FEL Theory for Pedestrians

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Introduction

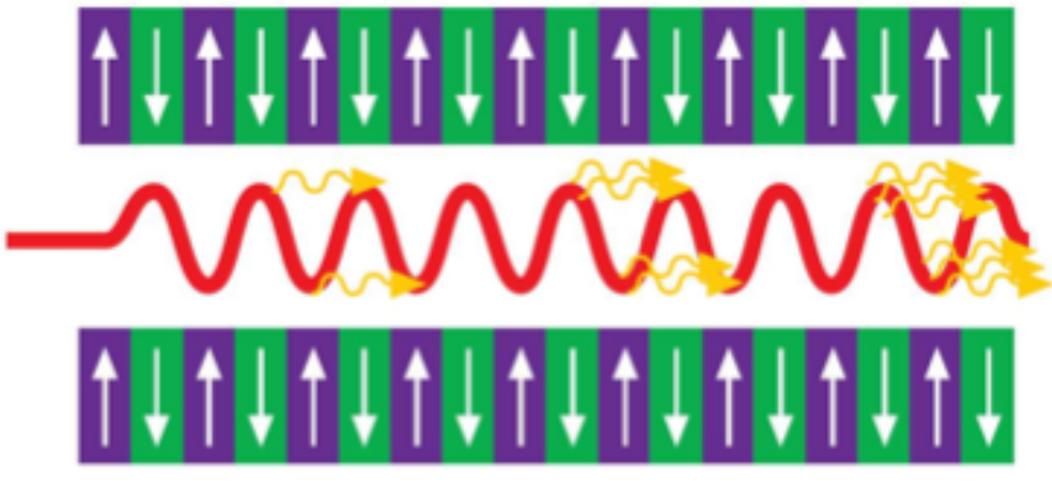
Undulator Radiation

Low-Gain Free Electron Laser

One-dimensional theory of the high-gain FEL

Applications of the high-gain FEL equations

Undulator radiation



assume undulator period λ_u = 25 mm

We consider an electron that was accelerated by 500 million volts
(Lorentz factor $\gamma = 1000$)

Electron moves on a wavelike curve through the undulator
(curve is perpendicular to magnetic field)

To estimate wavelength of undulator radiation, apply Theory of Relativity twice:

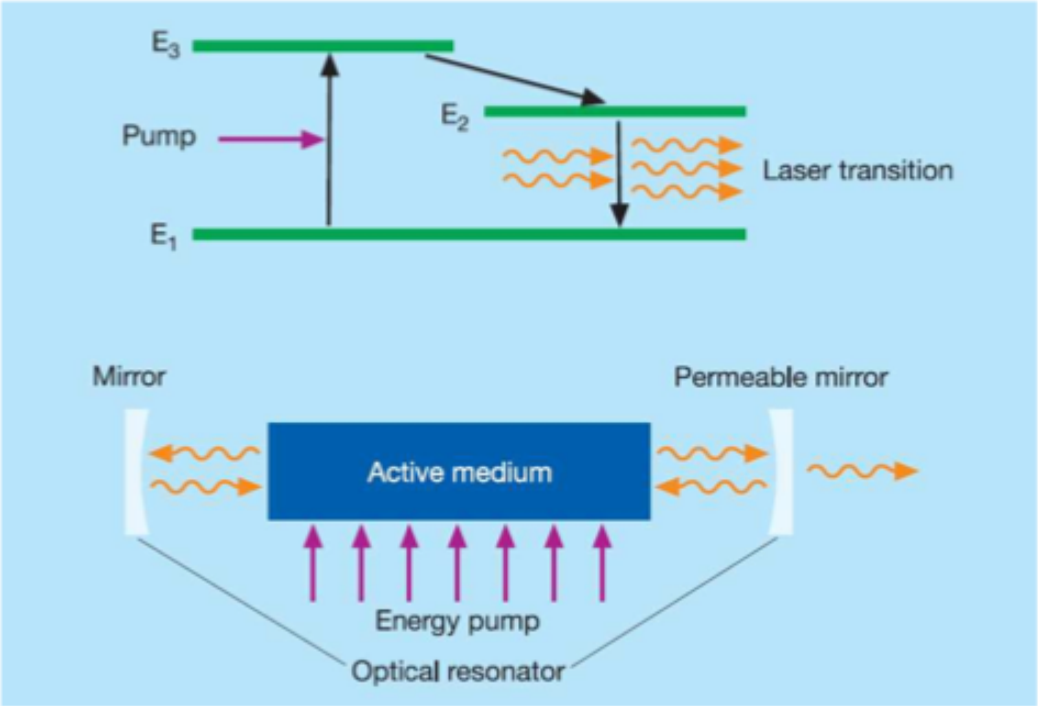
(1) Moving system: undulator period appears shortened by length contraction
$\lambda^* = \lambda_u / \gamma$. Electron emits radiation of wavelength λ^* (about 25 μm)

(2) Doppler effect reduces wavelength by another factor of $1 / \gamma$ (about 25 nm)

Result:
radiation wavelength is about a million times shorter than undulator period

Reduction from 25 mm to about 25 nm

Comparison of Quantum Laser and Free-Electron Laser (FEL)

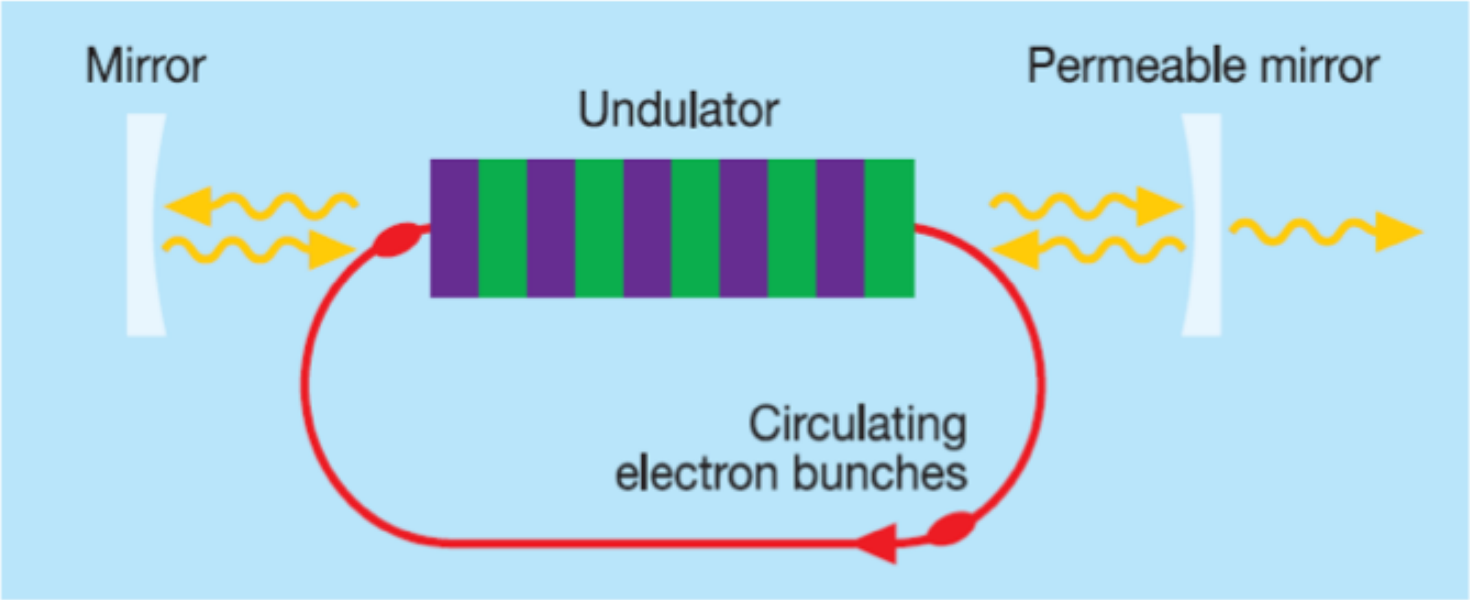


bound-electron laser

Conventional laser

3 main components:

(1) active laser medium
(2) energy pump
(3) optical resonator



Free-electron laser

(1) Role of active medium and energy pump are both taken over by relativistic electrons

(2) Optical resonator possible for visible and infrared light (not for UV and X rays)

Sinusoidal electron trajectory in undulator

Transverse acceleration by Lorentz force

$$\gamma m_e \dot{\boldsymbol{v}} = -e\boldsymbol{v} \times \boldsymbol{B} \quad \text{with} \quad \boldsymbol{B} = -B_0 \sin(k_u z)\, \boldsymbol{e}_y$$

Yields two coupled equations

$$\ddot{x} = \frac{e}{\gamma m_e} B_y \dot{z} \qquad\qquad \ddot{z} = -\frac{e}{\gamma m_e} B_y \dot{x}$$

First-order solution

$$x(t) \approx \frac{eB_0}{\gamma m_e \beta c k_u^2} \sin(k_u \beta c t)\,, \quad z(t) \approx \beta c t\,, \quad \beta = v/c$$

Undulator parameter

$$K = \frac{eB_0}{m_e c k_u} = \frac{eB_0 \lambda_u}{2\pi m_e c} \quad K \approx 1..2$$

Second-order solution

small longitudinal oscillation
leads to odd higher harmonics

$$x(t) = \frac{K}{\gamma k_u} \sin(\omega_u t) \quad z(t) = \bar{v}_z t - \frac{K^2}{8\gamma^2 k_u} \sin(2\omega_u t)$$

Average longitudinal speed

$$\bar{v}_z = \bar{\beta}\, c \quad \text{with} \quad \bar{\beta} = \left(1 - \frac{1}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)\right)$$

