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MHD turbulence in shear flows and astrophysical discs — specific anisotropy of nonlinear processes, nonmodal growth and sustenance

George Mamatsashvili

Helmholtz-Zentrum Dresden-Rossendorf

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Outline

- Types of nonlinear cascades in (spectraly/modally stable) shear flows:
  direct, inverse and, so-called, **transverse**

- Origin of the nonlinear transverse cascade and its role in the onset and sustenance of subcritical turbulence

- Realisation of the transverse cascade in specific flows:

  – 2D MHD turbulence in spectrally stable shear flows with parallel magnetic field

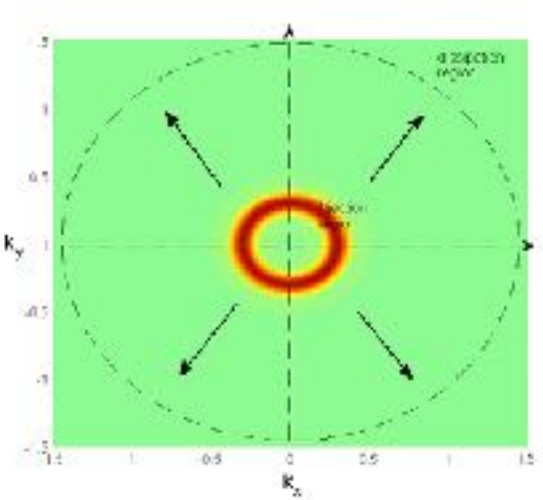
  – 3D MHD magnetorotational (MRI) - turbulence in Keplerian disc flows with net azimuthal magnetic field

- Summary and future extensions

Types of nonlinear cascades in Fourier space in spectrally stable shear flows

- linear processes (dynamics) in many flow systems usually depend on a certain combination of wavenumbers (e.g., on $k^2 = k_x^2 + k_y^2 + k_z^2$ in the isotropic case).

- Nonlinear processes often appear to depend also on the same combination of wavenumbers and change the latter, leading to usual **direct** or/and **inverse** cascades



- **Classical examples include:** Kolmogorov's isotropic HD turbulence or Iroshnikov-Kraichnan theory of MHD turbulence, where nonlinear cascades change only wavenumber magnitude k (i.e., proceed along $\boldsymbol{k}$) of modes

- **Non-normal (non-self-adjoint) nature of linear dynamics of spectrally/modally stable (i.e. without exponentially growing modes) shear flows and its consequences were extensively analysed by the HD community in the 1990s (**e.g., reviews by *Schmid & Henningson 01, Schmid 07*)

Implications of non-normality in (spectrally stable) shear flows

- Eigenfunctions of linearized equations in modal analysis of shear flows are non-orthogonal/non-normal and strongly interfere

- Perturbations exhibit significant finite-time transient, or nonmodal phenomena (amplification), which are missed out in the classical modal/spectral analysis and is grasped by appealing to a different nonmodal analysis

- This transient growth is "imperfect" — it is unable to independently ensure the permanent growth/sustenance of perturbations

- This "imperfection" of the transient growth is compensated by nonlinear positive feedback: subcritical transition and maintenance of turbulence is determined by a subtle interplay of linear *nonmodal* amplification and nonlinear feedback

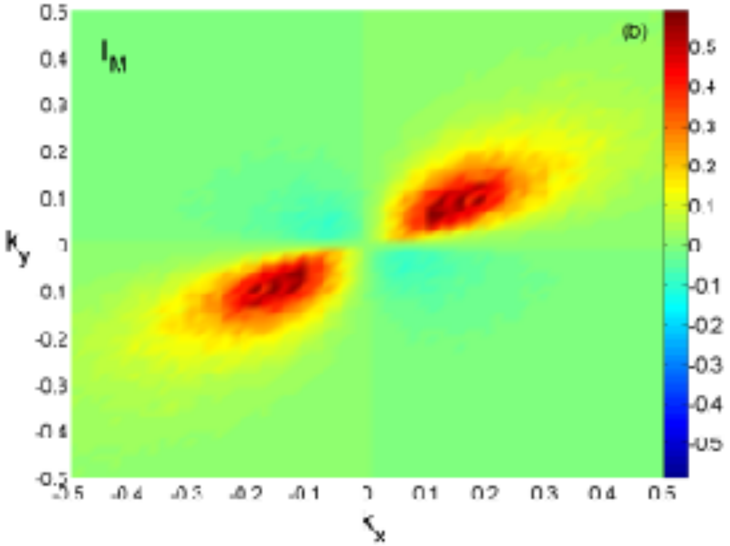
- linear nonmodal processes are **anisotropic** in Fourier (**k-**)space due to the shear.

- **This strong anisotropy of linear processes in shear flows, in turn, leads to anisotropy of nonlinear processes (transfers, or cascades) in k-space as well**
(e.g., Chagelishvili et al 2002, Horton et al. 2010, Mamatsashvili et al. 2014)

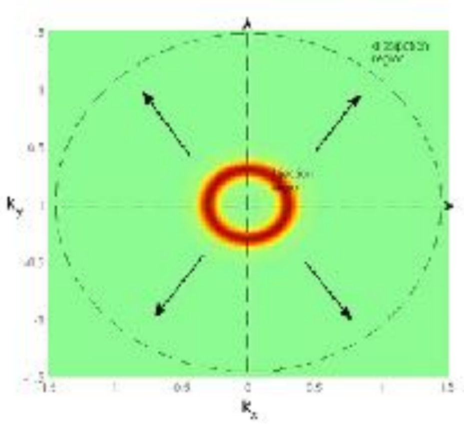
- We refer to this shear-induced anisotropic redistribution of modes over wavevector orientations (angles) as a **nonlinear transverse cascade – a new type of nonlinear cascade existing in shear flows**

- Mean magnetic field itself also causes anisotropy of nonlinear cascades (e.g., Goldreich & Sridhar 1995, Boldyrev 2006), however, **the transverse cascade, originating primarily from shear, is fundamentally different from these anisotropic cascades due to magnetic field**

linear (injection, red) and the nonlinear (transverse cascade from blue to yellow areas) processes in Fourier space and their interplay in the presence of the shear



Classical case without shear flow. External forcing is at narrow wavenumber range and the cascade is isotropic



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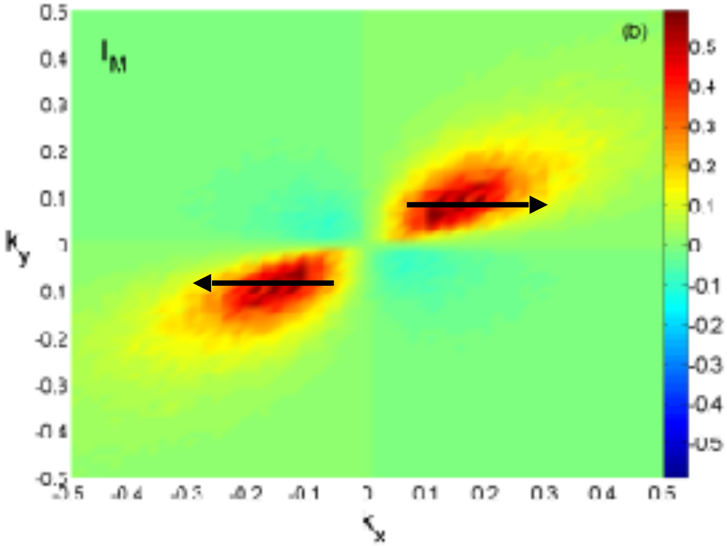

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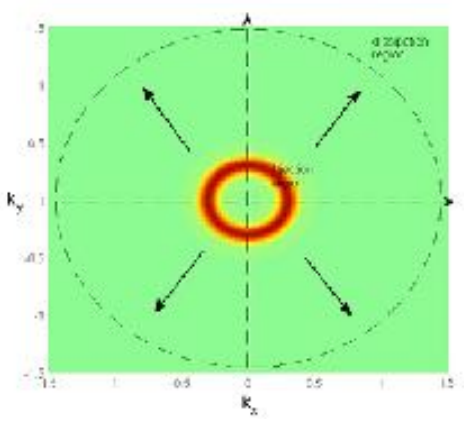

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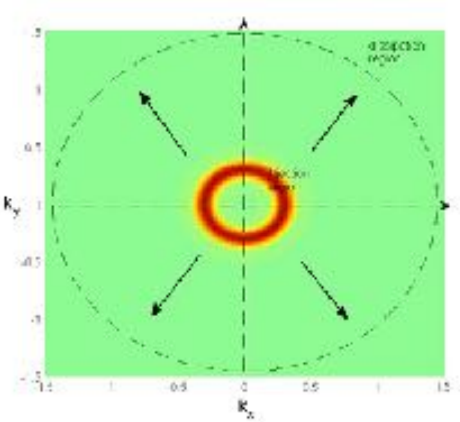

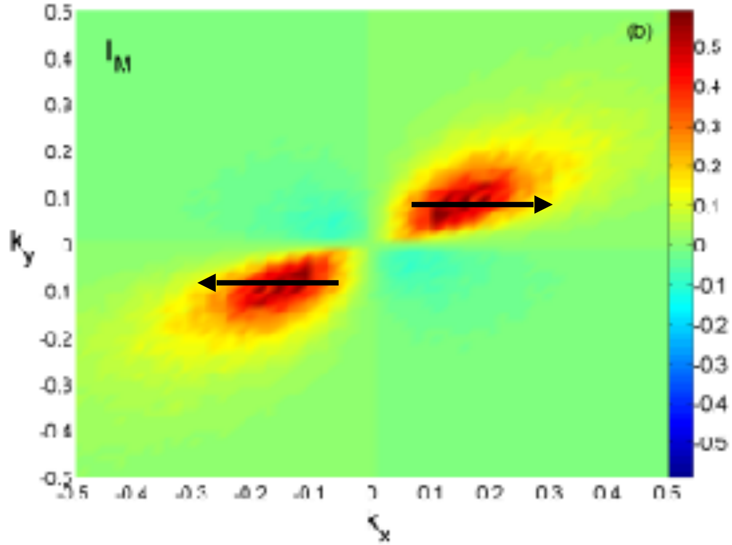

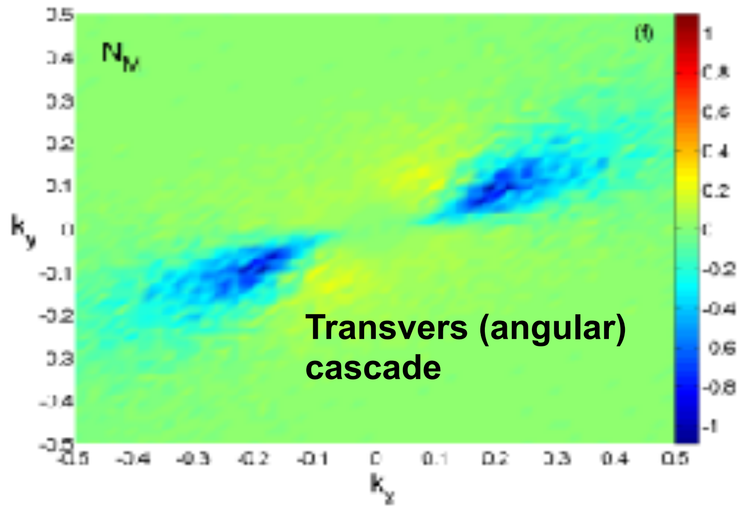

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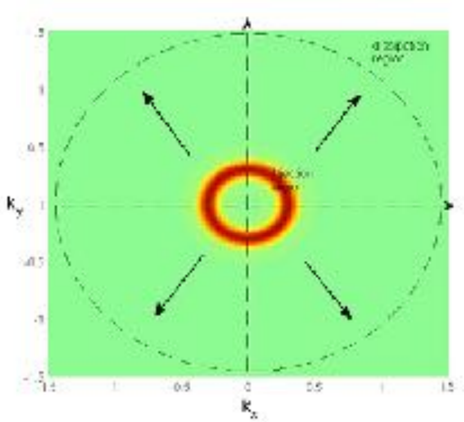

