh [6], an equivalence mapping [7] between PTQM in the sector of unbroken $\mathcal{PT}$-symmetry and conventional quantum mechanics (CQM) leaves the passage time of a brachistochrone invariant $\tau_h = \tau_{\mathcal{PT}}$. This implies that a vanishing passage time $\tau_{\mathcal{PT}} \to 0$ in the $\mathcal{PT}$-symmetric system is necessarily connected with a vanishing distance between initial and finite states in the equivalent Hermitian system—an effect geometrically analyzed in [8]. In the case of the Hermitian equivalent of the BBJM brachistochrone, initial and final states will nearly coincide (coincidence problem) so that the brachistochrone effect in such an interpretation would lose any physical relevance.

In this Letter, we propose a realization of the BBJM brachistochrone [1] which resolves the switching problem between PTQM and CQM regimes [1] and avoids the coincidence problem [6,8] and which can be considered as a starting point for a direct experimental implementation. The key idea consists in a reinterpretation of the BBJM brachistochrone as a $\mathcal{PT}$-symmetric subsystem of a larger CQM system living in a higher-dimensional Hilbert space. For this purpose we use a Naimark dilation (extension) technique [9] as it is widely used in quantum information theory [5]. We will demonstrate that the resulting large system will have the structure of an entangled two-spin (two-qubit) system so that an experimental realization of the BBJM-brachistochrone effect should be feasible, e.g., in a suitably designed system of entangled polarized photons [10].

Technically, the construction of the large Hermitian system will be accomplished by a three-step procedure: (i) by building a suitable positive operator valued measure (POVM) [5,9,11] over the nonorthogonal eigenstates of the $\mathcal{PT}$-symmetric Hamiltonian H and its adjoint $H^\dagger$, (ii) by Naimark dilating (extending) [9] this POVM into an orthogonal projector set in a higher-dimensional Hilbert space, and (iii) by constructing from it a corresponding Hermitian Hamiltonian $\mathbf{H} = \mathbf{H}^\dagger$ and a unitary evolution operator $\mathbf{U}(t) = e^{-it\mathbf{H}}$.

BBJM brachistochrone.—The BBJM brachistochrone [1] that we are going to Hermitianly dilate (extend) describes the evolution from an initial state $|\psi_I\rangle$ to a final state $|\psi_F\rangle$ governed by a $\mathcal{PT}$-symmetric Hamiltonian H whose structure is chosen in such a way that the time τ required for the evolution becomes minimal. As shown in Ref. [2], such a minimal-passage-time solution follows a minimal geodesic in projective Hilbert space, and it is therefore located in the two-dimensional subspace $\mathcal{H}_2 = \mathbb{C}^2$ spanned by $|\psi_I\rangle$ and $|\psi_F\rangle$. In this $\mathcal{H}_2$, the $\mathcal{PT}$-symmetric Hamiltonian H can be chosen as [1,4]

$$H = E_0 I_2 + s\begin{pmatrix} i\sin(\alpha) & 1 \\ 1 & -i\sin(\alpha) \end{pmatrix}, \quad E_0, s \in \mathbb{R}, \qquad (1)$$

where $\mathcal{P} = \sigma_x$ denotes the parity operator, $\mathcal{T}$ is the antilinear operator of time reflection and complex conjugation [12], E_0 denotes an irrelevant offset energy, and s is a general scaling factor of the matrix. (As usual, σ_x, σ_y, and σ_z are Pauli matrices.) The angle $\alpha \in (-\pi/2, \pi/2)$ characterizes the non-Hermiticity of the Hamiltonian: $H(\alpha = 0)$ is Hermitian, whereas in the limit $\alpha \to \pm\pi/2$ the Hamiltonian H becomes strongly non-Hermitian and similar to a Jordan block, i.e., its eigenvectors

$$|E_+(\alpha)\rangle = \frac{e^{i\alpha/2}}{\sqrt{2\cos(\alpha)}}\begin{pmatrix} 1 \\ e^{-i\alpha} \end{pmatrix},$$
$$|E_-(\alpha)\rangle = \frac{ie^{-i\alpha/2}}{\sqrt{2\cos(\alpha)}}\begin{pmatrix} 1 \\ -e^{i\alpha} \end{pmatrix} \quad (2)$$