Co-moving coordinate system

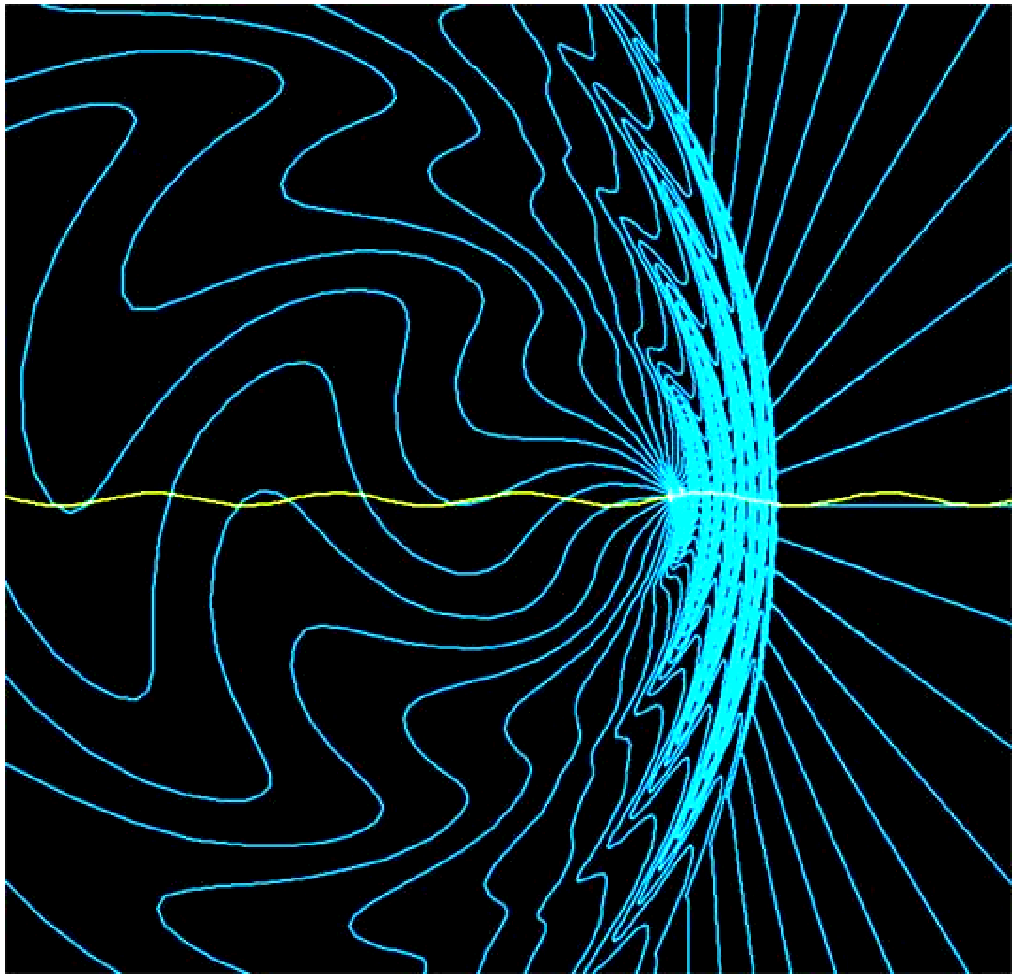
In moving system: electron emits dipole radiation $\omega^* = \bar{\gamma}\omega_u$, $\lambda_u^* = \lambda_u/\bar{\gamma}$

Lorentz transformation of photon energy into laboratory system

$$\hbar\omega^* = \bar{\gamma}\hbar\omega_\ell(1-\bar{\beta}\cos\theta) \quad \Rightarrow \quad \lambda_\ell = \frac{2\pi c}{\omega_\ell} = \frac{2\pi c\bar{\gamma}}{\omega^*}(1-\bar{\beta}\cos\theta) = \lambda_u(1-\bar{\beta}\cos\theta)$$

Use $\bar{\beta} = \left[1-(1+K^2/2)/(2\gamma^2)\right]$ and $\cos\theta \approx 1-\theta^2/2$

$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}\left(1+\frac{K^2}{2}+\gamma^2\theta^2\right)}$$



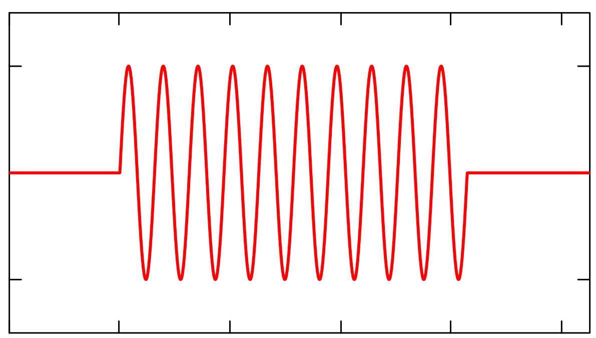
Computation by Shintake

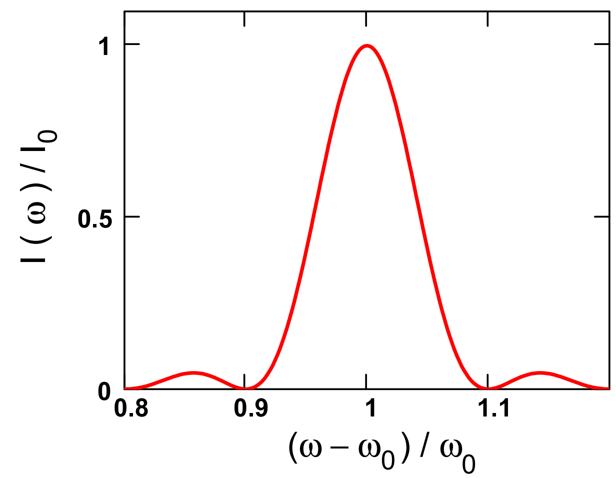
Radiation power in laboratory system

$$P = -\frac{dW}{dt} = -\frac{dW^*}{dt^*} = P^* \quad \Rightarrow \quad \boxed{P = \frac{e^2 c\, \gamma^2 K^2 k_u^2}{12\pi\varepsilon_0 (1 + K^2/2)^2}}$$

Line shape of undulator radiation

Electron passing an undulator with N_u periods produces wave train with N_u oscillations.



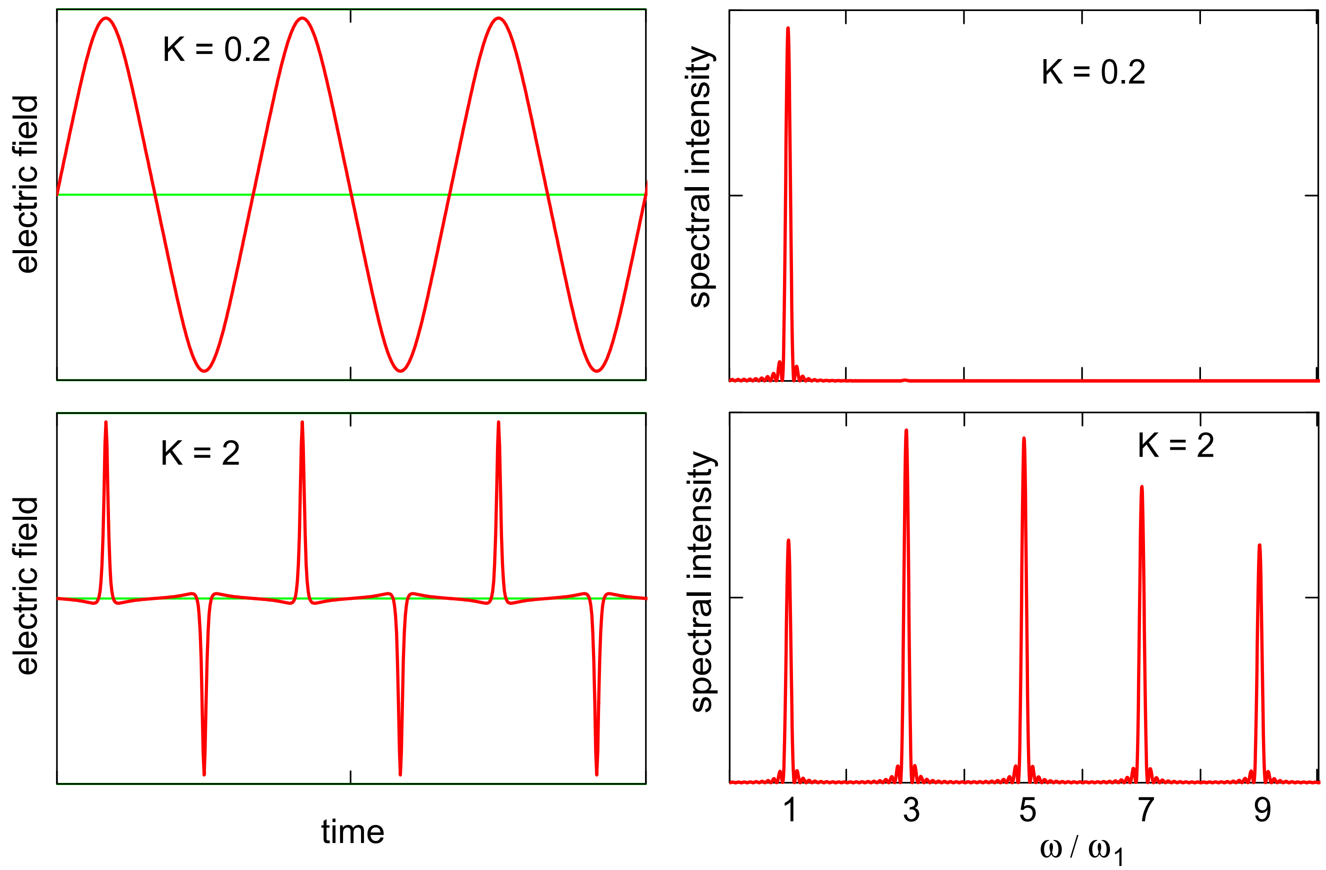


$$\text{Spectral intensity :} \qquad I(\omega) \propto \left(\frac{\sin\xi}{\xi}\right)^2 \quad \text{with} \quad \xi = \frac{\pi\, N_u(\omega - \omega_0)}{\omega_0}$$