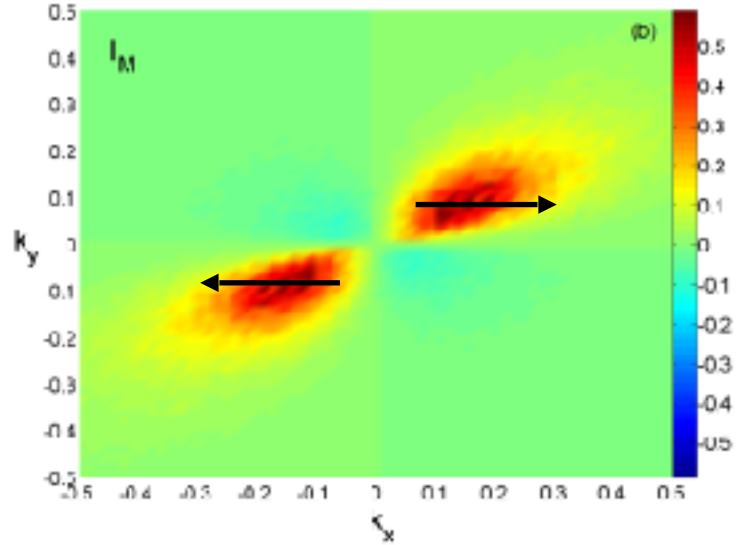

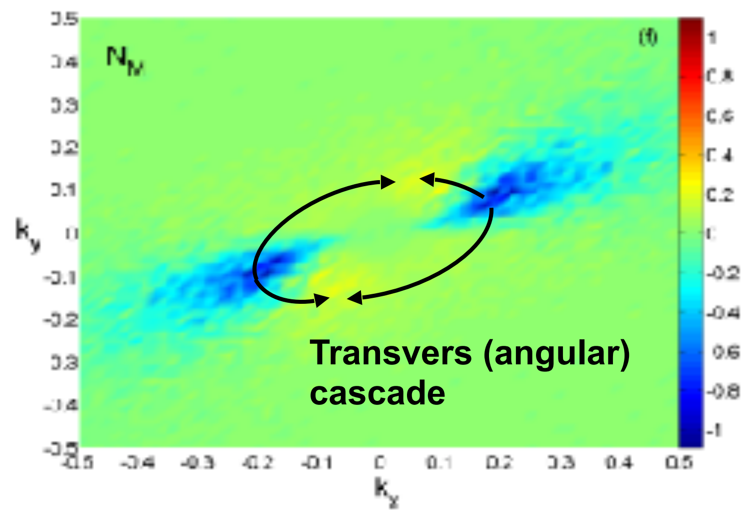

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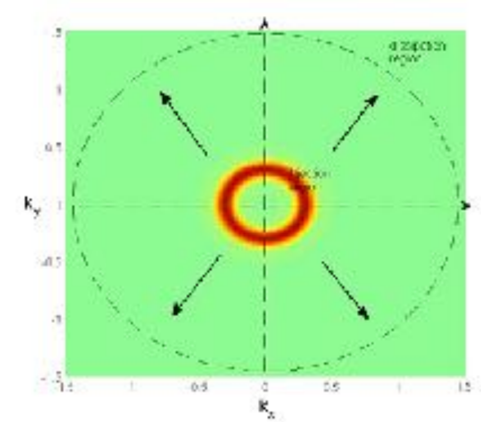
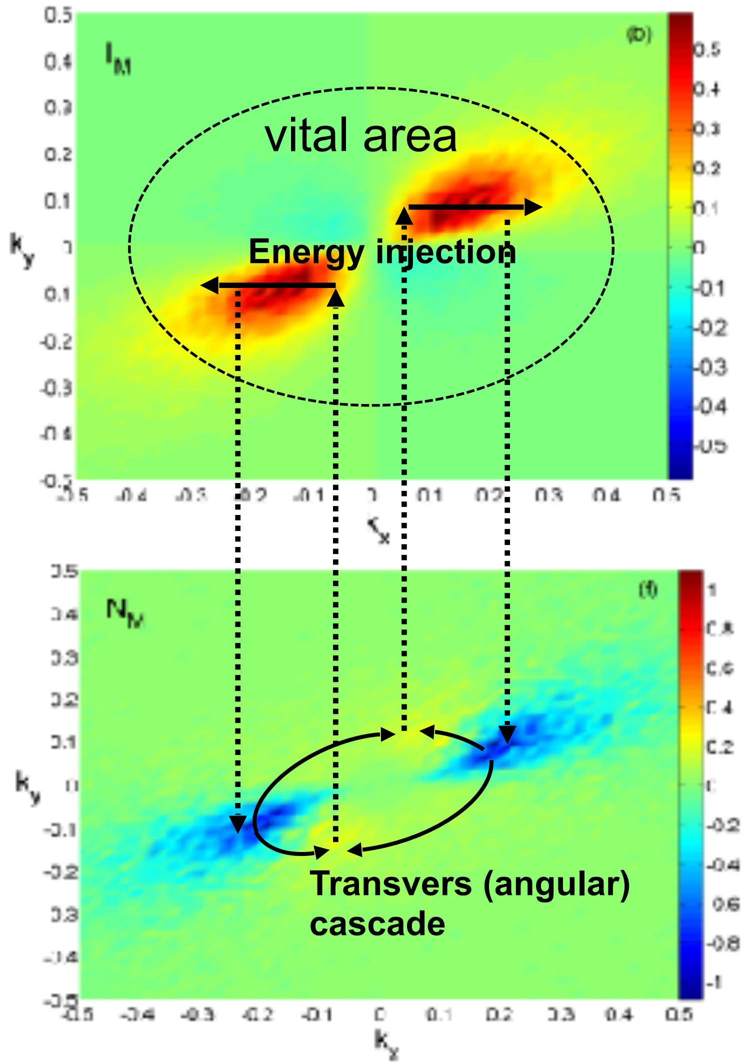
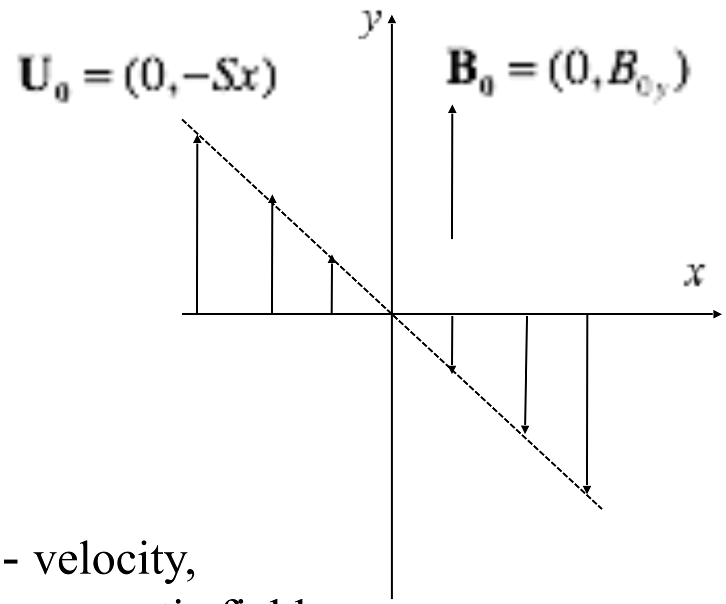




Sustenance of subcritical turbulence in 2D MHD constant shear flow

(*Mamatsashvili et al. 2014, PRE*)

Equilibrium: unbounded flow with constant shear (S) of velocity threaded by a parallel uniform magnetic field



This equilibrium is spectrally stable in the linear regime, i.e., has no exponentially growing modes

Equations of non-ideal MHD

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{\nabla P}{\rho} + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi\rho} + \nu\nabla^2\mathbf{U},$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B},$$

$$\nabla\cdot\mathbf{U} = 0,$$

$$\nabla\cdot\mathbf{B} = 0,$$

$\mathbf{U}$ - velocity,
$\mathbf{B}$ - magnetic field
P - total (thermal +magnetic) pressure
ρ density
ν viscosity
η resistivity

Units and characteristic parameters

$$St \to t, \quad \frac{\mathbf{r}}{\ell} \to \mathbf{r}, \quad \frac{\mathbf{u}}{u_A} \to \mathbf{u}, \quad \frac{p}{\rho_0 u_A^2} \to p, \quad \frac{\mathbf{b}}{B_{0y}} \to \mathbf{b},$$

where $u_A = \dfrac{B_{0y}}{\sqrt{4\pi\rho_0}}$ is the Alfvén speed and $\ell = \dfrac{u_A}{S}$ is the corresponding length

Dynamics is characterised by

1. Ratio of flow domain size L_x, L_y to ℓ : $\dfrac{L_x}{\ell}, \dfrac{L_y}{\ell}$ which sets the minimum wavenumbers in the domain.

2. Non-dimensional Elsasser-like numbers defined in terms of u_A and shear S

$$\Lambda_\nu = \frac{u_A^2}{\nu S}, \quad \Lambda_\eta = \frac{u_A^2}{\eta S}$$

for usually Reynolds numbers in terms of domain size L_x and the flow velocity SL_x , these correspond to

$$\mathrm{Re} = \frac{SL_x^2}{\nu} = \frac{S^2 L_x^2}{u_A^2}\Lambda_\nu, \quad \mathrm{R}m = \frac{SL_x^2}{\nu} = \frac{S^2 L_x^2}{u_A^2}\Lambda_\eta,$$

where $\beta = \dfrac{S^2 L_x^2}{u_A^2}$ plays a role of plasma beta and characterises the strength of the imposed field

Spectral representation of the variables and equations

Fourier transform of perturbations $f \equiv (\mathbf{u}, \mathbf{b})$ about the above equilibrium

$$f(\mathbf{r},t) = \int \bar{f}(\mathbf{k},t)\exp(i\mathbf{k}\cdot\mathbf{r})\,dk_x dk_y$$

Dynamical equations in Fourier (k-)space

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{u}_x = -\mathrm{i}k_x\bar{p} + \mathrm{i}k_y\bar{b}_x - \frac{k^2}{\mathrm{Re}}\bar{u}_x + \mathrm{i}k_y N_1 + \mathrm{i}k_x N_2,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{u}_y = \bar{u}_x - \mathrm{i}k_y\bar{p} + \mathrm{i}k_y\bar{b}_y - \frac{k^2}{\mathrm{Re}}\bar{u}_y + \mathrm{i}k_x N_1 + \mathrm{i}k_y N_3,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{b}_x = \mathrm{i}k_y\bar{u}_x - \frac{k^2}{\mathrm{Rm}}\bar{b}_x + \mathrm{i}k_y N_4,$$

$$\left(\frac{\partial}{\partial t} + k_y \frac{\partial}{\partial k_x}\right)\bar{b}_y = -\bar{b}_x + \mathrm{i}k_y\bar{u}_y - \frac{k^2}{\mathrm{Rm}}\bar{b}_y - \mathrm{i}k_x N_4,$$

$$k_x\bar{u}_x + k_y\bar{u}_y = 0,$$

$$k_x\bar{b}_x + k_y\bar{b}_y = 0,$$

where the terms $N_1(\mathbf{k},t)$, $N_2(\mathbf{k},t)$, $N_3(\mathbf{k},t)$ and $N_4(\mathbf{k},t)$ describe transfers of modes over wavenumbers via **nonlinear triad interactions and** are given by convolutions in Fourier space

$$N_1(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_x(\mathbf{k}',t)\bar{u}_y(\mathbf{k}-\mathbf{k}',t)\right]$$

$$N_2(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_x(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t) - \bar{u}_x(\mathbf{k}',t)\bar{u}_x(\mathbf{k}-\mathbf{k}',t)\right]$$

$$N_3(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{b}_y(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_y(\mathbf{k}',t)\bar{u}_y(\mathbf{k}-\mathbf{k}',t)\right]$$

and the **Electromotive Force (EMF) of perturbations**

$$N_4(\mathbf{k},t) = \int d^2\mathbf{k}' \left[\bar{u}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_y(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t)\right]$$

Evolution equations for the kinetic $\bar{E}_M \doteq |\bar{b}_x|^2 + |\bar{b}_y|^2$, and magnetic $\bar{E}_K \doteq |\bar{u}_x|^2 + |\bar{u}_y|^2$ spectral energy densities

$$\frac{\partial \bar{E}_K}{\partial t} + \frac{\partial}{\partial k_x}(k_y \bar{E}_K) = I_K + I_{K-M} + D_K + N_K,$$

Drift due to flow

Injection

cross terms

Dissipation

nonlinear transfers

$$\frac{\partial \bar{E}_M}{\partial t} + \frac{\partial}{\partial k_x}(k_y \bar{E}_M) = I_M + I_{M-K} + D_M + N_M,$$

Injection terms are responsible for the nonmodal growth – the only source of energy for the perturbations

