The connection of the Busse solution in spheroids with the precession-driven fluid flow in a cylinder

André Giesecke
Helmholtz-Zentrum Dresden-Rossendorf

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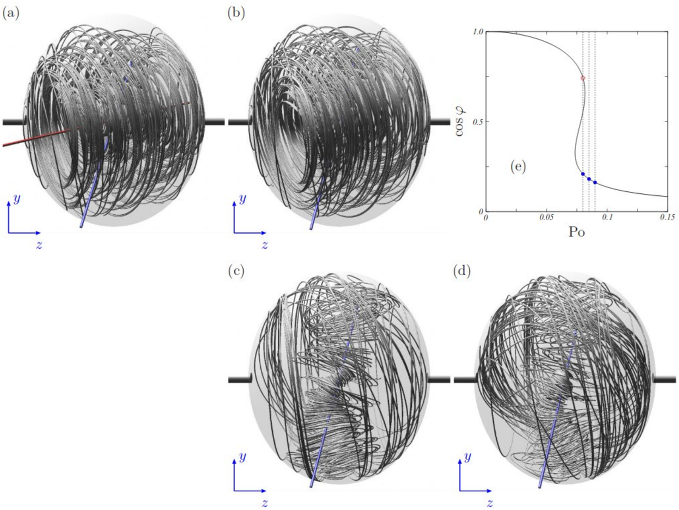
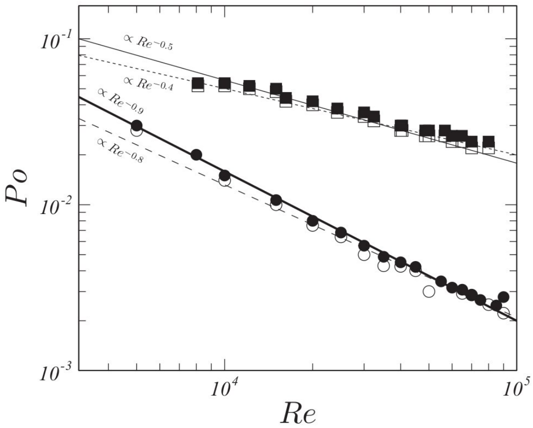
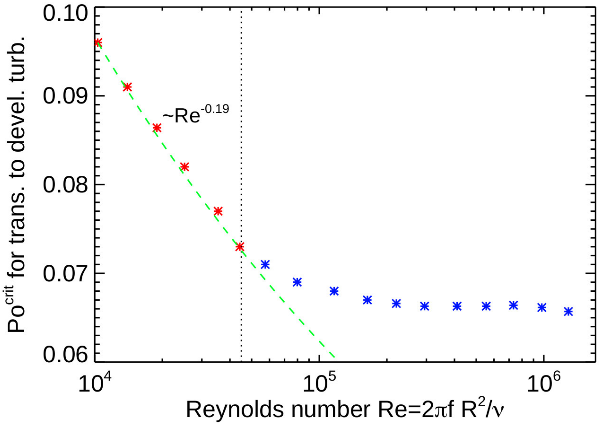
- **motivation for precession dynamo:**
  magnetic field generation
  in planets and/or moons
  ⇒ spheres, spheroids, ellipsoids
- **technical challenges:**
  – liquid sodium
  – mechanical demands
  ⇒ cylindrical container
- **nevertheless:**
  experiments with water in a precessing spheroid show qualitative similarities with the fluid flow evolution in the precession water experiment at HZDR
  (Goto 2014, 2017, 2019)

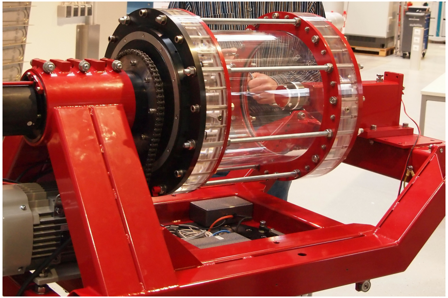


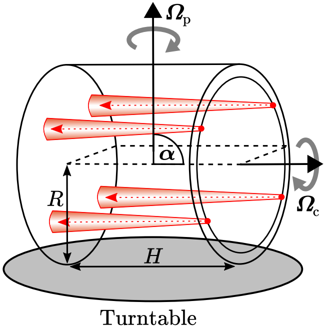
Is the cylinder equivalent to the spheroid?