and eigenvalues $E_\pm = E_0 \pm s\cos(\alpha) =: E_0 \pm \omega_0/2$ coalesce for fixed $|s| < \infty$ [4]. The Hamiltonian is restricted to purely real eigenvalues, i.e., the parameter sector of exact $\mathcal{PT}$ symmetry [12]. The operator $U(t) = e^{-itH}$ of the nonunitary evolution induced by H has the explicit form

$$U(t) = \frac{e^{-iE_0 t}}{\cos(\alpha)}\begin{pmatrix} \cos(y-\alpha) & -i\sin(y) \\ -i\sin(y) & \cos(y+\alpha) \end{pmatrix}, \quad (3)$$

with $y := \omega_0 t/2$ (we set $\hbar = 1$). In the BBJM-brachistochrone setup [1], this $U(t)$ is used to evolve an initial state $|\psi_I\rangle = (1, 0)^T$ into a final state $|\psi_F\rangle = \mu_F(0, 1)^T$, $\mu_F := -ie^{-iE_0\tau}$. The time τ required for this evolution follows from the condition $y = \alpha + \pi/2$ as $\tau = \frac{\alpha+\pi/2}{s\cos(\alpha)}$ and tends for

$$\alpha = \varepsilon - \pi/2, \qquad \varepsilon \to +0 \quad (4)$$

and fixed $s\cos(\alpha) = \omega_0/2$ to zero: $\tau \to 0$. In this way the evolution from $|\psi_I\rangle$ to the orthogonal $|\psi_F\rangle$ induced by the $\mathcal{PT}$-symmetric Hamiltonian H with eigenstates of fixed energy difference $E_+ - E_- = \omega_0$ appears faster than an evolution between these states induced by any Hermitian Hamiltonian with the same energy difference ω_0 between its eigenstates. This is due to the fact that the evolution time τ between orthogonal states in Hermitian systems has to be larger than the Anandan-Aharonov lower bound $\tau \geq \tau_h = \pi/\omega_0$ [2,13].

Before we embed the BBJM brachistochrone into a larger Hermitian model, we briefly collect the required setup information. The eigenvectors (2) of H are normalized with regard to the $\mathcal{PT}$ inner product $(u, v) = \mathcal{PT}u \cdot v$ as $(E_\pm, E_\pm) = \pm 1$, $(E_\pm, E_\mp) = 0$ [12], and for $\alpha \neq 0$ they are nonorthogonal with regard to the standard inner product in the Hilbert space $\mathcal{H}_2 = \mathbb{C}^2$: $\langle E_\pm | E_\mp\rangle \neq 0$. We supplement them via $H^\dagger(\alpha) = H(-\alpha)$ with the eigenvectors $|E_+(-\alpha)\rangle$ and $|E_-(-\alpha)\rangle$ of the adjoint operator $H^\dagger$ and arrange them as columns in the matrices

$$\Psi := [|E_+(\alpha)\rangle, |E_-(\alpha)\rangle], \quad \Xi := [|E_+(-\alpha)\rangle, |E_-(-\alpha)\rangle]. \quad (5)$$

With $\tilde{E} := \mathrm{diag}(E_+, E_-)$, the eigenvalue problems for H and $H^\dagger$ then take the compact matrix form

$$H\Psi = \Psi\tilde{E}, \qquad H^\dagger\Xi = \Xi\tilde{E}. \quad (6)$$

Apart from the biorthonormality relation $\Xi^\dagger\Psi = I_2$, it holds that $\Psi\Psi^\dagger H^\dagger = H\Psi\Psi^\dagger$ and $\Xi\Xi^\dagger H = H^\dagger\Xi\Xi^\dagger$ so that one identifies $(\Psi\Psi^\dagger)^{-1} = \Xi\Xi^\dagger = \eta$ as a metric operator in the pseudo-Hermiticity condition $\eta H = H^\dagger\eta$ [7]. Additionally to its obvious Hermiticity $\eta = \eta^\dagger$, the metric can be suitably scaled to be an element of the hyperbolic ("boost") sector of the complex orthogonal group $SO(2, \mathbb{C})$ [8]