There is no external forcing

Energy injection: $I_K = \bar{u}_x \bar{u}_y^* + \bar{u}_x^* \bar{u}_y = -\frac{2k_x k_y}{k^2}\bar{E}_K,$ — **Reynolds stress spectrum**

$I_M = -\bar{b}_x \bar{b}_y^* - \bar{b}_x^* \bar{b}_y = \frac{2k_x k_y}{k^2}\bar{E}_M,$ — **Maxwell stress spectrum**

Kinetic-magnetc Cross terms: $I_{K-M} = \mathrm{i}k_y \left(\bar{u}_x^* \bar{b}_x + \bar{u}_y^* \bar{b}_y - \bar{u}_x \bar{b}_x^* - \bar{u}_y \bar{b}_y^*\right), \quad I_{M-K} = -I_{K-M},$

Dissipation: $D_K = -\frac{2k^2}{\mathrm{Re}}\bar{E}_K, \quad D_M = \frac{2k^2}{\mathrm{Re}_m}\bar{E}_M$

Nonlinear transfers: $N_K(\mathbf{k},t) = \mathrm{i}(k_y \bar{u}_x^* + k_x \bar{u}_y^*)N_1(\mathbf{k},t) + \mathrm{i}k_x \bar{u}_x^*[N_2(\mathbf{k},t) - N_3(\mathbf{k},t)] + \mathrm{c.c.}$

$N_M(\mathbf{k},t) = \mathrm{i}(k_y \bar{b}_x^* - k_x \bar{b}_y^*)N_4(\mathbf{k},t) + \mathrm{c.c.}\ .$

The net effect of nonlinear transfers is zero $\int [N_K(\mathbf{k},t) + N_M(\mathbf{k},t)] d^2\mathbf{k} = 0,$

— **no net energy gain for turbulence by nonlinearity**

Nonlinear evolution – self-sustained 2D MHD turbulence

Simulations of the nonlinear dynamics in 2D (in (x,y)-plane) with pseudo-spectral code **SNOOPY** (by G. Lesur)

Random noise finite amplitude perturbations of velocity and magnetic

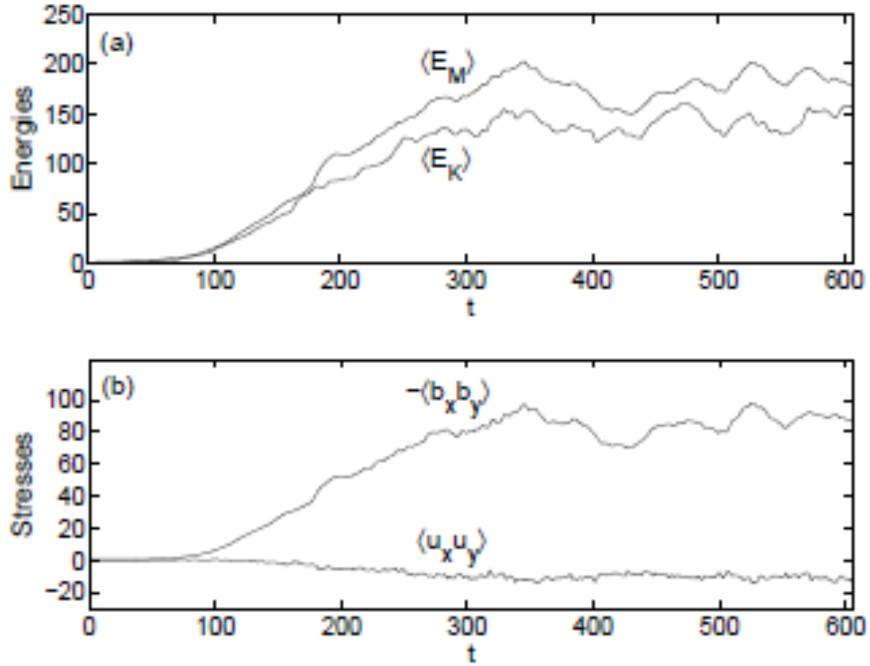
Rectangular domain with sizes: $L_x = L_y = 400 u_A / S$

Resolution: 512×512

Shear-periodic boundary conditions

Elsasser-like numbers as defined above are: $\Lambda_\nu = \Lambda_\eta = 5$

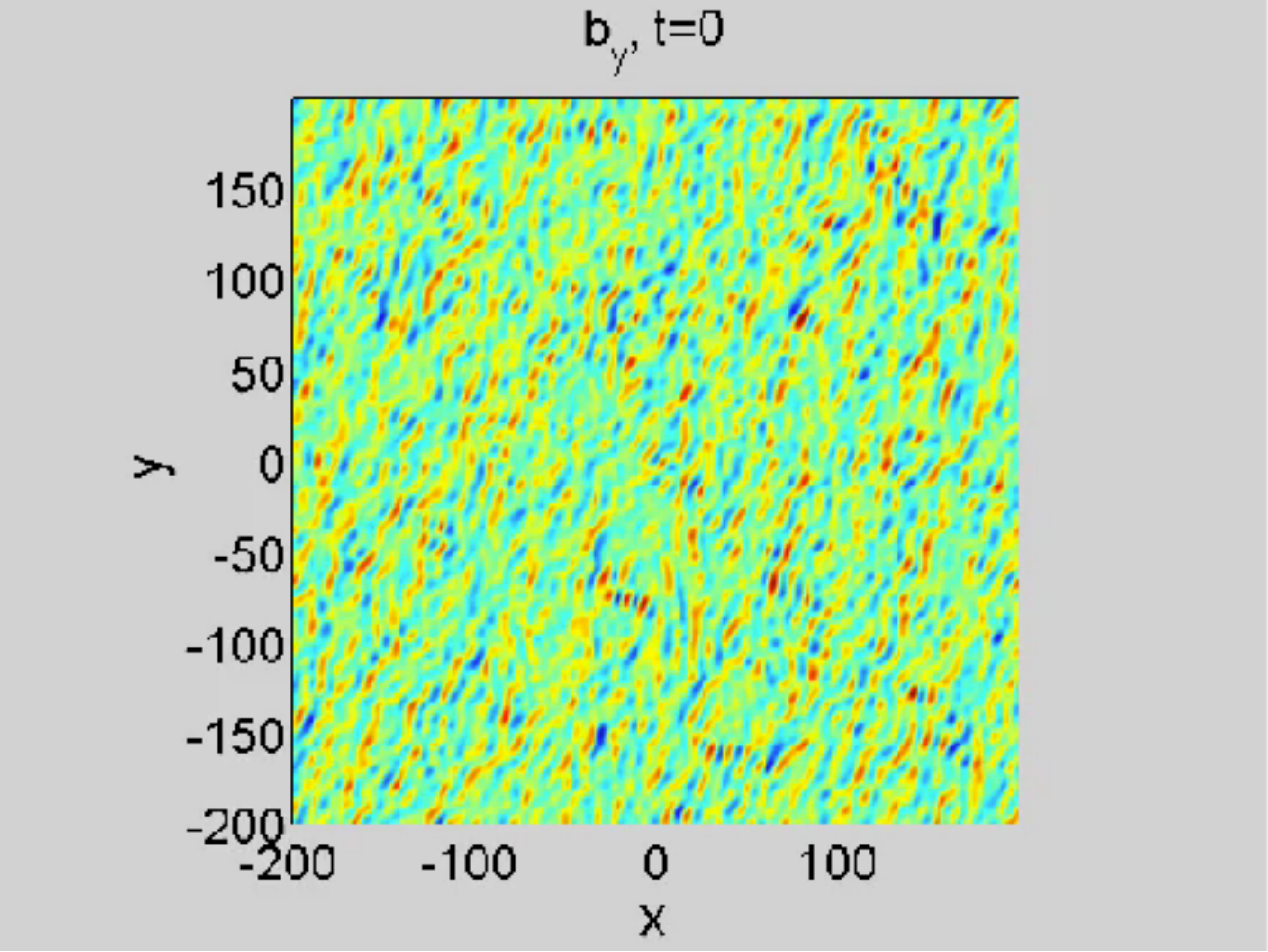
Evolution of the domain-averaged energies and stresses



These quantities grow as a result of nonmodal amplification of separate modes and then saturate to a statistically-steady turbulent state.

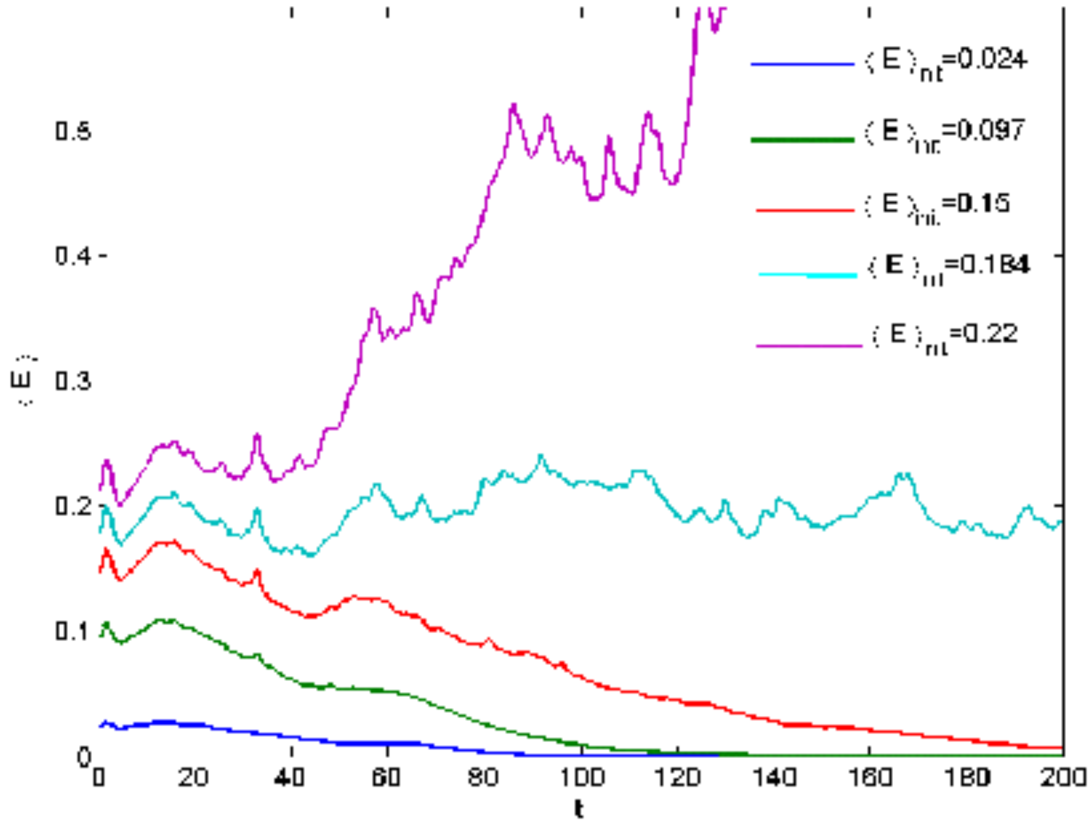
Positive Maxwell stress dominates over negative Reynolds stress and supplies the turbulence with energy.