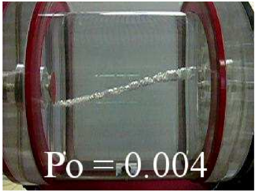


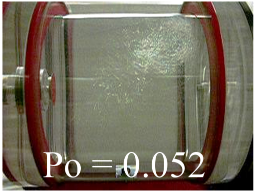
- different scaling for transition to turbulent regime $\Rightarrow$ change ellipticity $\eta = (a-b)/a$?
- no asymptotic behavior in spheroidal geometry (Re to small?)



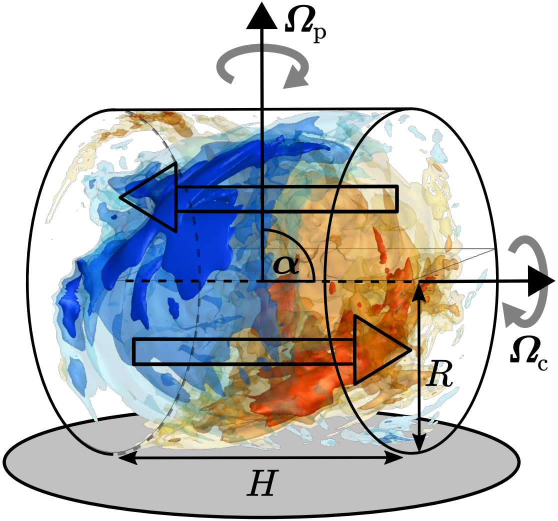


- flow measurements with **UDV** at down-scaled water experiment ($R = 0.163\,\mathrm{m}$)
- **simulations** with SEMTEX Blackburn & Sherwin 2004
  maximum $\mathrm{Re} = 10^4$
- **behaviour of gas bubbles as auxiliary quantity for flow state**
- accumulation along minimum pressure line
- supposed to be representative for fluid rotation axis





Fundamental Flow Structure from Numerical Simulations



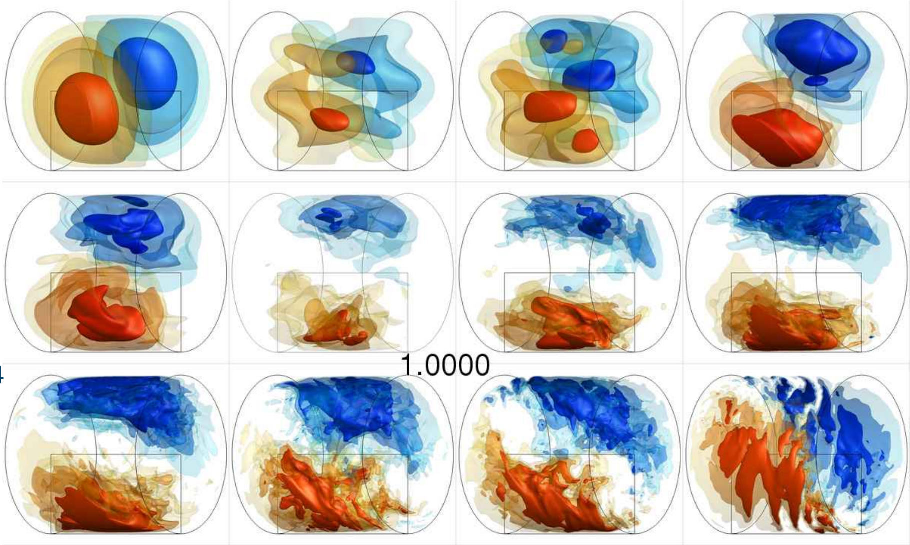
axial velocity

- **structure of forcing:**
  $\boldsymbol{F}_{\rm p} = -\Omega_{\rm p}\Omega_{\rm c} r \sin\alpha \cos(\Omega_{\rm c} t + \varphi)\boldsymbol{e}_z$
  $\Rightarrow$ antisymmetric w.r.t. equatorial plane
  $\Rightarrow$ inertial modes with $m = 1$ and k odd
  $\Rightarrow$ standing waves in turntable frame
- fundamental $m = 1$ mode reflects fluid's tendency to align axis of rotation and axis of precession (like a gyroscope)
- **turntable reference frame:**
  superimposition of $m = 1$ mode and (modified) solid body rotation
  $\Rightarrow$ fluid rotates around axis different from $\Omega_{\rm p}$ and $\Omega_{\rm c}$ $\Rightarrow$ uniform vorticity solution