$$\eta = \frac{1}{\cos(\alpha)}\begin{pmatrix} 1 & -i\sin(\alpha) \\ i\sin(\alpha) & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cosh(\beta) & -i\sinh(\beta) \\ i\sinh(\beta) & \cosh(\beta) \end{pmatrix} = e^{\beta\sigma_y}, \quad (7)$$

with parameter identification $\sin(\alpha) =: \tanh(\beta)$ and $\cos(\alpha) = 1/[\cosh(\beta)]$. As a final ingredient, we fix the notation for the one-to-one similarity mapping between the $\mathcal{PT}$-symmetric Hamiltonian H and its isospectral Hermitian counterpart $h = h^\dagger$ [7], $H = \rho^{-1}h\rho$, $H^\dagger = \rho h\rho^{-1}$, $\rho^2 = \eta$, as well as for the unitary eigenvector matrix Φ

$$h\Phi = \Phi\tilde{E}, \qquad \Phi^\dagger = \Phi^{-1}. \quad (8)$$

The eigenvectors of H and $H^\dagger$ can be regarded as $\alpha \rightleftharpoons -\alpha$, i.e., $\beta \rightleftharpoons -\beta$, mirror symmetrically distorted versions of the eigenvectors of the Hamiltonian h

$$\Psi = \rho^{-1}\Phi, \qquad \Xi = \rho\Phi, \qquad \rho^{-1}(\beta) = \rho(-\beta). \quad (9)$$

The orthogonal initial and final vectors $|\psi_I\rangle$ and $|\psi_F\rangle$, respectively, in the BBJM-brachistochrone model on their turn can be considered as eigenstates of a Hermitian spin operator $S_z = \sigma_z$ (a von Neumann observable with orthogonal projector decomposition), whereas the $\mathcal{PT}$-symmetric (non-Hermitian) Hamiltonian H has nonorthogonal eigenvectors $|E_\pm\rangle$ and is not a von Neumann observable. Under the one-to-one equivalence mapping [7] from H to the Hermitian Hamiltonian h, the spin operator S_z maps into a non-Hermitian operator $s_z = \rho S_z\rho^{-1} \neq s_z^\dagger$. Hence, the BBJM-brachistochrone system in both representations (H, S_z) and (h, s_z) contains operators which are not von Neumann observables, and therefore the system cannot be considered as fundamental.

Naimark dilation.—In order to give the BBJM system with evolution $\psi(t) = U(t)\psi_I$ a meaning in CQM, we embed it into a larger purely Hermitian system

$$\hat{\boldsymbol{\psi}}(t) = \mathbf{U}(t)\hat{\boldsymbol{\psi}}_I, \qquad \hat{\boldsymbol{\psi}}(t) = \begin{pmatrix} \psi(t) \\ \chi(t) \end{pmatrix}, \quad (10)$$

with unitary evolution operator $\mathbf{U}(t) = [\mathbf{U}^\dagger(t)]^{-1}$ and additional ancilla wave function component $\chi(t)$. For this purpose we construct an auxiliary POVM [5,9,11] $\sum_{k=1}^4 A_k = I_2$ over the nonorthogonal eigenvectors (2) and (5) of H and its adjoint $H^\dagger$ with rank-one operators $A_{1,2} = f^2|E_\pm(\alpha)\rangle\langle E_\pm(\alpha)|$ and $A_{3,4} = f^2|E_\pm(-\alpha)\rangle\langle E_\pm(-\alpha)|$. For symmetry reasons, all A_k are scaled with the same normalization factor

$$f := \sqrt{\frac{\cos(\alpha)}{2}} = \frac{1}{\sqrt{2\cosh(\beta)}}. \quad (11)$$