Turbulence’s structure in the (x,y)-plane (shown is streamwise magnetic field)



Subcritical transition

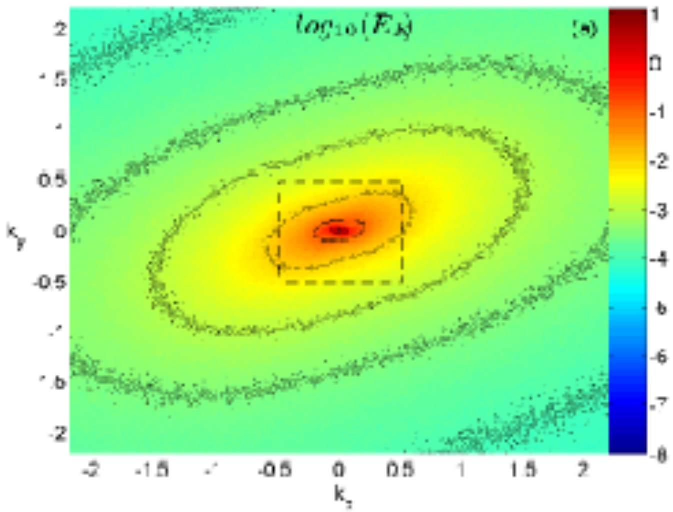
Total energy evolution at different initial perturbations — transition to the sustained state occurs at sufficiently high initial amplitude of perturbations



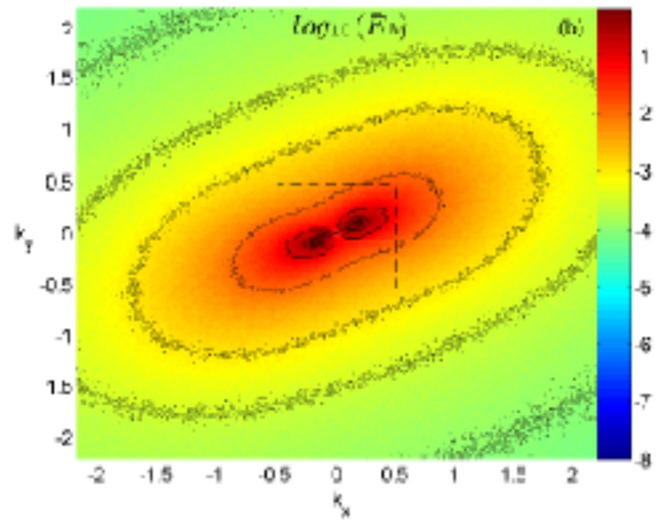
Using the simulation data, we calculate both energy spectra as well as individual energy injection I_K, I_M and nonlinear transfer terms N_K, N_M in spectral equations above, thereby getting essential understanding about the roles of different linear and nonlinear processes and their interplay in the sustenance of the turbulence.

Time-averaged 2D spectra of the kinetic and magnetic energies

kinetic energy



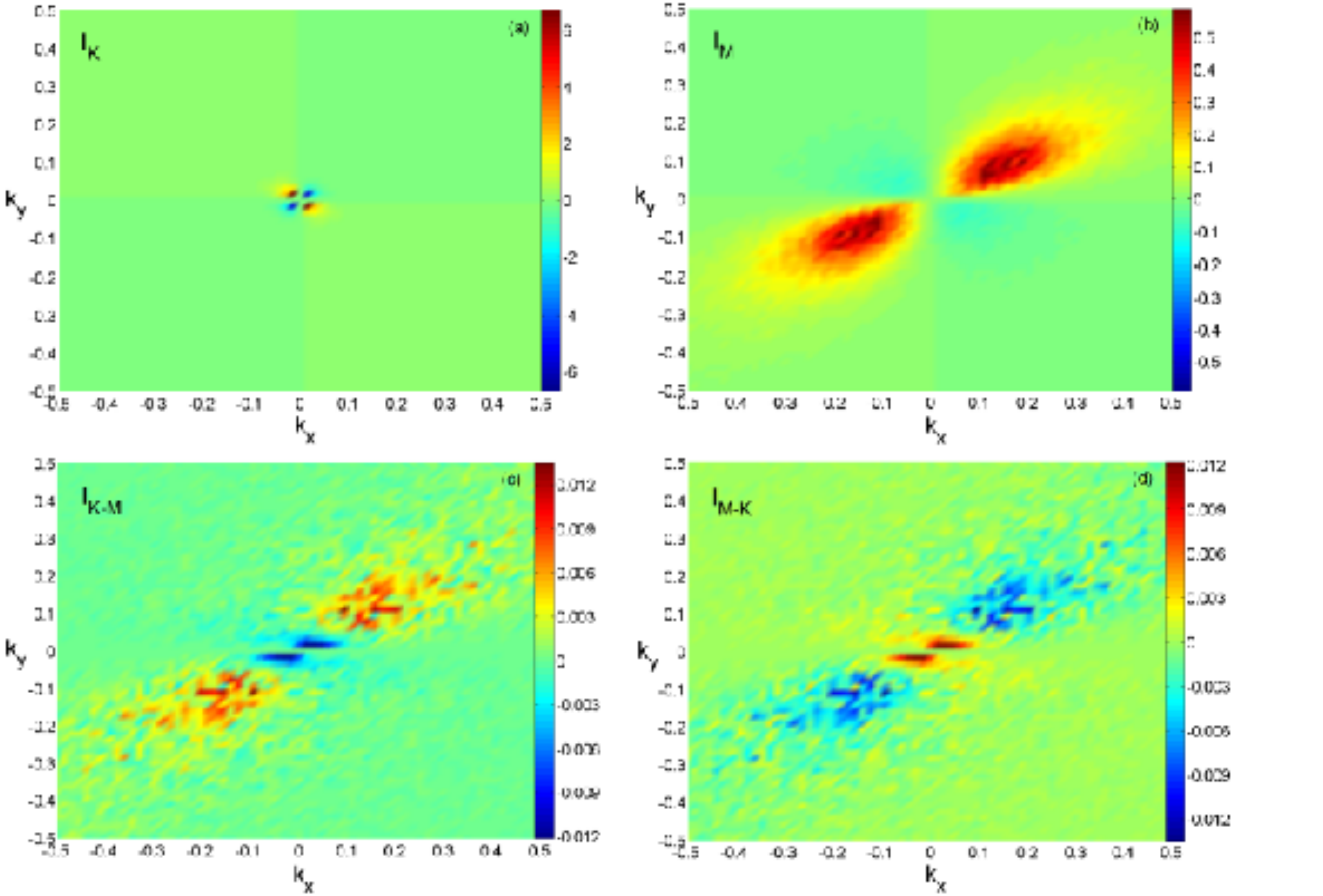
magnetic energy



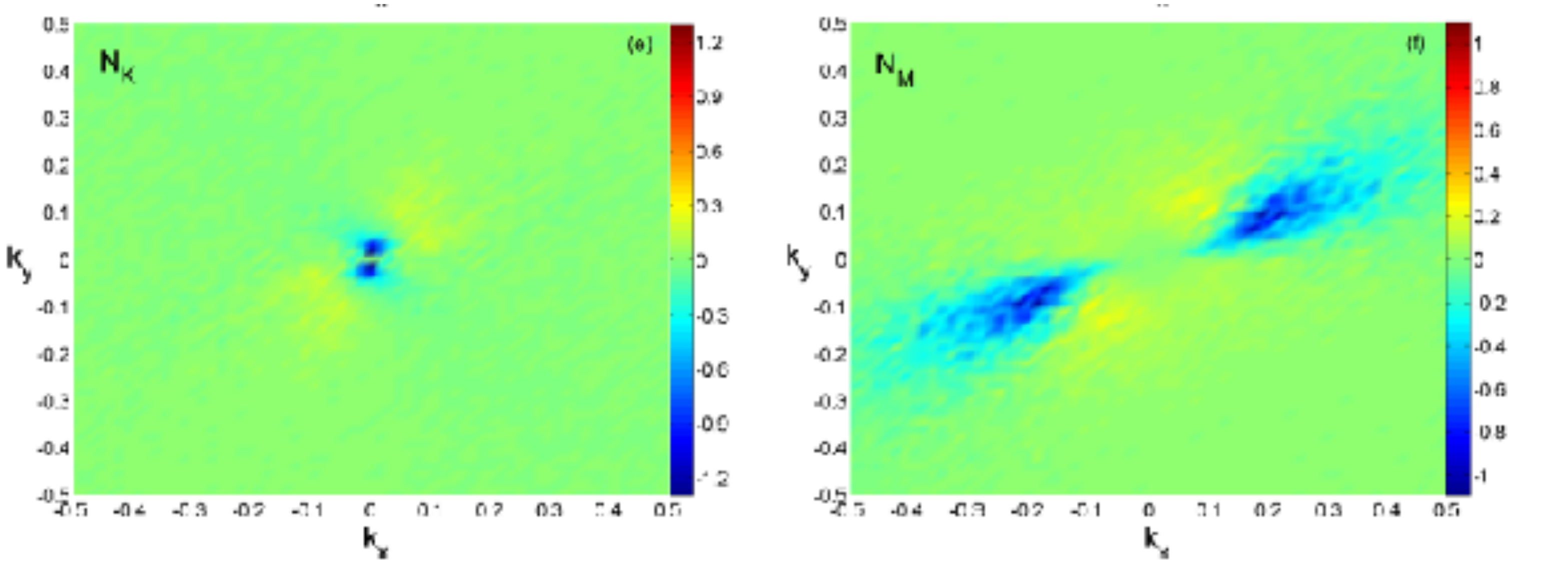
Both spectra are clearly anisotropic, having more power at the $k_x / k_y > 0$ side

This anisotropy is due to shear

Action of linear-origin injection I_K, I_M and cross terms I_{K-M}, I_{M-K} in Fourier space (averaged in time over quasi-steady state)



Action of nonlinear transfer terms N_K, N_M in k-plane in the turbulent state and the essence of the transverse cascade



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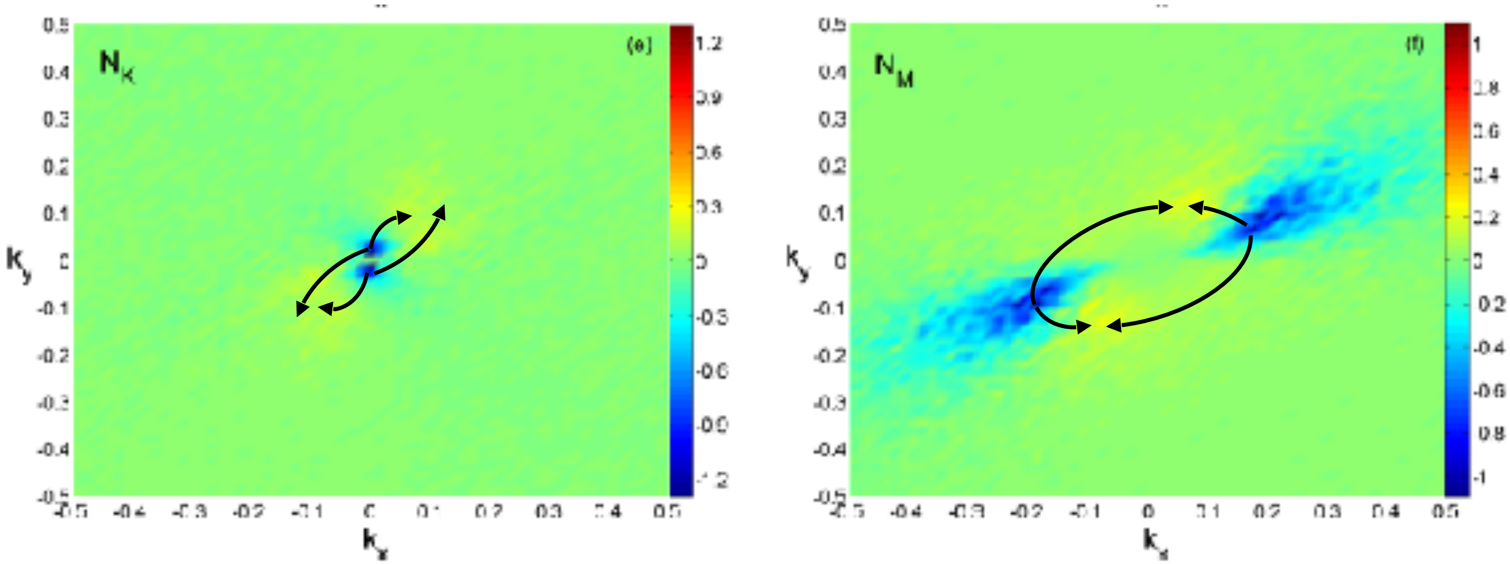


These terms transfer respectively kinetic (left panel) and magnetic (right panel) energies from blue regions where they are negative to yellow regions where they are positive as well as between each other.

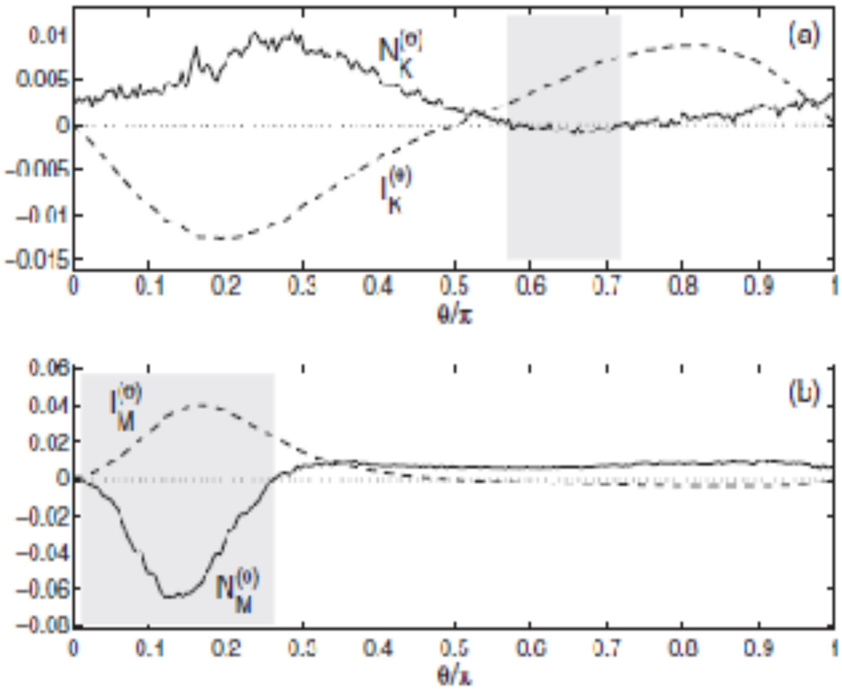
Both energy transfers are anisotropic – strongly depend on the azimuthal angle in k-plane.