Structure of Precession Driven Flows in a Cylinder

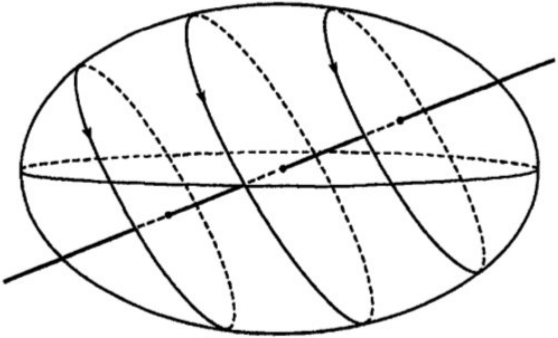
axial velocity in the turntable reference frame

$\mathrm{Re} = 10^4$



$$\begin{aligned}\mathrm{Po} &= 0.001\\ &= 0.0015\\ &= 0.002\\ &= 0.005\end{aligned}$$

$$\begin{aligned}\mathrm{Po} &= 0.01\\ &= 0.03\\ &= 0.05\\ &= 0.075\end{aligned}$$



- assume ellipsoidal fluid state with spatially constant vorticity and non-penetrating boundary conditions $\boldsymbol{q} = \boldsymbol{\omega}_\mathrm{f} \times \boldsymbol{r}$

$$\frac{\partial}{\partial t}\omega_\mathrm{f}^x - \frac{a^2-c^2}{a^2+c^2}\omega_\mathrm{f}^x\omega_\mathrm{f}^y - \frac{2a^2}{a^2+c^2}\omega_\mathrm{f}^y\Omega_x = 0$$

$$\frac{\partial}{\partial t}\omega_\mathrm{f}^y + \frac{a^2-c^2}{a^2+c^2}\omega_\mathrm{f}^x\omega_\mathrm{f}^z - \omega_\mathrm{f}\Omega_x + \frac{2a^2}{a^2+c^2}\omega_\mathrm{f}^x\Omega_z = 0$$

$$\frac{\partial}{\partial t}\omega_\mathrm{f}^z + \frac{2c^2}{a^2+c^2}\omega_\mathrm{f}^y\Omega_x = 0$$

- solution for small Po is rotation about axis slightly rotated away from z-axis
- viscous coupling tries to generate constant vorticity flow aligned with rotation axis of container but container axis is continually moving due to precession

$\Rightarrow$ **alignment never reached**

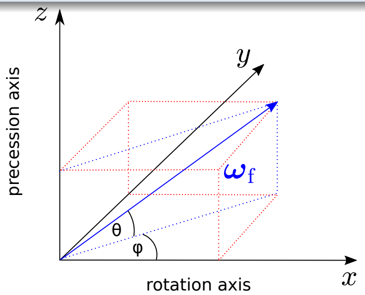
- spheroidal solution derived by Busse 1968 based on torque balance, steady flow, no spin up condition and small deviation from solid body rotation (see also Noir et al GJI 2003):

$$2\int \boldsymbol{r}\times(\boldsymbol{\Omega}\times\boldsymbol{q})\,d\mathcal{V} = -\oint p\boldsymbol{r}\times\hat{\boldsymbol{n}}d\mathcal{S} + \mathrm{Ek}\int \boldsymbol{r}\times\nabla^2\boldsymbol{q}d\mathcal{V} \quad \text{with} \quad \boldsymbol{q}=\boldsymbol{\omega}_{\mathrm{f}}\times\boldsymbol{r}$$

$$\boldsymbol{\omega}_{\mathrm{f}} = \omega_{\mathrm{f}}^2\left(\hat{\mathbf{e}}_z + \frac{\left[0.259(\mathrm{Ek}/\omega_{\mathrm{f}})^{1/2}+\eta\omega_{\mathrm{f}}^2+\Omega\cos\alpha\right]\hat{\mathbf{e}}_z\times(\boldsymbol{\Omega}\times\hat{\mathbf{e}}_z)+2.62(\mathrm{Ek}\omega_{\mathrm{f}})^{1/2}(\hat{\mathbf{e}}_z\times\boldsymbol{\Omega})}{2.62^2\mathrm{Ek}\omega_{\mathrm{f}}+\left[0.259(\mathrm{Ek}/\omega_{\mathrm{f}})^{1/2}+\eta\omega_{\mathrm{f}}^2+\Omega\cos\alpha\right]^2}\right)$$

$$cos\theta = \frac{\hat{\mathbf{e}}_z\cdot\boldsymbol{\omega}_f}{\omega_f}$$

$$cos\varphi = -\frac{\hat{\mathbf{e}}_x\cdot[\hat{\mathbf{e}}_z\times(\hat{\mathbf{e}}_z\times\boldsymbol{\omega}_f)]}{\omega_f\sin\theta}$$