Following standard techniques [9,11], the nonorthogonal POVM elements A_k, $A_jA_k \neq \delta_{jk}A_k$ in the two-dimensional Hilbert space $\mathcal{H}_2 \cong \mathbb{C}^2$ can be Naimark-dilated (extended) into orthoprojectors $P_1, \ldots, P_4$, $P_jP_k = \delta_{jk}P_k$, $\sum_{k=1}^4 P_k = I_4$ in a four-dimensional Hilbert space $\mathcal{H}_4$. Under this Naimark dilation the POVM-normalized nonorthogonal eigenvectors $f|E_\pm(\alpha)\rangle$ and $f|E_\pm(-a)\rangle$ of H

and its adjoint $H^\dagger$ are extended into a set of orthonormal vectors $|v_1\rangle, \ldots, |v_4\rangle \in \mathcal{H}_4$, $\langle v_j|v_k\rangle = \delta_{jk}$, so that $P_k = |v_k\rangle\langle v_k|$ and the 4×4 matrix $\mathbf{V} := [|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle]$ is unitary [9]. The explicit form of $\mathbf{V}$ can be easily found from the block matrix ansatz

$$\mathbf{V} = f\begin{pmatrix} \rho^{-1} & \rho \\ X & Y \end{pmatrix}\mathbf{\Phi},$$

which naturally follows from relations (5) and (9) and the notation $\mathbf{\Phi} := \mathrm{diag}(\Phi, \Phi) = I_2 \otimes \Phi$. The auxiliary condition $f^2[\rho^2 + \rho^{-2}] = f^2[\eta + \eta^{-1}] = I_2$ together with the unitarity constraint $\mathbf{V}\mathbf{V}^\dagger = \mathbf{V}^\dagger\mathbf{V} = \mathbf{I}_4$ fixes the nonsingular matrices X and Y up to an irrelevant unitary rotation as $X = \rho$, $Y = -\rho^{-1}$, so that

$$\begin{aligned}\mathbf{V} &= f[\sigma_z \otimes \rho^{-1} + \sigma_x \otimes \rho](I_2 \otimes \Phi) \\ &= f[\sigma_z \otimes \Psi + \sigma_x \otimes \Xi]. \end{aligned} \tag{12}$$

We use this matrix $\mathbf{V}$ to construct a self-consistent CQM in $\mathcal{H}_4$. For this purpose we require the original eigenvalue problems for H and $H^\dagger$ to be recovered when the model is restricted to the first two rows of $\mathbf{V}$. From relations (6) and an ansatz $f[H\Psi, H^\dagger\Xi] = f[\Psi\tilde{E}, \Xi\tilde{E}] = f[\Psi, \Xi]\mathbf{E}$, the eigenvalue matrix $\mathbf{E}$ for the dilated problem can be read off as $\mathbf{E} := \mathrm{diag}(\tilde{E}, \tilde{E}) = I_2 \otimes \tilde{E}$. This means that the corresponding dilated Hamiltonian $\mathbf{H}$ will have the two eigenvalues $E_\pm$ of H and its isospectral adjoint $H^\dagger$ as double degenerate eigenvalues. The Hamiltonian $\mathbf{H}$ itself can be built from Eq. (8) and $\mathbf{HV} = \mathbf{VE}$ as $\mathbf{H} = \mathbf{VEV}^\dagger$ so that

$$\begin{aligned}\mathbf{H} &= f^2[I_2 \otimes (H\eta^{-1} + \eta H) + i\sigma_y \otimes (H - H^\dagger)] \\ &= I_2 \otimes \Lambda + i\sigma_y \otimes \Omega, \\ \Lambda &:= f^2(H\eta^{-1} + \eta H) = E_0 I_2 + \frac{\omega_0}{2}\cos(\alpha)\sigma_x, \\ \Omega &:= f^2(H - H^\dagger) = i\frac{\omega_0}{2}\sin(\alpha)\sigma_z. \end{aligned}$$

This $\mathbf{H}$ is Hermitian by construction. In the Hermitian limit of the original $\mathcal{PT}$-symmetric Hamiltonian H, i.e., for $\alpha = \beta = 0$, it holds that $\eta = I_2$ and $\mathbf{H}$ reduces to $\mathbf{H} = I_2 \otimes h$—a trivially doubled h. In contrast to the PTQM Hamiltonian H, its dilation $\mathbf{H}$ remains well defined also in the strongly non-Hermitian vanishing-passage-time regime (4) where the matrix components of H diverge for fixed ω_0 as $s \to \infty$. This regularization effect is due to the normalization factor f^2 induced in $\mathbf{H}$ via the auxiliary POVM construction.