This energy redistribution is fundamentally different from that found in classical cases (direct/inverse along k) without shear

Angular dependence of the injection and transfer terms in (k_x ,k_y)-plane – illustrating the transversal nature of the transfer

k-integrated injection and transfer terms represented as a function of polar angle θ

$$I^{(\theta)}_{K,M} = \int_{k_{\min}}^{k_{\max}} I_{K,M} k dk, \quad N^{(\theta)}_{K,M} = \int_{k_{\min}}^{k_{\max}} N_{K,M} k dk$$

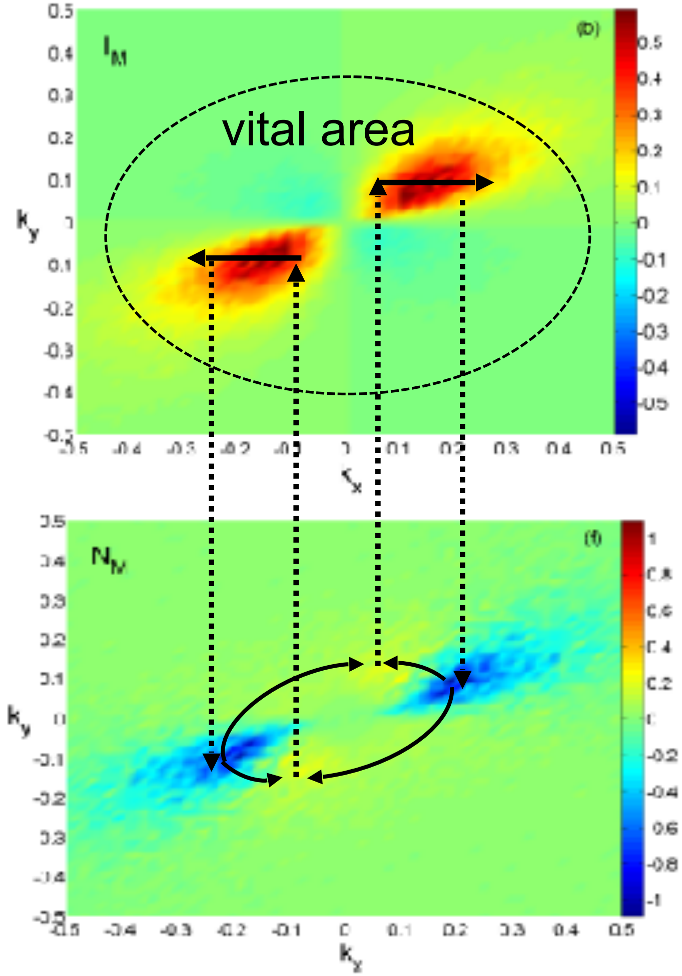


Note the opposite relative trends between $I_K^{(\theta)}, N_K^{(\theta)}$ and $I_M^{(\theta)}, N_M^{(\theta)}$

Interplay of transient growth and nonlinear transverse cascade – basic subcycle of the sustenance scheme

$$\frac{\partial \bar{E}_M}{\partial t} + \frac{\partial}{\partial k_x}\left(k_y \bar{E}_M\right) = I_M + I_{M-K} + D_M + N_M,$$

This mechanism is concentrated and works at low wavenumbers $k < S / u_A$

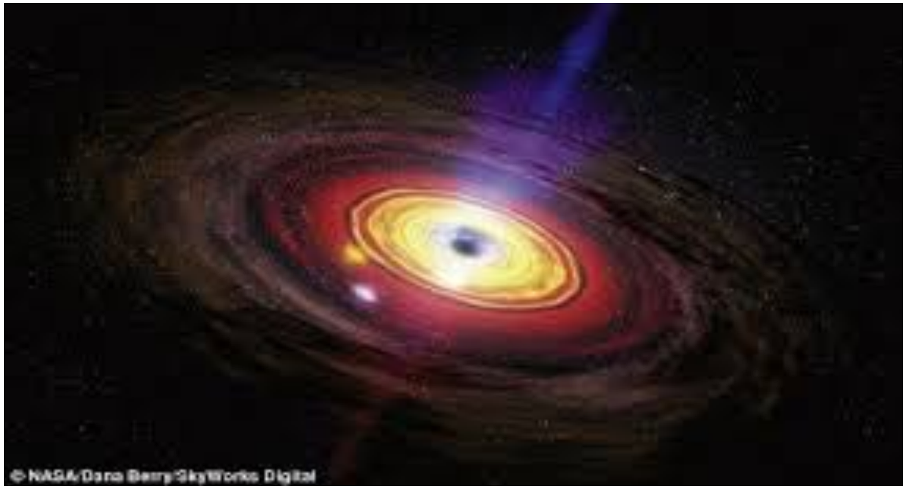


Shear–induced drift and transient/nonmodal amplification of the modes

Nonlinear transverse cascade, replenishing new active modes in the vital area

Magnetorotational instability (MRI) in astrophysical discs - a short overview

- MRI is a powerful dynamical instability arising in the presence of differentially rotating, or shear flows threaded by background magnetic field (Velikhov 1959, Chandrasekhar 1960, Balbus & Hawley 1991,1998)



- **MRI is considered today as the most promising candidate driving accretion process in astrophysical disc shear flows – it gives rise to MHD turbulence and associated enhanced turbulent transport of angular momentum outward and mass inward**

Linear MRI

– exponential growth of axisymmetric perturbations
(e.g., Balbus & Hawley 1991, Goodman & Xu 1994, Latter et al. 2009, 2010,2015)

– transient/nonmodal growth of non-axisymmetric perturbations
(e.g., Balbus & Hawley 1992, Mamatsashvili et al 2013, Squire & Bhattacharjee 2014)

Magnetic field configuration

– Vertical field MRI (e.g., Balbus & Hawley 1991, Goodman & Xu 1994, Bodo et al. 2008, Pessah & Goodman 2009, Latter et al. 2009, 2010, 2015)

– Azimuthal field MRI (e.g.,Ogilvie & Pringle 1996, Papaloizou &Terquem 1997)

– Zero-net flux MRI (e.g., Fromang & Papaloizou 2007, Lesur & Ogilvie 2008, Davis et al. 2010, Bodo et al. 2011, 2014, Shi et al. 2016, Riols et al. 2017)

Effects of dissipation

Dependence of the turbulence transport level and dynamo action on the Reynolds number **Re** and magnetic Prandtl number **Pm** (e.g., Fromang et al. 2007, Fromang 2010, Lesur & Longaretti 2007, Longaretti & Lesur 2010, Meheut et al. 2015, Pessah & Chan 2008, Nauman & Pessah 2016, Riols et al. 2015, 2017)

Effects of nonlinear transfers in MRI-driven turbulence - state-of-the-art

— Anisotropic spectrum of MRI-turbulence (Hawley et al. 1995, Lesur & Longaretti 2011 Nauman & Blackman 2014, Murphy & Pessah 2015)

— Non-locality of transfers (Simon et al 2009, Lesur & Longaretti 2011), using, however, **shell-averaging in Fourier space, which wipes out spectral anisotropy, sacrificing the essential part of the turbulence dynamics !**

— A novel – **transverse** – type of transfers in shear flows and discs (Horton et al 2010, Mamatsashvili et al. 2014, Gogichaishvili et al. 2017)

Analysis of the MRI-turbulence dynamics in Fourier space is key for understanding (to name but a few):

1. Spectra and sustenance of MRI-turbulence with a net azimuthal field as well as with zero net flux, where azimuthal field is generated via magnetic dynamo action

2. Dominant scales, or active modes in the vital area mainly responsible for the turbulent transport

4. Non-locality — nonlinear interaction between large and small (up to dissipation) scales responsible for Pm dependence (e.g., Riols et al. 2015, 2017)

5. Turbulent and numerical dissipation

Transverse cascade, active modes and dynamical balances in MRI-turbulence of Keplerian discs — a net azimuthal field case

(*Gogichaishvili*, ***Mamatsashvili***, *et al. 2017*)

Incompressible conducting fluid with constant viscosity, thermal diffusivity and resistivity in the shearing box with vertically constant stratification N^2 (Boussinesq approximation)

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\cdot\nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P + \frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{4\pi\rho} - 2\mathbf{\Omega}\times\mathbf{U} + 2q\Omega^2 x\,\mathbf{e_x} - \Lambda N^2\theta\,\mathbf{e_z} + \nu\nabla^2\mathbf{U},$$

$$\frac{\partial \theta}{\partial t} + \mathbf{U}\cdot\nabla\theta = \frac{u_z}{\Lambda} + \chi\nabla^2\theta, \quad \nabla\cdot\mathbf{U} = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{U}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}, \quad \nabla\cdot\mathbf{B} = 0,$$

where $\Lambda = -g/N^2$ with gravitation acceleration $\boldsymbol{g}$ and Brunt-Väisälä frequency N

Ω - is the angular velocity of the local rotating reference frame

time is normalized by Ω^{-1}, distances by disc scale height $\boldsymbol{H}$, velocity by ΩH and the magnetic field by $\Omega H\sqrt{4\pi\rho}$

Equilibrium

Keplerian shear flow- $\mathbf{U}_0 = (0, -q\Omega x, 0)$ with q=**1.5**

Uniform azimuthal/toroidal magnetic field $\mathbf{B}_0 = (0, B_{0y}, 0)$,

with magnitude $B_{0y} = \sqrt{2/\beta} = 0.1$, where the plasma $\beta = 8\pi\rho\Omega^2 H^2 / B_{0y}^2 = 200$

Reynolds number, Re, magnetic Reynolds number, Rm, and Péclet number, Pe

$$\mathrm{Re} = \frac{\Omega H^2}{\nu}, \quad Rm = \frac{\Omega H^2}{\eta}, \quad Pe = \frac{\Omega H^2}{\chi}, \quad \mathrm{Re} = Rm = Pe = 3200$$

We use shearing-periodic in x and periodic in y and z boundary conditions

Main goals:

In the local shearing-box model, the MRI in the presence of azimuthal field has a transient nature, which eventually decays, i.e., there is no exponential instability in the classical sense (e.g., Balbus & Hawley 1992, Brandenburg & Dintrans 2006).