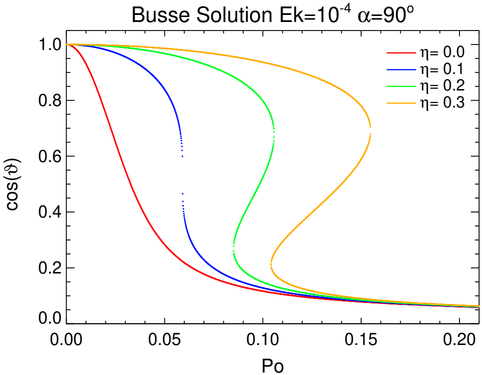
$$\boldsymbol{\Omega} = \mathrm{Po}\,(0,0,1)$$

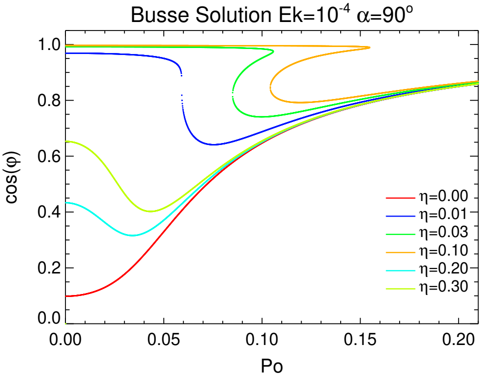
$$\hat{\mathbf{e}}_x = (\cos\alpha, 0, \sin\alpha)$$

$$\hat{\mathbf{e}}_z = (\sin\alpha, 0, \cos\alpha)$$

$$\boldsymbol{\omega}_\mathrm{f} = \omega_\mathrm{f}^2 \left(\hat{\mathbf{e}}_z + \frac{X}{X^2+Y^2} \hat{\mathbf{e}}_z \times (\boldsymbol{\Omega} \times \hat{\mathbf{e}}_z) + \frac{Y}{X^2+Y^2} (\hat{\mathbf{e}}_z \times \boldsymbol{\Omega}) \right)$$

with $X = 0.259(\mathrm{Ek}/\omega_\mathrm{f})^{1/2} + \eta\omega_\mathrm{f}^2 + \mathrm{Po}\cos\alpha$ and $Y = 2.62(\mathrm{Ek}\,\omega_\mathrm{f})^{1/2}$





- simulations performed at $\mathrm{Ek} = 1 \times 10^{-4}, \alpha = 90^\circ, \mathrm{Po} \in [0.001, 0.2]$
- calculate time averaged velocity in precession frame and corresponding vorticity

$$\langle \boldsymbol{u}(\boldsymbol{r}) \rangle = \frac{1}{T} \int \boldsymbol{u}(\boldsymbol{r}) dt \qquad \langle \boldsymbol{\omega}(\boldsymbol{r}) \rangle = \nabla \times \langle \boldsymbol{u}(\boldsymbol{r}) \rangle$$

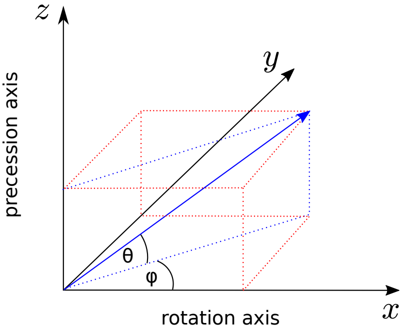
- calculate uniform vorticity applying integration over sphere embedded within cylinder origin at $r = 0, z = 0$, radius $R = 0.3$

$$\langle \omega_{x,y,z} \rangle = \frac{1}{\mathcal{V}} \int \omega_{x,y,z} d\mathcal{V}$$