The $\mathbf{H}$-induced unitary evolution in $\mathcal{H}_4$ is governed by the operator $\mathbf{U}(t) = e^{-it\mathbf{H}} = \mathbf{V}e^{-i\mathbf{E}t}\mathbf{V}^\dagger$, which with $U(t) = e^{-itH} = \rho^{-1}e^{-ith}\rho$ and $y = \omega_0 t/2$ can be represented as

$$\begin{aligned}\mathbf{U}(t) &= f^2\{I_2 \otimes [U(t)\eta^{-1} + \eta U(t)] \\ &\quad + i\sigma_y \otimes [U(t) - \eta U(t)\eta^{-1}]\} \\ &= (I_2 \otimes F + i\sigma_y \otimes G) = \begin{pmatrix} F & G \\ -G & F \end{pmatrix}, \end{aligned} \tag{13}$$

$$\begin{aligned} F &:= e^{-iE_0 t}[I_2\cos(y) - i\sigma_x\sin(y)\cos(\alpha)], \\ G &:= e^{-iE_0 t}[\sin(y)\sin(\alpha)\sigma_z]. \end{aligned}$$

Physically, $\mathbf{U}(t)$ describes the time evolution of the coupled brachistochrone-ancilla system (10) in a Hilbert space $\mathcal{H}_4 = \mathcal{H}_2 \oplus \tilde{\mathcal{H}}_2$, with $\psi(t) \in \mathcal{H}_2$ and $\chi(t) \in \tilde{\mathcal{H}}_2$. In order to exactly reproduce the $\mathcal{H}_2$ evolution (3) of the BBJM-brachistochrone subsystem

$$\psi(t) = U(t)\psi_I = \frac{e^{-iE_0 t}}{\cos(\alpha)}\begin{pmatrix} \cos(y-\alpha) \\ -i\sin(y) \end{pmatrix}, \tag{14}$$

the initial vector $\chi_I \in \tilde{\mathcal{H}}_2$ of the ancilla subsystem should be chosen appropriately. To obtain χ_I, we represent $\hat{\boldsymbol{\psi}}(t) \in \mathcal{H}_4$ as

$$\hat{\boldsymbol{\psi}}(t) = \begin{pmatrix} \psi(t) \\ \chi(t) \end{pmatrix} = e_+ \otimes \psi(t) + e_- \otimes \chi(t), \tag{15}$$

with $e_+ := (1,0)^T$ and $e_- := (0,1)^T$, define $P_\pm := e_\pm \otimes e_\pm^\dagger$, and introduce the projectors $\mathbf{P}_\pm = P_\pm \otimes I_2$ on the brachistochrone ($\mathbf{P}_+$) and the ancilla ($\mathbf{P}_-$) subspace. The identification rule (10) then takes the form $\mathbf{P}_+\hat{\boldsymbol{\psi}}(t) = \mathbf{P}_+\mathbf{U}(t)\hat{\boldsymbol{\psi}}_I = e_+ \otimes \psi(t) = e_+ \otimes U(t)\psi_I$. After taking the time derivative, one finds from

$$\mathbf{P}_+\mathbf{H}\hat{\boldsymbol{\psi}}(t) \equiv e_+ \otimes [\Lambda\psi(t) + \Omega\chi(t)] = e_+ \otimes H\psi(t) \tag{16}$$

a synchronization link [14] between ancilla and brachistochrone evolution $\chi(t) = \Omega^{-1}(H - \Lambda)\psi(t) = \eta\psi(t) = \eta U(t)\eta^{-1}\chi_I$ as well as the explicit ancilla evolution

$$\chi(t) = \frac{e^{-iE_0 t}}{\cos(\alpha)}\begin{pmatrix} \cos(y) \\ -i\sin(y-\alpha) \end{pmatrix}.$$