Our main goal is to understand how nonlinear terms can provide feedback to transiently growing MRI, leading to the sustained turbulence, also observed in other studies of azimuthal field MRI-turbulence

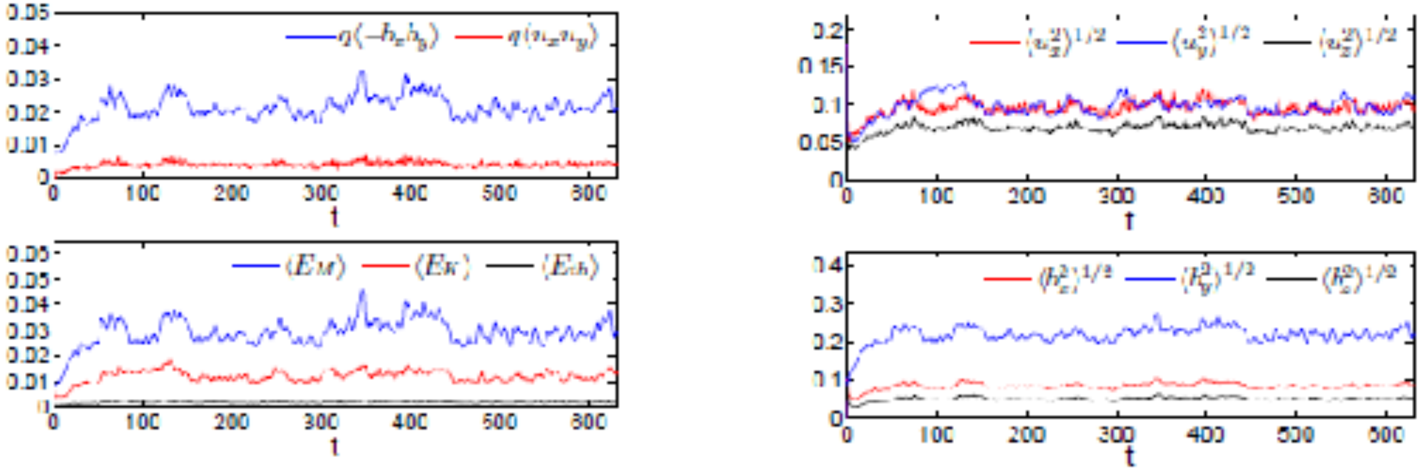
(e.g., Hawley et al. 95, Simon & Hawley 2009, Meheut et al. 2015)

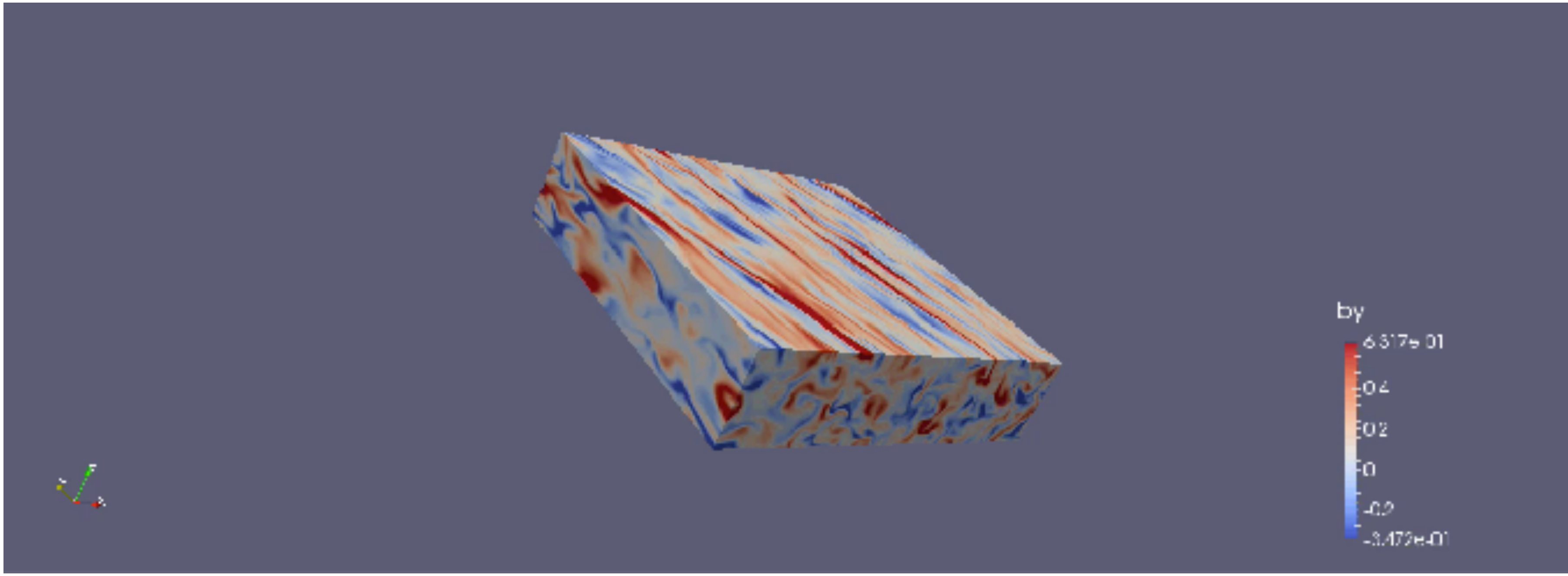
We investigate nonlinear transfers in Fourier space and the role of the transverse cascade in sustaining azimuthal field MRI-turbulence using the data of numerical simulations

Simulation parameters (simulations are done with SNOOPY code)

(L_x, L_y, L_z)	(N_x, N_y, N_z)	$\langle\langle E_K\rangle\rangle$	$\langle\langle E_M\rangle\rangle$	$\langle\langle E_u\rangle\rangle$	$\langle\langle b_x^2\rangle\rangle^{1/2}$	$\langle\langle b_y^2\rangle\rangle^{1/2}$	$\langle\langle b_z^2\rangle\rangle^{1/2}$	$\langle\langle u_x u_y\rangle\rangle$	$\langle\langle -b_x b_y\rangle\rangle$
(8, 8, 1)	(512, 512, 64)	0.0173	0.0422	0.0022	0.101	0.266	0.06	0.0037	0.0198
(4, 4, 1)	(256, 256, 64)	0.0125	0.03	0.0019	0.086	0.224	0.05	0.0028	0.0146
(2, 4, 1)	(128, 256, 64)	0.0116	0.0298	0.0019	0.085	0.223	0.05	0.0028	0.0144
(1, 4, 1)	(64, 256, 64)	0.0111	0.0295	0.0018	0.085	0.222	0.05	0.0027	0.0143
(4, 2, 1)	(256, 128, 64)	0.0056	0.012	0.0011	0.053	0.14	0.03	0.0013	0.0059

General characteristics of the saturated turbulent state – physical space





Main equations in Fourier k-space: $f(\mathbf{r},t) = \int \bar{f}(\mathbf{k},t)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\, d^3\mathbf{k} \qquad f \equiv (\mathbf{u}, p, \theta, \mathbf{b})$

Momentum equations

$$\frac{\partial}{\partial t}\frac{|\bar{u}_x|^2}{2} = qk_y\frac{\partial}{\partial k_x}\frac{|\bar{u}_x|^2}{2} + \mathcal{H}_x + \mathcal{I}_x^{(u\theta)} + \mathcal{I}_x^{(ub)} + \mathcal{D}_x^{(u)} + \mathcal{N}_x^{(u)},$$

$$\frac{\partial}{\partial t}\frac{|\bar{u}_y|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{u}_y|^2}{2} + \mathcal{H}_y + \mathcal{I}_y^{(u\theta)} + \mathcal{I}_y^{(ub)} + \mathcal{D}_y^{(u)} + \mathcal{N}_y^{(u)},$$

$$\frac{\partial}{\partial t}\frac{|\bar{u}_z|^2}{2} = qk_y\frac{\partial}{\partial k_x}\frac{|\bar{u}_z|^2}{2} + \mathcal{H}_z + \mathcal{I}_z^{(u\theta)} + \mathcal{I}_z^{(ub)} + \mathcal{D}_z^{(u)} + \mathcal{N}_z^{(u)},$$

the effects of rotation (Coriolis force) and shear:

$$\mathcal{H}_x = \left(1-\frac{k_x^2}{k^2}\right)(\bar{u}_x\bar{u}_y^* + \bar{u}_x^*\bar{u}_y) + 2(1-q)\frac{k_xk_y}{k^2}|\bar{u}_x|^2, \quad \mathcal{H}_y = \frac{1}{2}\left[q-2-2(q-1)\frac{k_y^2}{k^2}\right](\bar{u}_x\bar{u}_y^* + \bar{u}_x^*\bar{u}_y) - 2\frac{k_xk_y}{k^2}|\bar{u}_y|^2$$

$$\mathcal{H}_z = (1-q)\frac{k_yk_z}{k^2}(\bar{u}_x\bar{u}_z^* + \bar{u}_x^*\bar{u}_z) - \frac{k_xk_z}{k^2}(\bar{u}_y\bar{u}_z^* + \bar{u}_y^*\bar{u}_z).$$

Reynolds stress: $\mathcal{H} = \mathcal{H}_x + \mathcal{H}_y + \mathcal{H}_z = q(u_xu_y^* + u_x^*u_y)/2$

Kinetic-thermal exchange term: $\mathcal{I}_i^{(u\theta)} = N^2\left(\frac{k_ik_z}{k^2} - \delta_{iz}\right)\frac{\bar{\theta}\bar{u}_i^* + \bar{\theta}^*\bar{u}_i}{2}$, *(i=x,y,z)*

Kinetic-magnetic exchange term: $\mathcal{I}_i^{(ub)} = \frac{\mathrm{i}}{2}k_yB_{0y}(\bar{u}_i^*\bar{b}_i - \bar{u}_i\bar{b}_i^*)$,

Nonlinear transfer terms: $N_{ij}^{(u)}(\mathbf{k},t) = \int d^3\mathbf{k}'\left[\bar{b}_i(\mathbf{k}',t)\bar{b}_j(\mathbf{k}-\mathbf{k}',t) - \bar{u}_i(\mathbf{k}',t)\bar{u}_j(\mathbf{k}-\mathbf{k}',t)\right],$

$$\mathcal{N}_i^{(u)} = \frac{1}{2}(\bar{u}_iQ_i^* + \bar{u}_i^*Q_i), \quad Q_i = \mathrm{i}\sum_j k_jN_{ij}^{(u)} - \mathrm{i}k_i\sum_{m,n}\frac{k_mk_n}{k^2}N_{mn}^{(u)}, \qquad i,j,m,n = x,y,z.$$

Viscous dissipation term: $\mathcal{D}_i^{(u)} = -k^2|\bar{u}_i|^2/\mathrm{Re}$

Induction equations

$$\frac{\partial}{\partial t}\frac{|\bar{b}_x|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_x|^2}{2} + \mathcal{I}_x^{(bu)} + \mathcal{D}_x^{(b)} + \mathcal{N}_x^{(b)}$$

$$\frac{\partial}{\partial t}\frac{|\bar{b}_y|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_y|^2}{2} + \mathcal{M} + \mathcal{I}_y^{(bu)} + \mathcal{D}_y^{(b)} + \mathcal{N}_y^{(b)}$$

$$\frac{\partial}{\partial t}\frac{|\bar{b}_z|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_z|^2}{2} + \mathcal{I}_z^{(bu)} + \mathcal{D}_z^{(b)} + \mathcal{N}_z^{(b)},$$

Maxwell stress spectrum: $\mathcal{M} = -\frac{q}{2}(\bar{b}_x\bar{b}_y^* + \bar{b}_x^*\bar{b}_y)$,

Magnetic-kinetic exchange terms: $\mathcal{I}_i^{(bu)} = -\mathcal{I}_i^{(ub)}$

Nonlinear terms: $\mathcal{N}_x^{(b)} = \frac{\mathrm{i}}{2}\bar{b}_x^*[k_y\bar{\mathcal{E}}_z - k_z\bar{\mathcal{E}}_y] + c.c.$, $\mathcal{N}_y^{(b)} = \frac{\mathrm{i}}{2}\bar{b}_y^*[k_z\bar{\mathcal{E}}_x - k_x\bar{\mathcal{E}}_z] + c.c.$, $\mathcal{N}_z^{(b)} = \frac{\mathrm{i}}{2}\bar{b}_z^*[k_x\bar{\mathcal{E}}_y - k_y\bar{\mathcal{E}}_x] + c.c.$

where the Fourier transforms of the Electromotive Force (EMF) $\boldsymbol{\mathcal{E}} = \mathbf{u}\times\mathbf{b}$ are:

$$\bar{\mathcal{E}}_x(\mathbf{k},t) = \int d^3\mathbf{k}'\,[\bar{u}_y(\mathbf{k}',t)\bar{b}_z(\mathbf{k}-\mathbf{k}',t) - \bar{u}_z(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t)]$$

$$\bar{\mathcal{E}}_z(\mathbf{k},t) = \int d^3\mathbf{k}'\,[\bar{u}_x(\mathbf{k}',t)\bar{b}_y(\mathbf{k}-\mathbf{k}',t) - \bar{u}_y(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t)]$$

$$\bar{\mathcal{E}}_y(\mathbf{k},t) = \int d^3\mathbf{k}'\,[\bar{u}_z(\mathbf{k}',t)\bar{b}_x(\mathbf{k}-\mathbf{k}',t) - \bar{u}_x(\mathbf{k}',t)\bar{b}_z(\mathbf{k}-\mathbf{k}',t)]$$

Resistive dissipation terms: $\mathcal{D}_i^{(b)} = -k^2|\bar{b}_i|^2/\mathrm{Rm}$ $\quad i=x,y,z$