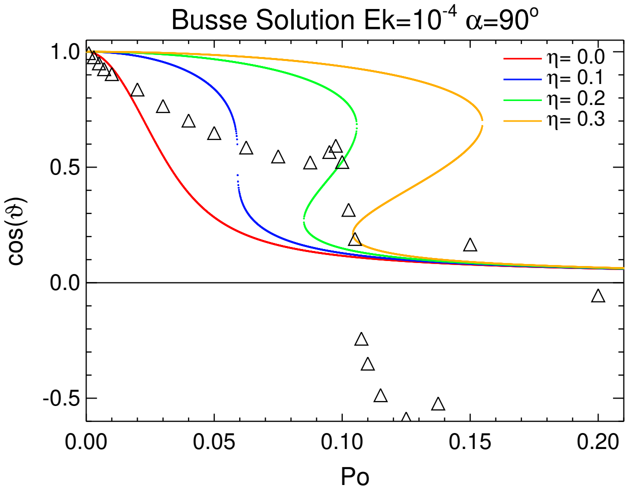
- orientation of fluid rotation axis is calculated as

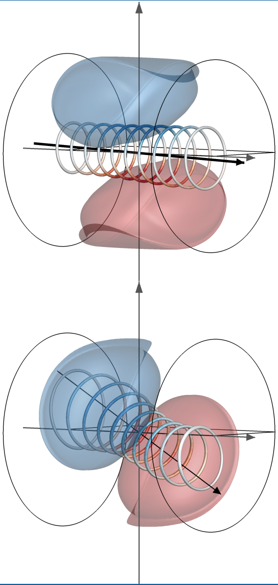
$$\cos\theta = \frac{\langle \omega_z \rangle}{\sqrt{\langle \omega_x \rangle^2 + \langle \omega_y \rangle^2 + \langle \omega_z \rangle^2}}$$