Initial and final ancilla components then take the form

$$\chi_I = \eta\psi_I = \frac{1}{\cos(\alpha)}\begin{pmatrix} 1 \\ i\sin(\alpha) \end{pmatrix},$$

$$\chi_F = \eta\psi_F = \frac{-\mu}{\cos(\alpha)}\begin{pmatrix} \sin(\alpha) \\ i \end{pmatrix},$$

respectively, with $\mu = e^{-iE_0\tau}$ an irrelevant phase factor, $\psi_I = (1,0)^T$, and $\psi_F = \mu(0,1)^T$. The relation $\chi_I = \eta\psi_I$ ensures a full synchronization

$$\hat{\boldsymbol{\psi}}(t) = \mathbf{U}(t)\hat{\boldsymbol{\psi}}_I = \begin{pmatrix} U(t) & 0 \\ 0 & \eta U(t)\eta^{-1} \end{pmatrix}\begin{pmatrix} \psi_I \\ \chi_I \end{pmatrix}$$

of the constructed unitary evolution $\mathbf{U}(t)$ of the Hermitian $\mathcal{H}_4$ system with the original nonunitary evolution $U(t)$ of the BBJM-brachistochrone subsystem in $\mathcal{H}_2$ for all parameter values $\alpha \in (-\pi/2, \pi/2)$ and times t, including the ultrafast evolution regime suggested in [1].

Discussion.—For a BBJM brachistochrone in the vanishing-passage-time regime (4), the ancilla vectors χ_I and χ_F become collinear, and their common denominator $\cos(\alpha) \approx \varepsilon$ makes them highly dominant compared to $\psi_{I,F}$. This χ dominance remains preserved for the normalized state vector $\hat{\boldsymbol{\phi}} := g\hat{\boldsymbol{\psi}}$, $\langle\hat{\boldsymbol{\phi}}|\hat{\boldsymbol{\phi}}\rangle = 1$, with $g := \cos(\alpha)/\sqrt{2}$, and leads to a very small brachistochrone component $|g\psi(t)|^2 \approx \varepsilon^2/2$ compared to the ancilla component $|g\chi(t)|^2 \approx 1 - \varepsilon^2/2$. As a result, the geodesic distance between the initial and final states in $\mathcal{H}_4$ becomes small $\delta_4 = 2\arccos(|\langle\hat{\boldsymbol{\phi}}_I|\hat{\boldsymbol{\phi}}_F\rangle|) \approx 2\varepsilon$. This means that the original geodesic distance $\delta_2 = 2\arccos(|\langle\psi_I|\psi_F\rangle|) = \pi$

between the initial and final states in the brachistochrone subsystem is strongly contracted by embedding the latter into the larger Hermitian $\mathcal{H}_4$ system. Geometrically, this follows from the fact that the geodesic distance is given by the angle spanned by the corresponding vectors on the Bloch sphere [2] and its generalization to higher dimensions. In the $\mathcal{H}_2$ subsystem, the vectors ψ_I and ψ_F are antipodal and span an angle $\delta_2 = \pi$. Adding a much longer vector χ orthogonal to ψ_I and ψ_F makes the resulting $\hat{\boldsymbol{\phi}}_I \approx (\psi_I, \chi)^T$ and $\hat{\boldsymbol{\phi}}_F \approx (\psi_F, \chi)^T$ almost collinear $\delta_4 \to 0$ in $\mathcal{H}_4$. In this way, the dilated model reconciles the Anandan-Aharonov lower bound [2,13] on minimal passage times in Hermitian systems with the vanishing-passage-time effect of the BBJM brachistochrone [1] for orthogonal states in the subsystem. The embedding of the BBJM system into a higher-dimensional Hilbert space can be regarded as a strengthening of the wormhole analogy drawn in Ref. [1] for the shortening of the passage time τ. A wormhole connection of two distant points on a given lower-dimensional manifold $\mathcal{M}$ can be best visualized by embedding $\mathcal{M}$ into a higher-dimensional surrounding $\mathcal{N} \supset \mathcal{M}$ so that the corresponding short distance in $\mathcal{N}$ becomes obvious [15].