Higher harmonics of undulator radiation

Complicated issue and not the topic of my FEL lecture
For details see J.A. Clarke, *The Science and Technology of Undulators and Wigglers*

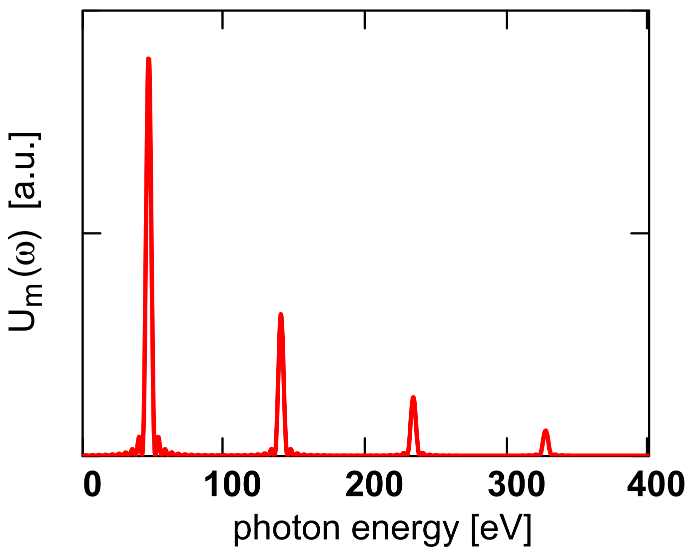
Model calculation for a detector at θ = 0 with very small aperture



small undulator parameter
only first harmonic

large undulator parameter
harmonics 1, 3, 5, 7...

radiation at angles θ > 0 has also even harmonics
(see Clarke)



detector of finite size centered at θ=0
aperture matched to bandwidth of n^{th} harmonic

Theory of the Low-Gain FEL

Energy transfer from electron to light wave

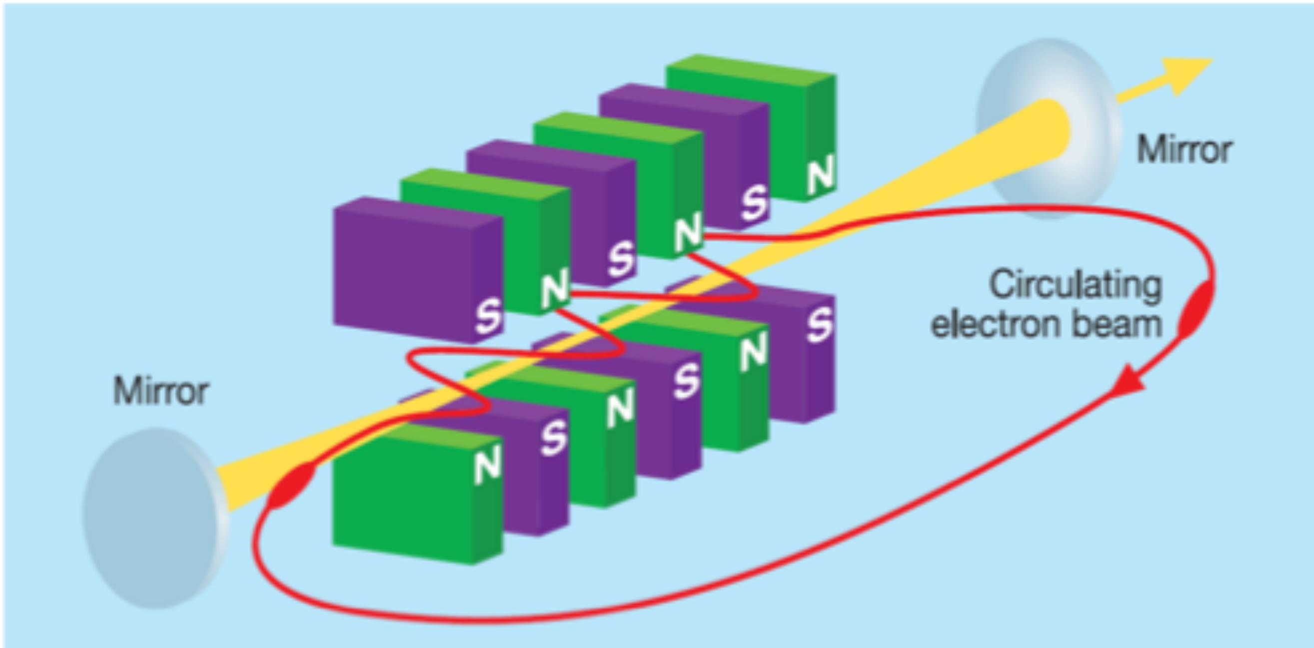
Differential equations of the low-gain FEL

The pendulum equations

FEL gain, Madey theorem

Higher harmonics

Principle of low-gain FEL (visible or infrared)



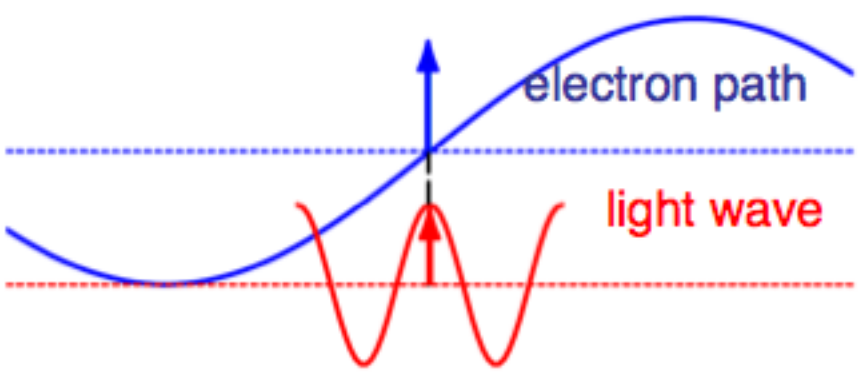
Light travels back and forth between two mirrors

Light is amplified by few % in each turn

Not possible in UV and X-ray range (no mirrors available)

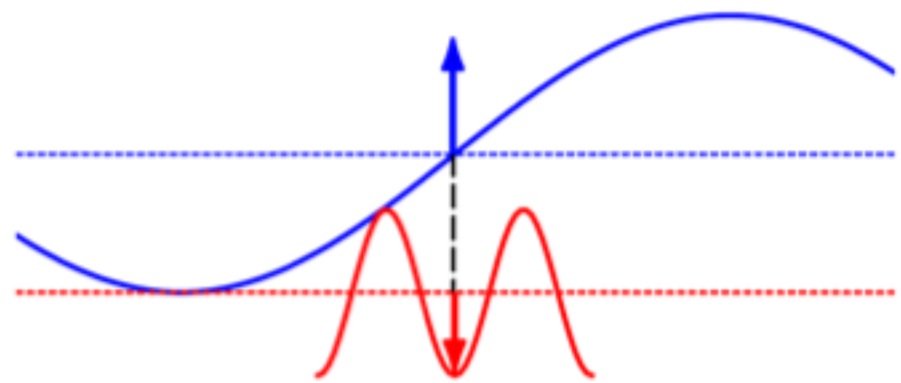
Correct phase of light wave: FEL case

energy transfer from electron to light wave



wrong phase

energy transfer from light wave to electron



Consider seeding by an external light source with wavelength λ_ℓ

$$E_x(z,t) = E_0 \cos(k_\ell z - \omega_\ell t + \psi_0) \quad \text{with} \quad k_\ell = \frac{\omega_\ell}{c} = \frac{2\pi}{\lambda_\ell}$$

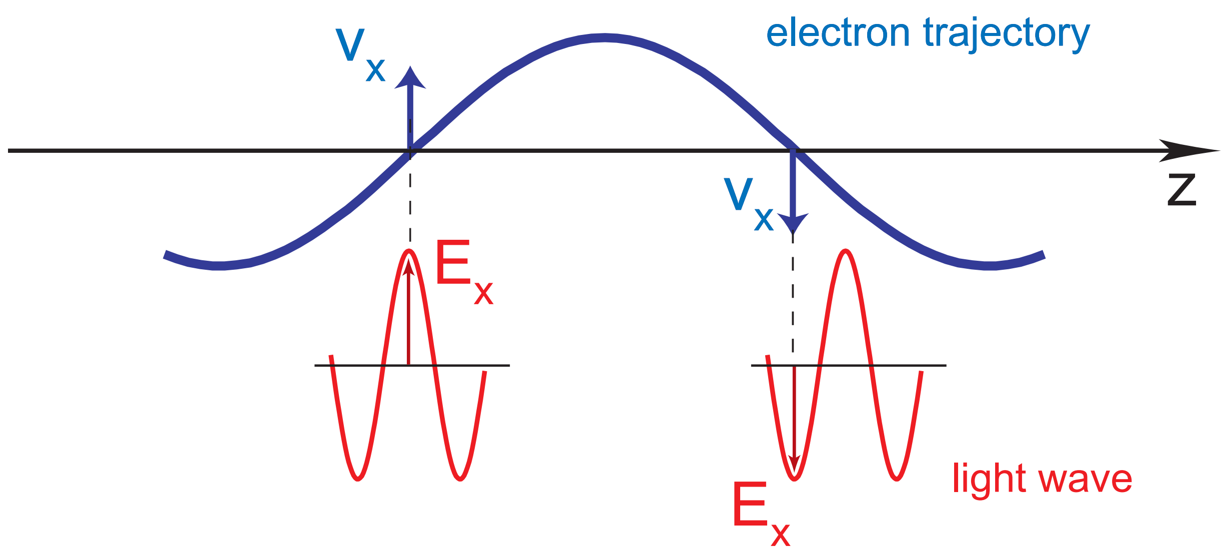
Question: can there be a continuous energy transfer from electron beam to light wave?
Electron energy $W = \gamma m_e c^2$ changes in time dt by

$$dW = \boldsymbol{v} \cdot \boldsymbol{F}\, dt = -e v_x(t) E_x(t)\, dt$$

Average electron speed in z direction $\bar{v}_z = c\left(1 - \frac{1}{2\gamma^2}\left(1 + K^2/2\right)\right) < c$
Electron and light travel times for half period of undulator:

$$t_{el} = \lambda_u/(2\bar{v}_z)\,, \quad t_{light} = \lambda_u/(2c)$$

Continuous energy transfer happens if $\quad \omega_\ell(t_{el} - t_{light}) = \pi$



slippage of light wave

1 optical wavelength
per undulator period

From the condition $\omega_\ell(t_{el} - t_{light}) = \pi$ compute light wavelength