Spectral energy equations

for the kinetic energy density: $\mathcal{E}_K = (|u_x|^2 + |u_y|^2 + |u_z|^2)/2,$

$$\frac{\partial \mathcal{E}_K}{\partial t} = -qk_y \frac{\partial \mathcal{E}_K}{\partial k_x} + \mathcal{H} + \mathcal{I}^{(u\theta)} + \mathcal{I}^{(ub)} + \mathcal{D}^{(u)} + \mathcal{N}^{(u)},$$

$$\mathcal{H} = \sum_i \mathcal{H}_i = \frac{q}{2}(u_x u_y^* + u_x^* u_y), \quad \mathcal{I}^{(u\theta)} = \sum_i \mathcal{I}_i^{(u\theta)}, \quad \mathcal{I}^{(ub)} = \sum_i \mathcal{I}_i^{(ub)}, \quad \mathcal{D}^{(u)} = \sum_i \mathcal{D}_i^{(u)} = -\frac{2k^2}{\mathrm{Re}}\mathcal{E}_K, \quad \mathcal{N}^{(u)} = \sum_i \mathcal{N}_i^{(u)}$$

for the magnetic energy density: $\mathcal{E}_M = (|\bar{b}_x|^2 + |\bar{b}_y|^2 + |\bar{b}_z|^2)/2,$

$$\frac{\partial \mathcal{E}_M}{\partial t} = -qk_y \frac{\partial \mathcal{E}_M}{\partial k_x} + \mathcal{M} + \mathcal{I}^{(bu)} + \mathcal{D}^{(b)} + \mathcal{N}^{(b)},$$

$$\mathcal{M} = \frac{q}{2}(b_x b_y^* + b_x^* b_y), \quad \mathcal{I}^{(bu)} = \sum_i \mathcal{I}_i^{(bu)} = \mathcal{I}^{(ub)},$$

$$\mathcal{D}^{(u)} = \sum_i \mathcal{D}_i^{(u)} = -\frac{2k^2}{\mathrm{Rm}}\mathcal{E}_M, \quad \mathcal{N}^{(b)} = \sum_i \mathcal{N}_i^{(b)}.$$

Energy drift due to shear In **k**-space

Energy Injection by Maxwell stress

Kinetic-magnetic energy exchange

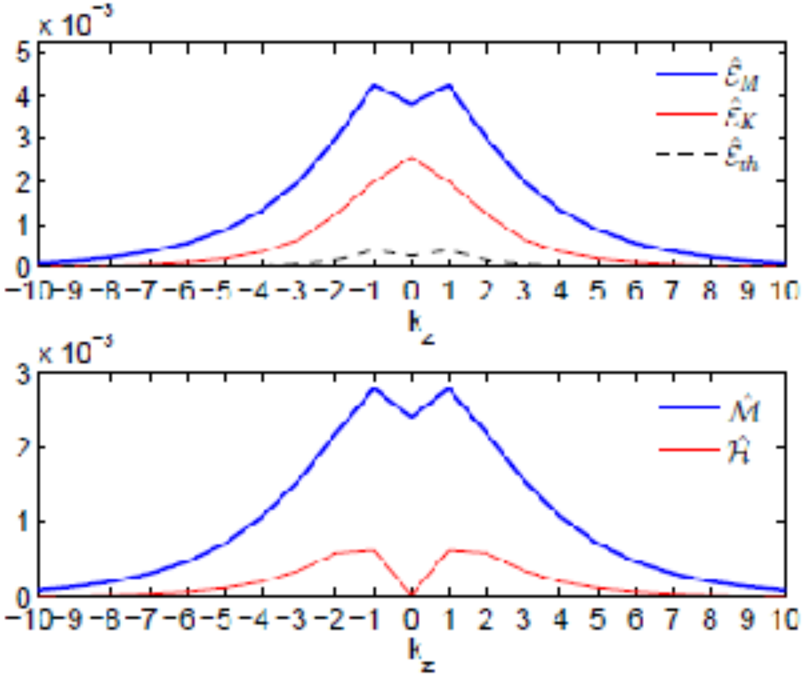
Dissipation due to resistivity

Nonlinear transfers

1. Magnetic energy is dominant among kinetic and thermal energies
2. Maxwell stress is larger than Reynolds stress and primarily determines turbulence's energy supply and sustenance
3. Nonlinear terms do not gain energy, but only redistribute it among modes

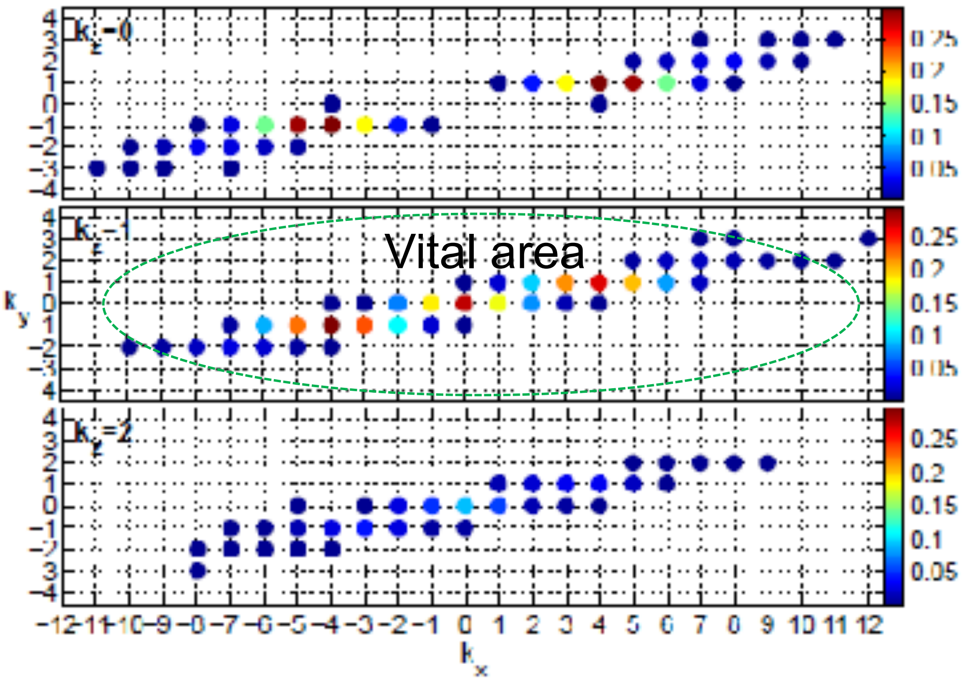
$$\int [\mathcal{N}^{(u)}(\mathbf{k},t) + N^2 \mathcal{N}^{(\theta)}(\mathbf{k},t) + \mathcal{N}^{(b)}(\mathbf{k},t)] d^3\mathbf{k} = 0,$$

Spectra of the energies and stresses integrated in (k_x, k_y) – plane vs. k_z for the box (4,4,1)



The modes with $|k_z| = 0, 1, 2$ (in units of $2\pi / L_z$) carry most of the energy and stress and hence are dynamically more important

Active modes at $k_z = (0,1,2)2\pi / L_z$ which play a main role in the sustaining process

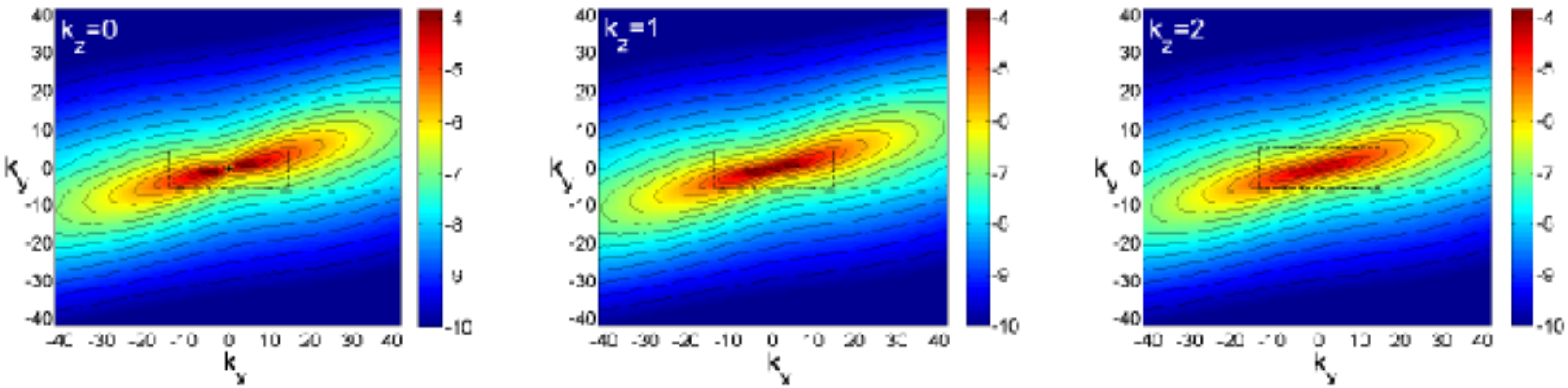


"Active" modes (in colour) whose magnetic energy reaches more than 50% of $\max(\mathcal{E}_M)$

The vital area – region in Fourier space where these active modes are concentrated

Magnetic energy spectra $\log_{10}\mathcal{E}_M$ in (k_x, k_y)-plane at different $k_z = 0, 1, 2$

(everywhere below wavenumbers are normalized as: $(k_x L_x / 2\pi, k_y L_y / 2\pi) \to (k_x, k_y)$)

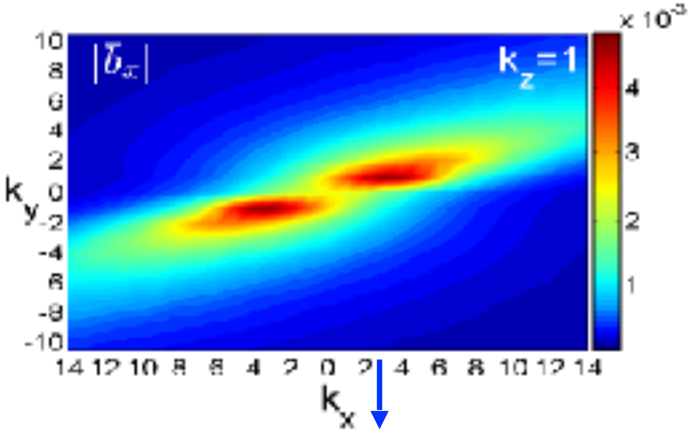


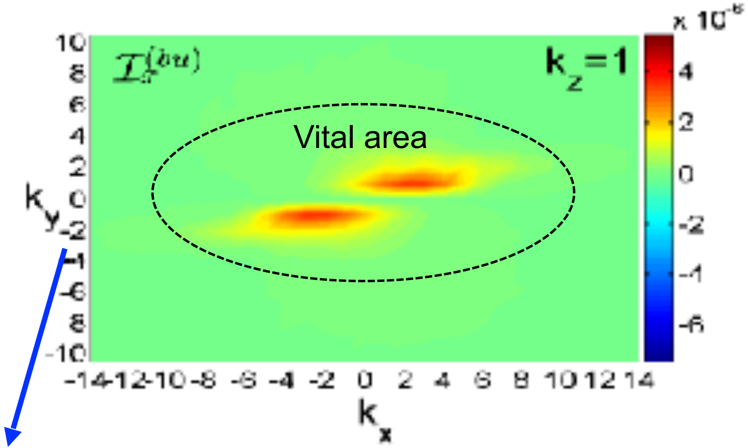
Energy spectra are strongly anisotropic in Fourier space due to shear

Averaging over shells of constant $|\mathbf{k}|$ in Fourier space, often done in the literature on MRI-turbulence, wipes out shear-induced anisotropy and hence the transverse cascade, which plays an essential role in the sustaining dynamics of turbulence

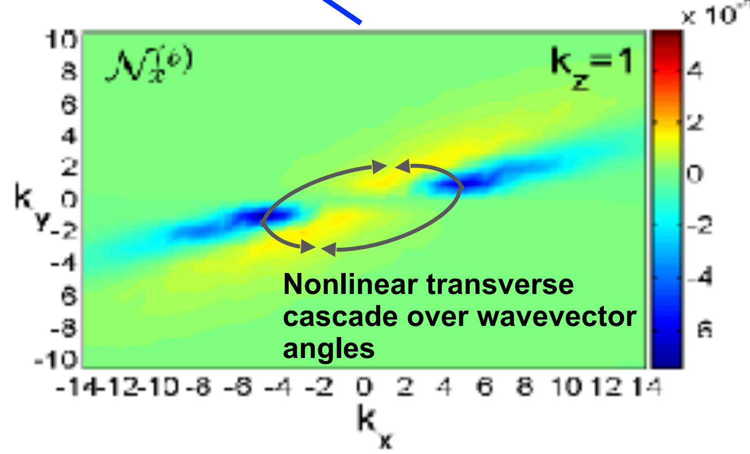
Essence of turbulence sustenance – interplay of linear nonmodal growth of MRI and nonlinear transverse cascade