The representation (15) indicates on a natural interpretation of the obtained Hermitian system $\hat{\boldsymbol{\phi}}(t) = e^{-it\mathbf{H}}\hat{\boldsymbol{\phi}}_I$ as a system of two entangled spin 1/2 particles, i.e., as a two-qubit system [5], with $\Sigma_1 = \sigma_z \otimes I_2$ and $\Sigma_2 = I_2 \otimes \sigma_z$ as spin operators of the two spin subsystems. In order to observe the BBJM-brachistochrone effect of the subsystem, one has to prepare an initial entangled state $\hat{\boldsymbol{\phi}}_I = e_+ \otimes \psi_I + e_- \otimes \chi_I$ to switch on the interaction Hamiltonian $\mathbf{H}$ during the passage time τ (assumed as smaller than the lower passage time bound $\tau_h = \pi/\omega_0$) and to evolve $\hat{\boldsymbol{\phi}}_I$ into the final state $\hat{\boldsymbol{\phi}}_F = e^{-i\tau\mathbf{H}}\hat{\boldsymbol{\phi}}_I$. This final state has to be analyzed in a two-step measurement. In a first (instantaneous) Σ_1 measurement, one selects (filters out) the up component e_+ of the first spin. This results in a state $\mathbf{P}_+\hat{\boldsymbol{\phi}}_F/\langle \mathbf{P}_+\hat{\boldsymbol{\phi}}_F | \mathbf{P}_+\hat{\boldsymbol{\phi}}_F\rangle^{1/2} = e_+ \otimes \psi_F/\langle \psi_F | \psi_F\rangle^{1/2}$ and separates the brachistochrone component from the ancilla component (connected with the down component e_- of the first spin). In a subsequent Σ_2 measurement, one analyzes the spin-up and spin-down states of the brachistochrone component $\psi_F/\langle \psi_F | \psi_F\rangle^{1/2}$ to recover the spin-flip effect from $\psi_I = (1, 0)^T$ to $\psi_F = \mu_F(0, 1)^T$.

A direct experimental test should be feasible with a suitably designed system of entangled photons passing an appropriately chosen arrangement of beam splitters, phase shifters, and mirrors as an implementation of the unitary operator $\mathbf{U}(\tau) = e^{-i\tau\mathbf{H}}$ [10].

Conclusions.—We have demonstrated that the quantum brachistochrone for a system with a $\mathcal{PT}$-symmetric Hamiltonian can be realized as a subsystem of a larger Hermitian system living in a higher-dimensional Hilbert space. The Hermitian system (constructed by Naimark dilating an auxiliary positive operator valued measure) has the structure of an entangled two-qubit system. This opens a way to direct experimental tests on the recently hypothesized "faster than Hermitian" evolution in $\mathcal{PT}$-symmetric quantum systems.

The work of U. G. has been supported by DFG within the Collaborative Research Center SFB 609. B. F. S. is partially supported by Grants No. RFBR-06-02-16719, No. SS-871.2008.2 (Russia), and No. 4-7531.50-04-844-07/5 (Saxon Ministry of Science).

*u.guenther@fzd.de
†samsonov@phys.tsu.ru

[1] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. **98**, 040403 (2007).
[2] D. J. Brody, J. Phys. A **36**, 5587 (2003); D. J. Brody and D. W. Hook, J. Phys. A **39**, L167 (2006).
[3] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Phys. Rev. Lett. **96**, 060503 (2006).
[4] U. Günther, I. Rotter, and B. Samsonov, J. Phys. A **40**, 8815 (2007).
[5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000); I. Bengtsson and K. Zyczkowski, *Geometry of Quantum States* (Cambridge University Press, Cambridge, England, 2006); L. Diósi, *A Short Course in Quantum Information Theory* (Springer, Berlin, 2007).
[6] A. Mostafazadeh, Phys. Rev. Lett. **99**, 130502 (2007).
[7] A. Mostafazadeh, J. Math. Phys. (N.Y.) **43**, 2814 (2002).
[8] U. Günther and B. Samsonov, Phys. Rev. A **78**, 042115 (2008).
[9] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982); *Statistical Structure of Quantum Theory* (Springer, Berlin, 2001).
[10] M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).
[11] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993).
[12] C. M. Bender, Rep. Prog. Phys. **70**, 947 (2007).
[13] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[14] Comparison with the relations (9) between the $\mathcal{PT}$-symmetric subsystem and its adjoint, i.e., with $\Xi = \eta\Psi$, shows that the ancilla $\chi(t)$ can be interpreted as adjoint to the BBJM brachistochrone.
[15] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988).