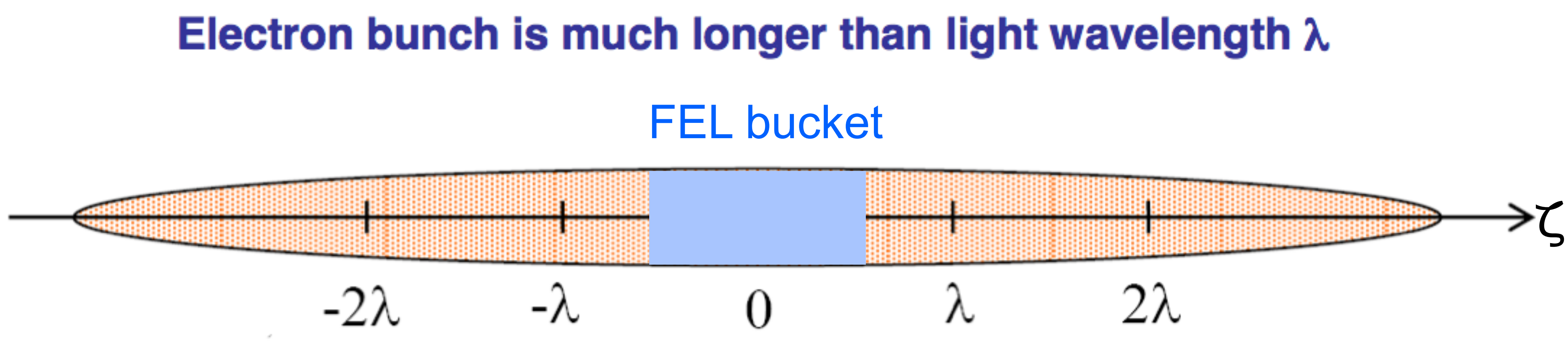
$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)}$$

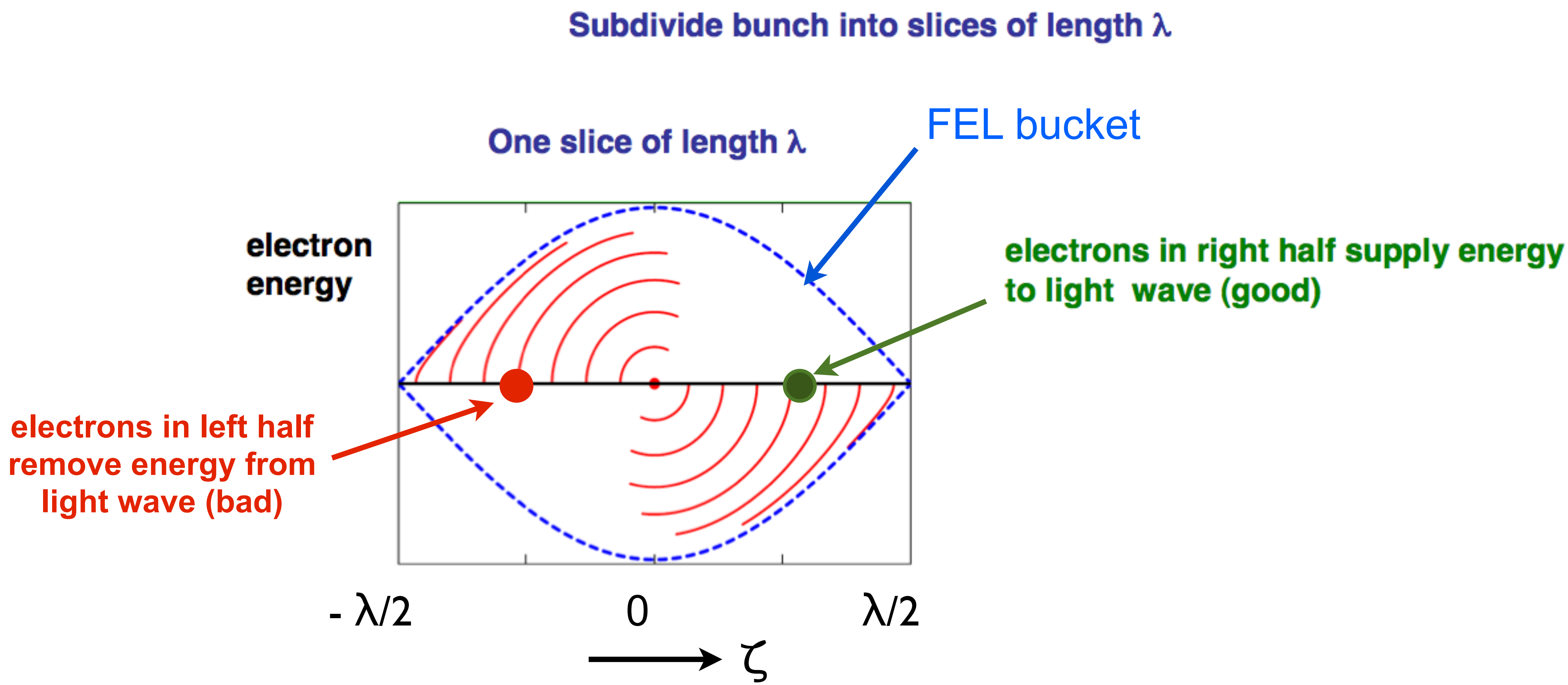
Identical with undulator radiation wavelength in forward direction ($\theta = 0$)

Remark: $\omega_\ell(t_{el} - t_{light}) = 3\pi, 5\pi \ldots$ also possible
$\Rightarrow$ generation of odd harmonics ($\lambda_\ell/3\,,\ \lambda_\ell/5\ldots$)

Note however: $\omega_\ell(t_{el} - t_{light}) = 2\pi, 4\pi \ldots$ yields zero net energy transfer from electron to light wave
$\Rightarrow$ even harmonics ($\lambda_\ell/2\,,\ \lambda_\ell/4\ldots$) are not present

Definition of FEL bucket





Differential equations of the low-gain FEL

Energy transfer from an electron to the light wave

$$\begin{aligned} \frac{dW}{dt} &= -ev_x(t)E_x(t) = -e\frac{cK}{\gamma}\cos(k_u z)E_0\cos(k_\ell z - \omega_\ell t + \psi_0) \\ &\equiv -\frac{ecKE_0}{2\gamma}\left[\cos\psi + \cos\chi\right] \end{aligned}$$

Ponderomotive phase ψ | rapidly oscillating phase χ

$$\psi = (k_\ell + k_u)z(t) - \omega_\ell t + \psi_0 \qquad \chi = (k_\ell - k_u)\bar{\beta}ct - \omega_\ell t + \psi_0$$

Continuous energy transfer from electron to light wave if ψ is constant
Optimum value $\psi = 0$

Neglect longitudinal oscillation, so $v_z \approx \bar{v}_z$

The condition $\psi = const$ can only be fulfilled for a certain wavelength

$$\psi(t) = (k_\ell + k_u)\bar{v}_z t - k_\ell c\, t + \psi_0 = const \quad \Leftrightarrow \quad \frac{d\psi}{dt} = (k_\ell + k_u)\bar{v}_z - k_\ell c = 0$$

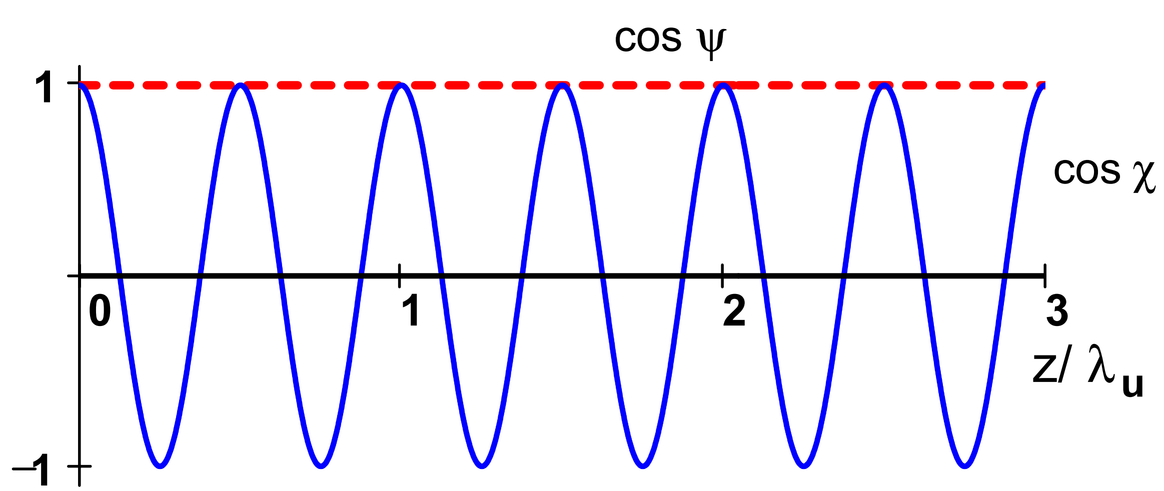
Insert $\bar{v}_z$ and use $k_u \ll k_\ell$ to compute light wavelength:

$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)}$$

Condition for sustained energy transfer yields wavelength of undulator radiation at $\theta = 0$
$\Rightarrow$ spontaneous undulator radiation can "seed" a SASE FEL

What about phase χ? The term $\cos\chi$ averages to zero

$$\chi(z) = \psi(z) - 2k_u z \qquad \Rightarrow \quad \cos\chi(z) \propto \cos(2k_u z)$$



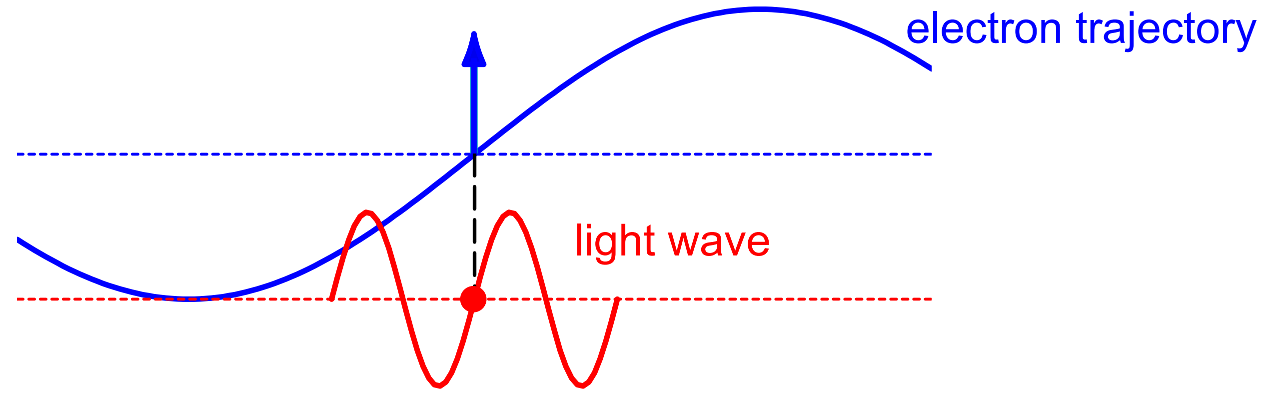
Internal bunch coordinate zeta and ponderomotive phase psi

$$\zeta = \lambda_\ell \cdot (\psi + \pi/2)/(2\pi)$$

bucket center at $\zeta = 0$, $\psi = -\pi/2$

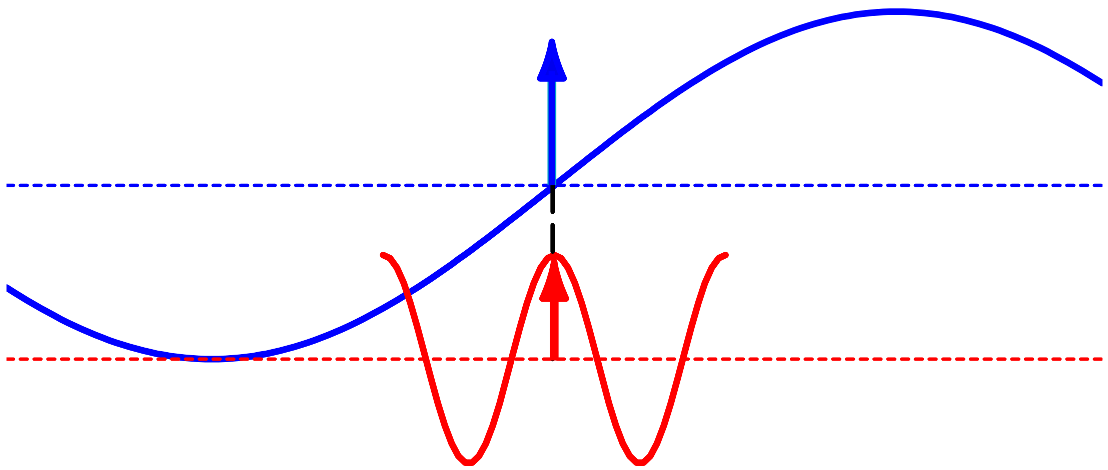
Reference particle: $\psi_0 = -\pi/2$

zero energy transfer between electron and light wave



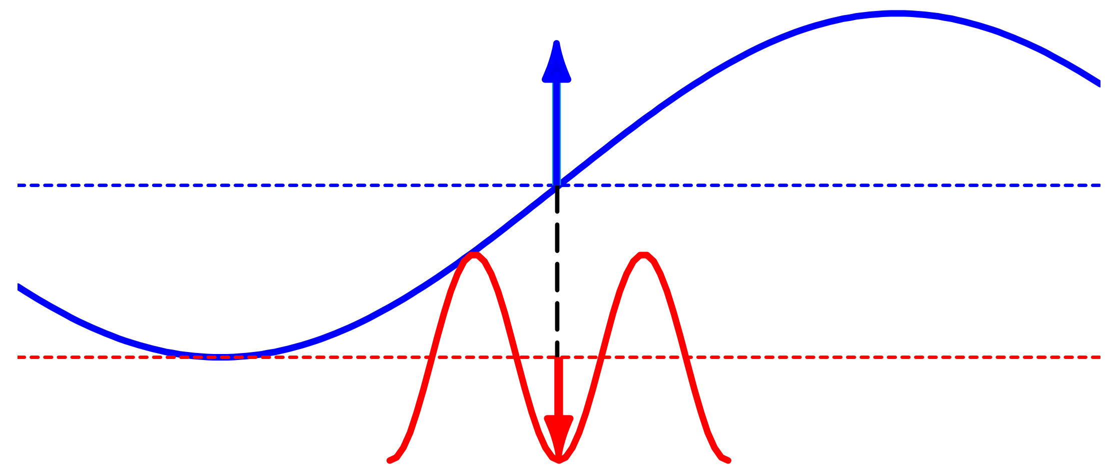
FEL case: $\psi_0 = 0$

energy transfer from electron to light wave



Laser-acceleration: $\psi_0 = -\pi$

energy transfer from light wave to electron



The pendulum equations

Lasing process in undulator is started by monochromatic light of wavelength λ_ℓ
Resonance electron energy $W_r = \gamma_r\, m_e c^2$ defined by

$$\lambda_\ell = \frac{\lambda_u}{2\gamma_r^2}\left(1+\frac{K^2}{2}\right) \quad \Rightarrow \quad \gamma_r = \sqrt{\frac{\lambda_u}{2\lambda_\ell}\left(1+\frac{K^2}{2}\right)}$$

(Electrons with energy $W = W_r$ emit undulator radiation with wavelength $\lambda = \lambda_\ell$)

Consider off-resonance electron $\gamma \neq \gamma_r$, define relative energy deviation

$$\eta = \frac{\gamma - \gamma_r}{\gamma_r} \qquad (0 < |\eta| \ll 1)$$

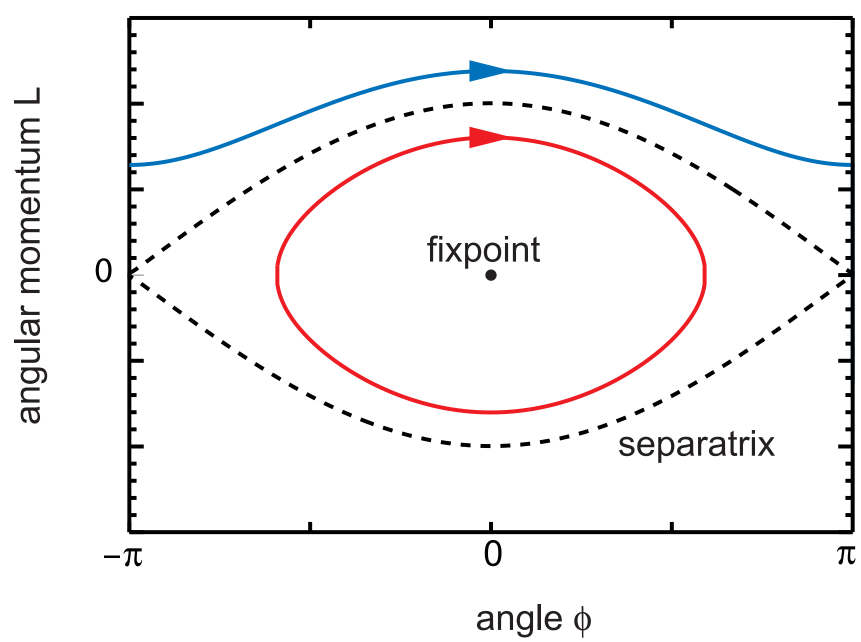
Ponderomotive phase no longer constant for $\eta \neq 0$. Also η changes due to interaction with radiation field

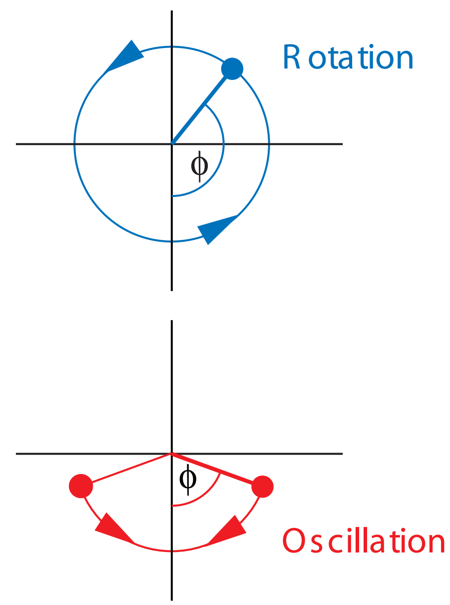
$$\frac{d\psi}{dt} = 2k_u c\,\eta \qquad\qquad \frac{d\eta}{dt} = -\frac{eE_0K}{2m_e c\gamma_r^2}\cos\psi$$