Dynamics of $|\bar{b}_x|$

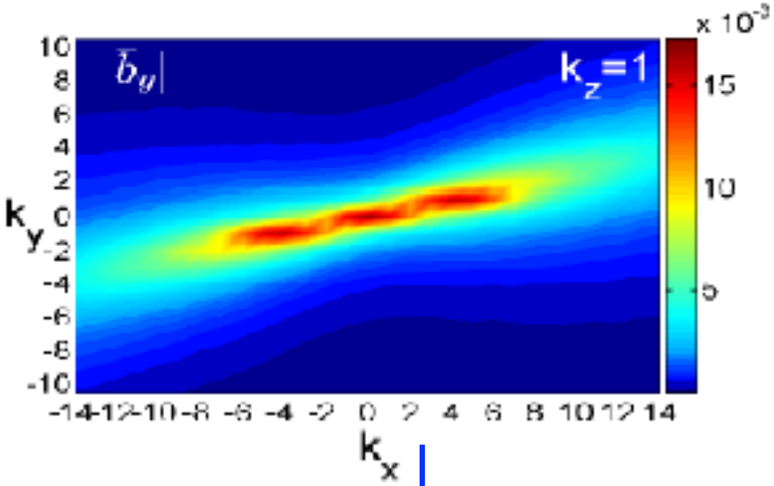


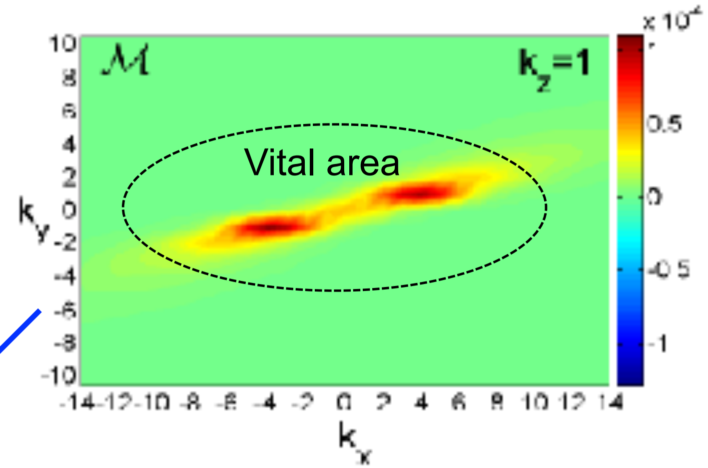


$$\frac{\partial}{\partial t}\frac{|\bar{b}_x|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_x|^2}{2} + \mathcal{I}_x^{(bu)} + \mathcal{D}_x^{(b)} + \mathcal{N}_x^{(b)}$$

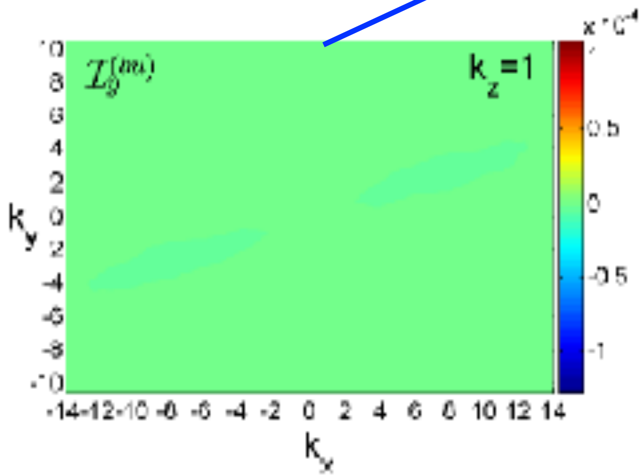


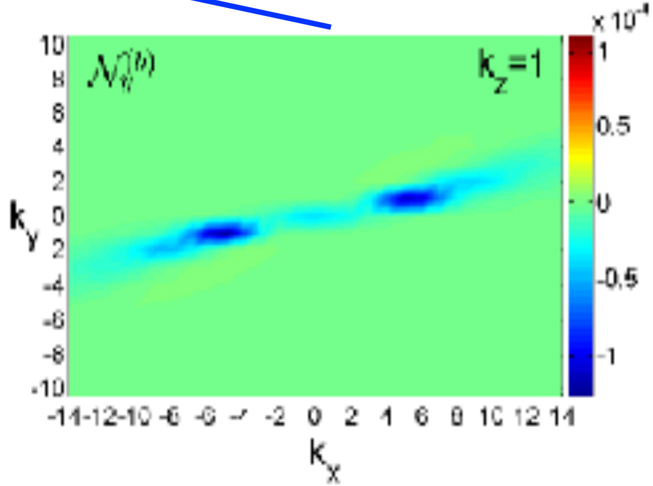
Dynamics of $|\bar{b}_y|$





$$\frac{\partial}{\partial t}\frac{|\bar{b}_y|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_y|^2}{2} + \mathcal{M} + \mathcal{T}_y^{(bu)} + \mathcal{D}_y^{(b)} + \mathcal{N}_y^{(b)}$$

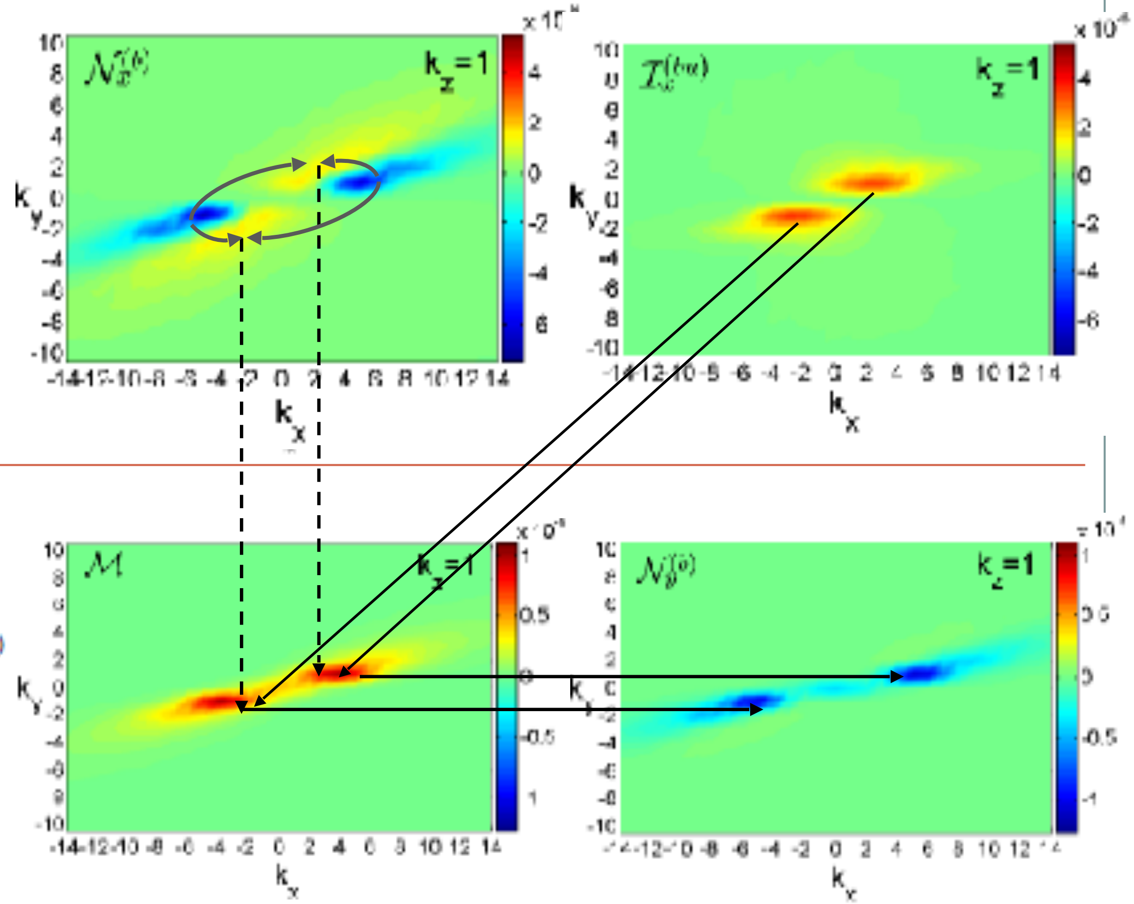




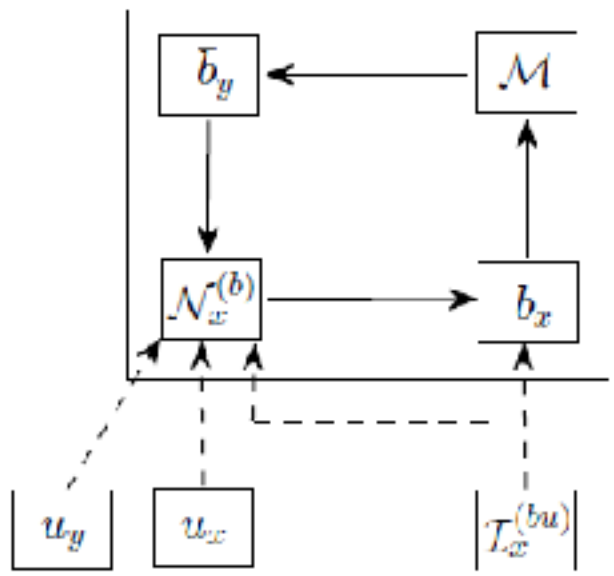
Basic subcycle of the sustenance of MRI-turbulence with azimuthal field

$$\frac{\partial}{\partial t}\frac{|\bar{b}_x|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_x|^2}{2} + \mathcal{I}_x^{(bu)} + \mathcal{D}_x^{(b)} + \mathcal{N}_x^{(b)}$$

$$\frac{\partial}{\partial t}\frac{|\bar{b}_y|^2}{2} = -qk_y\frac{\partial}{\partial k_x}\frac{|\bar{b}_y|^2}{2} + \mathcal{M} + \mathcal{T}_y^{(bu)} + \mathcal{D}_y^{(b)} + \mathcal{N}_y^{(b)}$$



Basic subcycle of the sustenance of MRI-turbulence with azimuthal field



The nonlinear term $\mathcal{N}_x^{(b)}$ plays a central role — it generates/maintains the seed radial fi $\bar{b}_x$ necessary for the turbulence sustenance

This mechanism of turbulence sustenance is analogous to recent ones proposed in connection with MRI-dynamo (Herault et al. 2011, Riols et al. 2015, 2017), except that it involves much more active non-axisymmetric modes

Summary

We analysed sustained MHD turbulence in 2D and 3D cases, spectrally stable and unstable, plane and astrophysical disc shear flows with parallel (azimuthal) mean magnetic field. Specifically,

- **sustained subcritical 2D MHD turbulence in a plane shear flow with a parallel magnetic field**

- **We investigated underlying sustaining mechanisms and dynamical balances of the turbulence in Fourier space by analysing individual linear and nonlinear terms and the refined interplay of these terms**

- **We identified a new type of nonlinear cascade over wavevector angles (orientations) — so-called, transverse cascade — a generic nonlinear process due to the shear**

Future extensions

Natural extension of this approach and results are:

- **sustenance of zero-net flux MRI-turbulence and dynamo action both in unstratified and stratified cases**

- **MRI turbulence in the presence of compressibility – generation of density waves by magnetic perturbations**

All these are relevant to the understanding of subcritical turbulence in other sheared and magnetized complex environments such as MHD winds, geophysical magnetic fields, etc.

Thank you

Please see our publications on this topic:

Horton et al. (2010), *Phys. Rev. E*, **81**, 066304
Mamatsashvili et al. (2014), *Phys. Rev. E*, **89**, 043101
Mamatsashvili et al. (2016), *Phys. Rev. E*, **94**, 023111
Gogichaishvili, **Mamatsashvili**, et al. (2017), *ApJ*, **845**, 70
Gogichaishvili, **Mamatsashvili**, et al. (2018), *ApJ,* **866***,* 134