$$\cos\varphi = \frac{\omega_x}{\sqrt{\omega_y^2 + \omega_z^2}}$$



Results and comparison with Busse solution

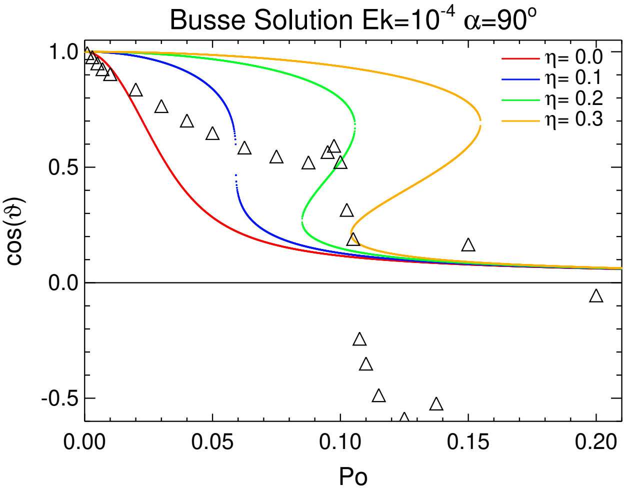


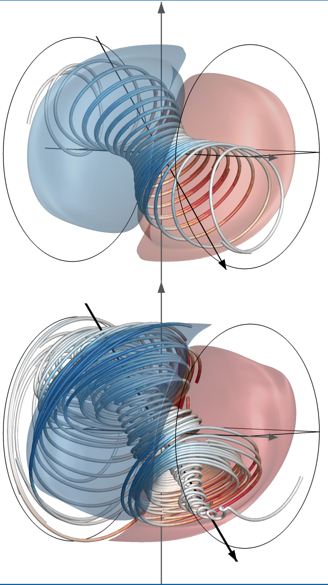


$\mathrm{Po} = 0.001$

$\mathrm{Po} = 0.01$

Results and comparison with Busse solution





$\mathrm{Po} = 0.05$

$\mathrm{Po} = 0.0875$

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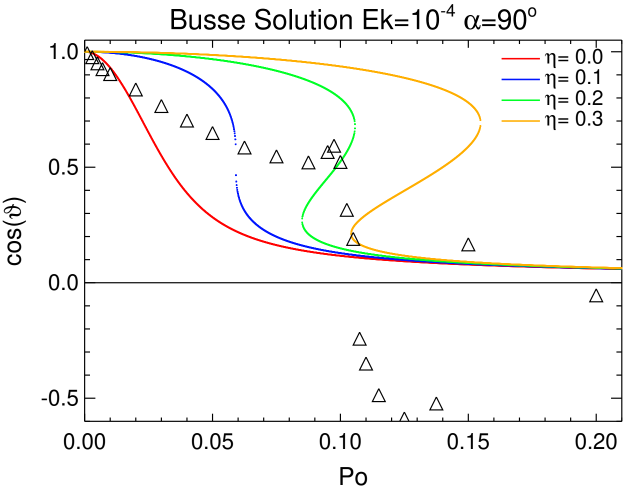
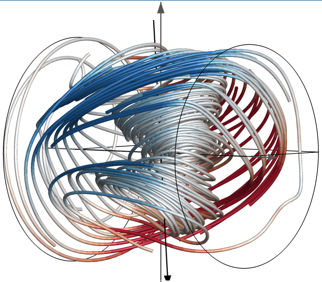
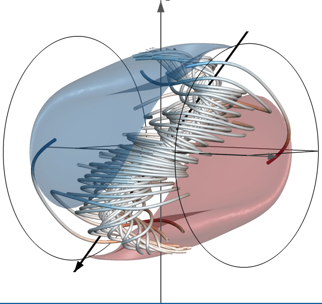




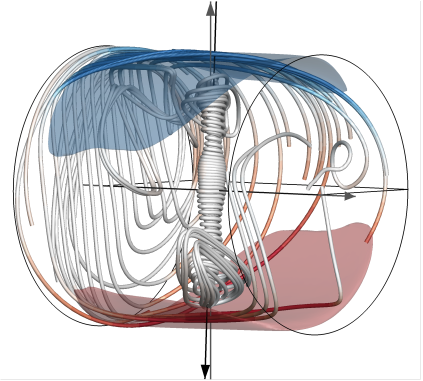
$\mathrm{Po} = 0.100$



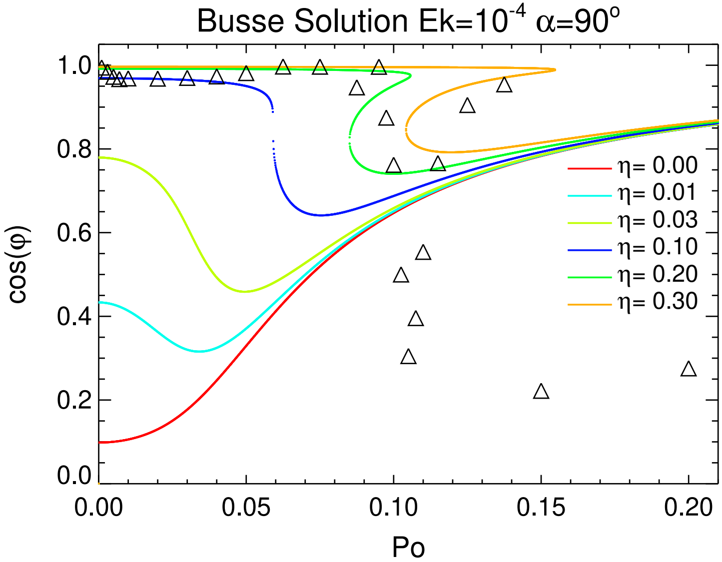
$\mathrm{Po} = 0.125$

Results and comparison with Busse solution

$\mathrm{Po} = 0.094, \mathrm{Re} \sim 10^6$



$\mathrm{Po} = 0.200, \mathrm{Re} = 10^4$



- inconsistent behavior: θ points out similarity with spheroid with small ellipticity η whereas behavior of azimuthal angle $\cos\varphi$ points out similarty with spheroids with large ellipticity η
- sharp change in "favored" regime around $\mathrm{Po} = 0.1$
- no relation to Busse solution in spheroids for $\mathrm{Po} \gtrsim 0.1$

Summary

- fluid rotation axis for cylindrical precession can be estimated from simulations for $\mathrm{Po}\lesssim 0.09$
- angle between cylinder rotation axis and fluid rotation axis can become larger than 90°
- more difficult for $\mathrm{Po} \approx 0.1$ and above because only weak flow in bulk
  $\Rightarrow$ goes along with small values for vorticity components (sometimes difficult to estimate sign)
- in this regime the prerequisites for Busse solution are no longer given

> - 2 different solutions resemble Busse model but quantitatively behavior differs
> - different torque balance in cylinder may require consideration of additional pressure torques from corners which prevents application of "simple" spheroidal model
> - relevance for dynamo action is not obvious:
>   - flow is dominated by large velocities (and high gradients) adjacent the side walls
>   - fluid motion in the bulk is weak (for sufficiently large Po)
>   - but: emergence of instability must consider rotation profile in fluid rotation frame