Define shifted phase $\varphi = \psi + \pi/2$ to see analogy with mathematical pendulum

$$\textbf{FEL} \qquad \frac{d\varphi}{dt} = 2k_u c \cdot \eta \qquad \frac{d\eta}{dt} = -\frac{eE_0 K}{2m_e c \gamma_r^2} \cdot \sin\varphi$$

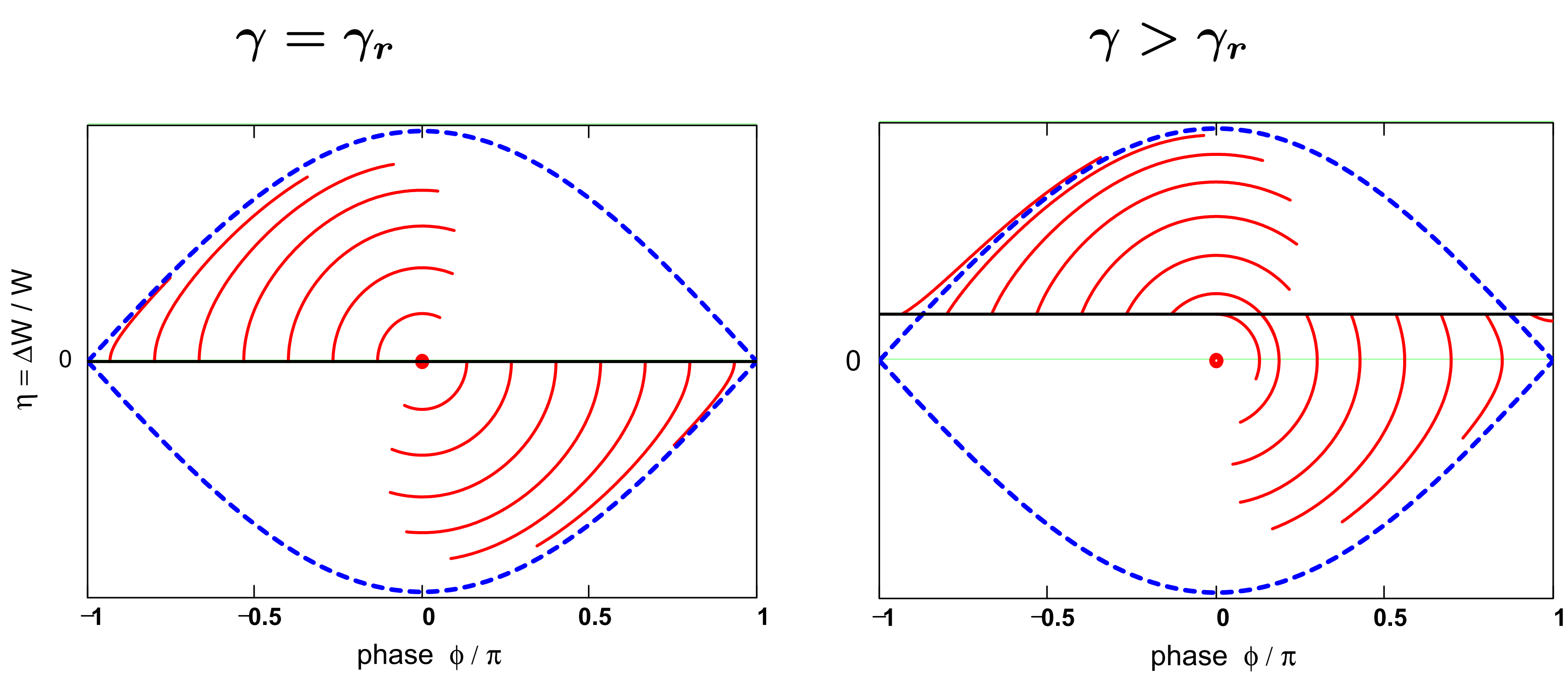
$$\textbf{pendulum} \qquad \frac{d\varphi}{dt} = \frac{1}{m\ell^2} \cdot L \qquad \frac{dL}{dt} = -m\,g \cdot \sin\varphi$$





Small angles: $\sin\varphi \approx \varphi$ pendulum carries out a harmonic oscillation:
$\varphi(t) = \varphi_0 \cos(\omega t)\,, \quad L(t) = -m\,\ell^2\omega\,\varphi_0 \sin(\omega t) \quad \Rightarrow$ elliptic phase space curves
Large angular momentum motion unharmonic. Very large angular momentum: rotation (unbounded motion)

FEL phase space curves



On resonance ($\gamma = \gamma_r$): net energy transfer zero

Above resonance ($\gamma > \gamma_r$): positive net energy transfer from electron beam to light wave

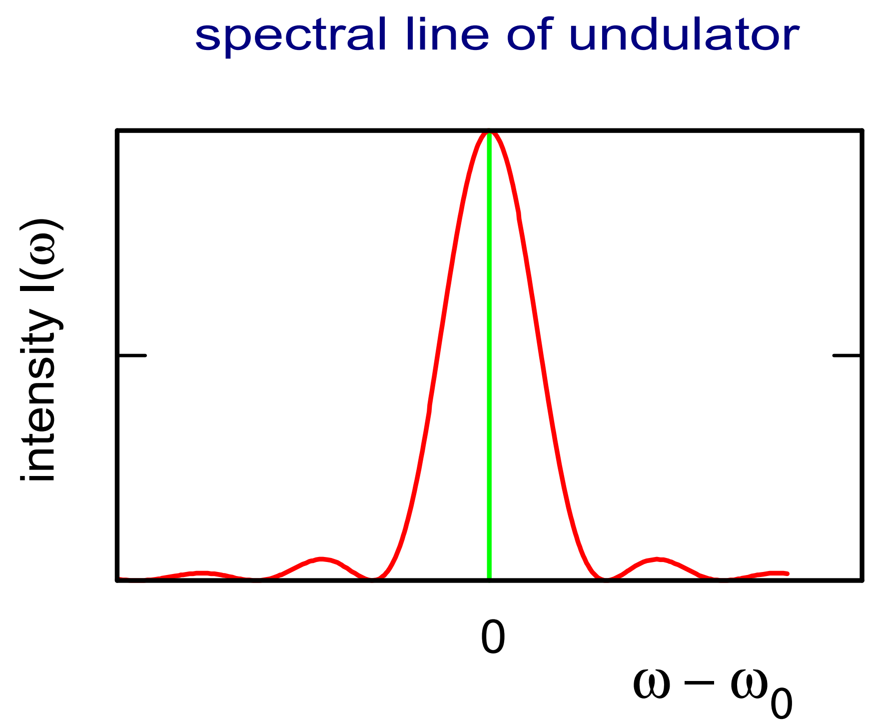
Resonance electron energy $W_r = \gamma_r \, m_e c^2$ defined by

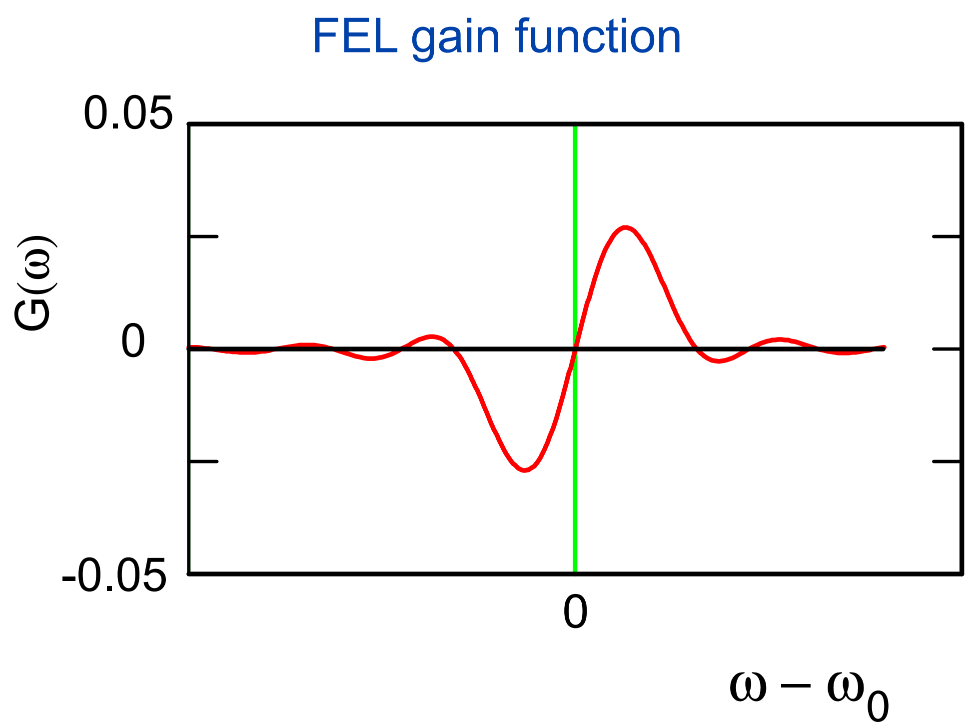
$$\lambda_\ell = \frac{\lambda_u}{2\gamma_r^2}\left(1 + \frac{K^2}{2}\right) \quad \Rightarrow \quad \gamma_r = \sqrt{\frac{\lambda_u}{2\lambda_\ell}\left(1 + \frac{K^2}{2}\right)}$$

(Electrons with energy $W = W_r$ emit undulator radiation with wavelength $\lambda = \lambda_\ell$)

Gain Function of Low-Gain FEL

Madey Theorem: the FEL gain curve is proportional to the negative derivative of the line-shape curve of undulator radiation





The normalized lineshape curve of undulator radiation and the gain curve of a typical low-gain FEL

Note: gain of FEL amplifier is $G(\omega)+1$

One-dimensional Theory of the High-Gain FEL

Microbunching

Basic Elements of the 1D FEL Theory

Radiation Field and Space Charge Field

The Coupled First-Order Differential Equations of the High-Gain FEL

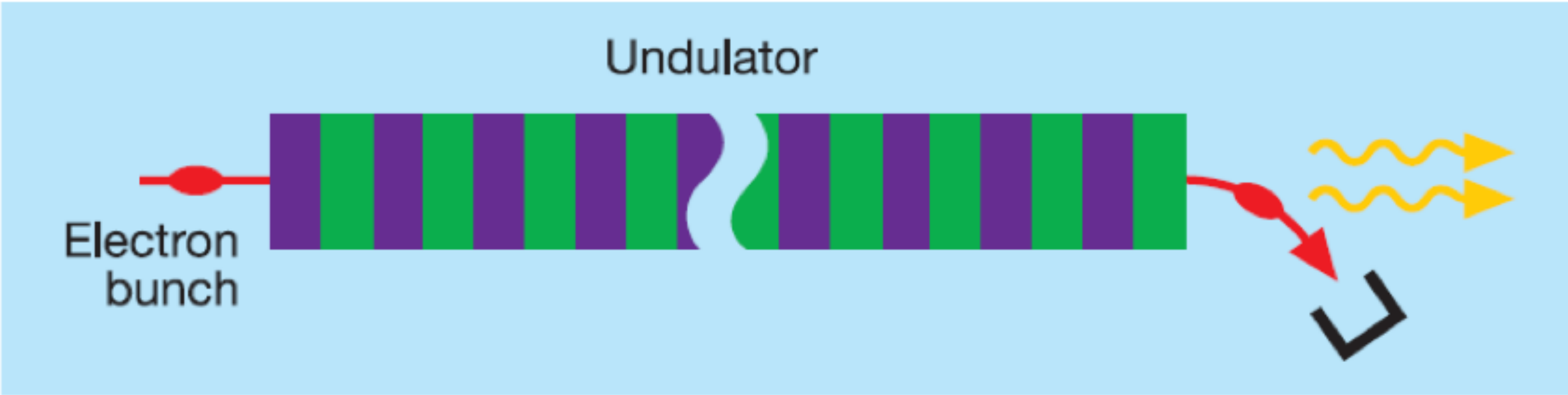
The Third-Order Equation of the High-Gain FEL

General analytic solution of the third-order equation

Ultraviolet and X-Ray FELs

No mirrors exist to build optical cavity for UV light and X rays

FEL gain must be achieved in single passage through a very long undulator



Important mechanism: Self-Amplified Spontaneous Emission SASE
(theory: Kondratenko, Saldin, Bonifacio, Pellegrini, Narducci ...)

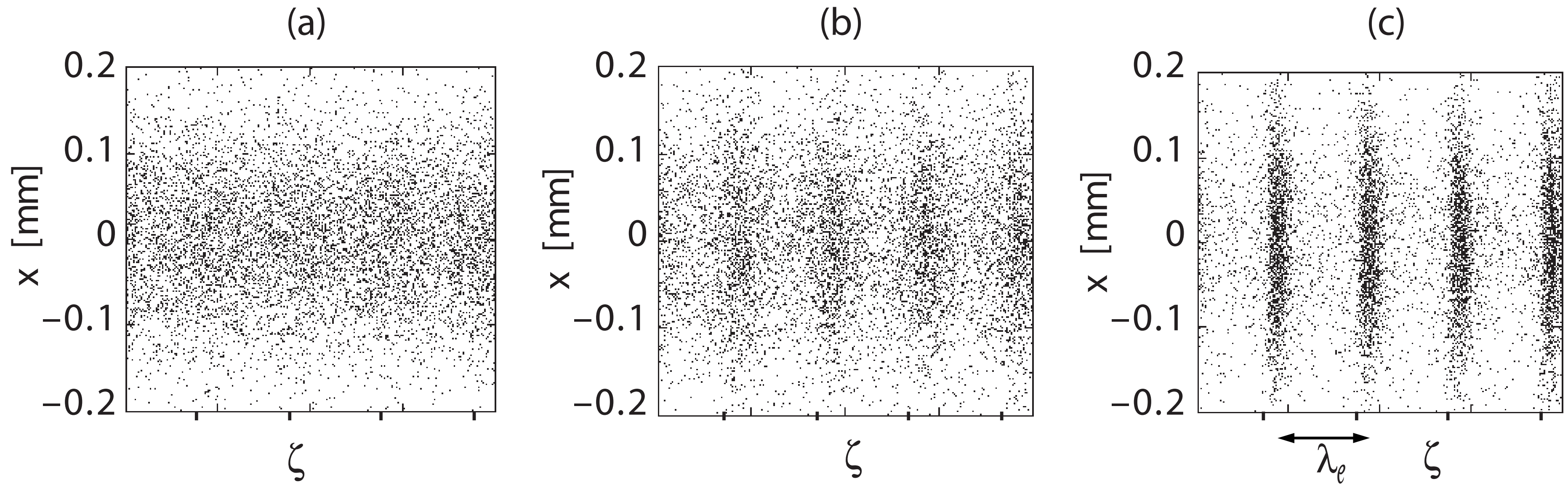
Undulator radiation is produced in the first section of the undulator and this radiation is amplified in the later sections

Microbunching

Essential feature of high-gain FEL: very many electrons radiate coherently
Radiation grows quadratically with the number of particles $P_N = N^2 P_1$

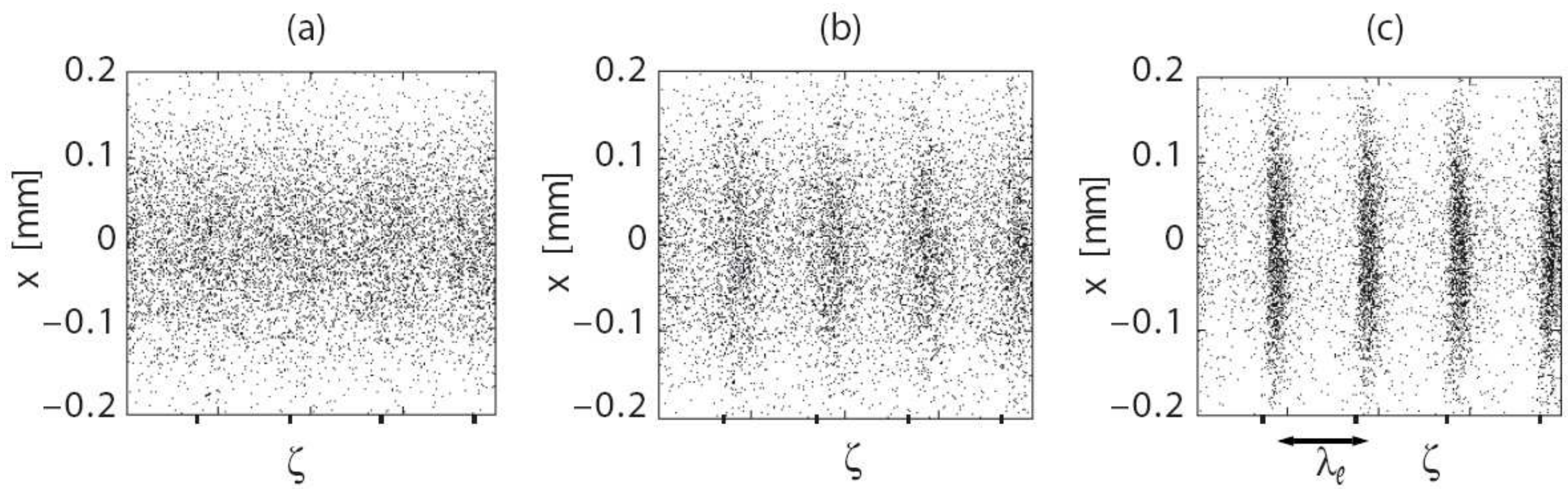
big problem: concentration of $\approx 10^9$ electrons into a tiny volume is impossible, $L_{bunch} \gg \lambda_\ell$

Simulation of microbunching by Sven Reiche, UCLA (code GENESIS)

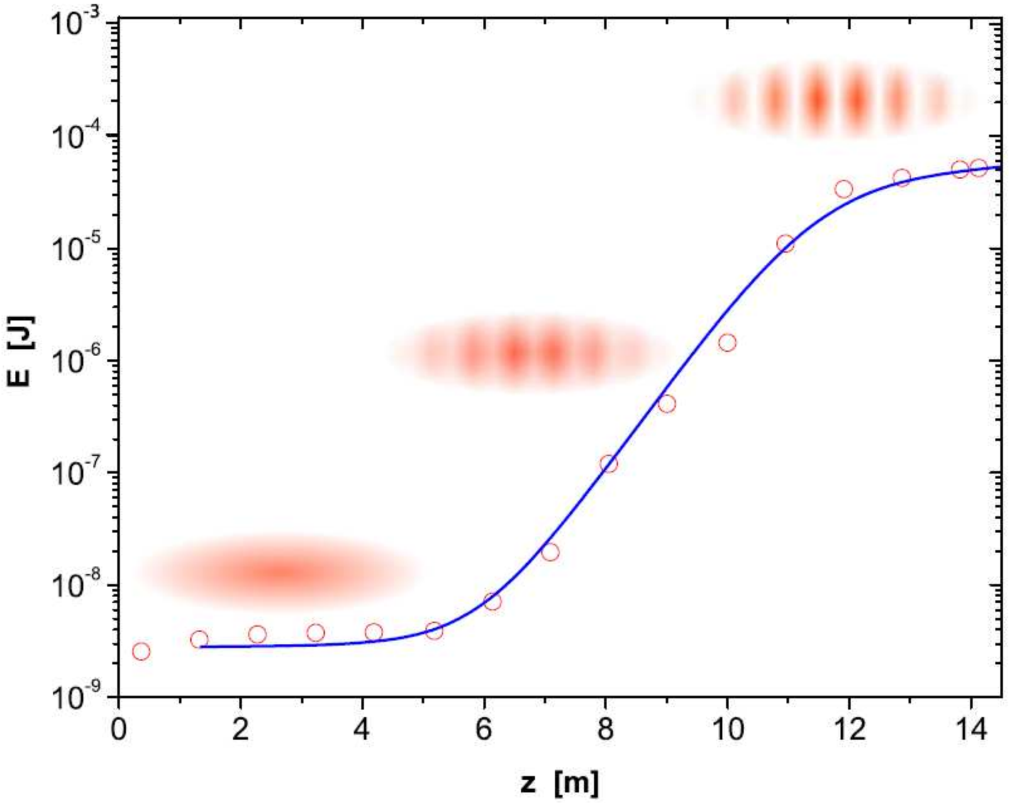


Electrons losing energy to light wave travel on a sinus orbit of larger amplitude than electrons gaining energy from light wave
Result: modulation of longitudinal velocity

Microbunching and exponential gain



Simulation of microbunching (Sven Reiche)



length of undulator

Measured FEL pulse energy at 98 nm wavelength

Basic elements of the one-dimensional FEL theory

1D FEL theory: dependency of bunch charge density and electromagnetic fields on transverse coordinates x, y is neglected. Also betatron oscillations and diffraction of the light wave are disregarded.

Complex notation

Note: this is a constant E_0 in the low-gain theory

$$\tilde{E}_x(z,t) = \tilde{E}_x(z)\exp[ik_\ell z - i\omega_\ell t] \qquad E_x(z,t) = \mathrm{Re}\left\{\tilde{E}_x(z)\exp[ik_\ell z - i\omega_\ell t]\right\}$$

Complex amplitude function $\tilde{E}_x(z)$, grows slowly with z

Analytic description of high-gain FEL

(1) coupled pendulum equations, describing phase-space motion of particles under the influence of electric field of light wave

(2) inhomogeneous wave equation for electric field of light wave

(3) evolution of a microbunch structure coupled with longitudinal space charge forces

Initial conditions:
uniform charge distribution in bunch at $z = 0$, lasing process started by seed laser

Interaction with periodic light wave gradually produces density modulation
periodic in ponderomotive phase ψ (resp. internal bunch coordinate ζ with period λ_ℓ)

$$\tilde{\rho}(\psi, z) = \rho_0 + \tilde{\rho}_1(z)e^{i\psi} \qquad\qquad \tilde{j}(\psi, z) = j_0 + \tilde{j}_1(z)e^{i\psi}$$

Oscillatory part in longitudinal velocity is neglected: $z(t) = \bar{\beta}c\,t$
Higher harmonics are ignored

Radiation field
Wave equation for E_x field

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] E_x(z,t) = \mu_0 \frac{\partial j_x}{\partial t} + \frac{1}{\varepsilon_0}\frac{\partial \rho}{\partial x}$$

1D FEL theory: charge density independent of x $\quad\Rightarrow\quad$ neglect $\partial\rho/\partial x$

High-gain FEL: complex amplitude $\tilde{E}_x(z)$ depends on path length z in undulator

$$E_x(z,t) = \tilde{E}_x(z)\exp[ik_\ell(z - ct)] \qquad \tilde{E}_x(0) = E_0$$

First goal: find differential equation for field amplitude $\tilde{E}_x(z)$

Slowly varying amplitude (SVA) approximation:
change of amplitude within one light wavelength (growth rate) is small
change of growth rate is negligible

$$\left|\tilde{E}_x'(z)\right| \lambda_\ell \ll \left|\tilde{E}_x(z)\right| \Rightarrow \left|\tilde{E}_x'(z)\right| \ll k_\ell \left|\tilde{E}_x(z)\right|$$

$$\left|\tilde{E}_x''(z)\right| \ll k_\ell \left|\tilde{E}_x'(z)\right| \Rightarrow \tilde{E}_x''(z) \text{ is negligible}$$

Result: Differential equation for slowly varying amplitude

$$\frac{d\tilde{E}_x}{dz} = -\frac{i\mu_0}{2k_\ell} \cdot \frac{\partial j_x}{\partial t} \cdot \exp[-ik_\ell(z-ct)]$$

Question: What is the transverse current j_x?

$$\boldsymbol{j} = \rho \boldsymbol{v} \quad \Rightarrow \quad j_x = j_z\, v_x/v_z \approx j_z\, \frac{K}{\gamma} \cos(k_u z)$$

$$\frac{d\tilde{E}_x}{dz} = -\frac{i\mu_0 K}{2k_\ell \gamma} \cdot \frac{\partial j_z}{\partial t} \exp[-ik_\ell(z-ct)] \cos(k_u z)$$

$$\frac{\partial \tilde{j}_z}{\partial t} = \frac{\partial \tilde{j}_z}{\partial \psi}\frac{\partial \psi}{\partial t} = -i\omega_\ell \, \tilde{j}_1 \, e^{i\psi} = -i\omega_\ell \, \tilde{j}_1 \, \exp[ik_\ell(z-ct)+ik_u z] \, .$$

The derivative of the transverse field becomes

$$\begin{aligned}\frac{d\tilde{E}_x}{dz} &= -\frac{\mu_0 c K}{2\gamma}\tilde{j}_1 \, \exp[ik_\ell(z-ct)+ik_u z]\exp[-i(k_\ell z - ct)] \, \frac{e^{ik_u z}+e^{-ik_u z}}{2} \\ &= -\frac{\mu_0 c K}{4\gamma}\tilde{j}_1 \, \{1+\exp(i2k_u z)\}\end{aligned}$$

The phase factor $\exp[i2k_u z]$ carries out two oscillations per undulator period λ_u and averages to zero

$$\boxed{\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c K}{4\gamma}\cdot \tilde{j}_1}$$

Space charge field (longitudinal field)

Electric field created by modulated charge density is computed using Maxwell equation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\varepsilon_0$

Rapidly oscillating field:

$$\frac{\partial E_z}{\partial z} = \frac{\tilde{\rho}_1(z)}{\varepsilon_0} \exp[i((k_\ell + k_u)z - \omega_\ell t)]$$

Amplitude of longitudinal electric field is

$$\tilde{E}_z = -\frac{i}{\varepsilon_0(k_\ell + k_u)}\,\tilde{\rho}_1 \approx -\frac{i\mu_0 c^2}{\omega_\ell} \cdot \tilde{j}_1$$

$$\boxed{\tilde{E}_z = -\frac{i\mu_0 c^2}{\omega_\ell} \cdot \tilde{j}_1}$$

The Coupled First-Order Differential Equations

Low-gain FEL:
Evolution of ponderomotive phase ψ and of relative energy deviation η described by pendulum equations (note that we use $z = \bar{\beta}\, c\, t$ as our quasi-time)

$$\frac{d\psi}{dz} = 2k_u\eta \; , \qquad \frac{d\eta}{dz} = -\frac{eE_0\hat{K}}{2m_e c^2\gamma_r^2}\cos\psi$$

High-gain FEL: field amplitude is z dependent

$$\left[\frac{d\eta}{dz}\right]_{light\ wave} = -\frac{e\hat{K}}{2m_e c^2\gamma_r^2}\mathrm{Re}(\tilde{E}_x e^{i\psi})$$

Add energy change due to interaction with space charge field:

$$\left[\frac{d\eta}{dz}\right]_{space\ charge} = -\frac{e}{m_e c^2\gamma_r}\mathrm{Re}(\tilde{E}_z e^{i\psi})$$

Combining the two effects yields

$$\frac{d\eta}{dz} = -\frac{e}{m_e c^2 \gamma_r} \mathrm{Re}\left\{ \left(\frac{\hat{K}\tilde{E}_x}{2\gamma_r} + \tilde{E}_z \right) e^{i\psi} \right\}$$

Goal: study phase space motion of electrons as in low-gain case, but take growth of field amplitude $\tilde{E}_x(z)$ into account and also evolution of space charge field $\tilde{E}_z(z)$. Both are related to modulation amplitude $\tilde{j}_1(z)$ of electron beam current density:

$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c K}{4\gamma} \cdot \tilde{j}_1(z) \qquad \tilde{E}_z(z) = -\frac{i\mu_0 c^2}{\omega_\ell} \cdot \tilde{j}_1(z)$$

Obvious task: compute $\tilde{j}_1$ for a given arrangement of electrons in phase space
Subdivide electron bunch into longitudinal slices of length λ_ℓ
corresponding to slices of length 2π in phase variable ψ

Distribution function for N particles per slice

$$S(\psi) = \sum_{n=1}^{N} \delta(\psi - \psi_n) \qquad \psi,\ \psi_n \in [0, 2\pi]$$

We consider first the special case of a perfectly uniform longitudinal distribution of the electrons in the bunch and continue the function $S(\psi)$ periodically. The more realistic case of a random longitudinal particle distribution is investigated later.

Fourier series

$$S(\psi) = \frac{c_0}{2} + \mathrm{Re}\left\{\sum_{k=1}^{\infty} c_k \exp(i\,k\,\psi)\right\}\ , \qquad c_k = \frac{1}{\pi}\int_0^{2\pi} S(\psi)\exp(i\,k\,\psi)d\psi$$

The modulated current density at the first harmonic is

$$\tilde{j}_1 = -e\,c\,n_e\,\frac{2}{N}\sum_{n=1}^{N}\exp(-i\psi_n)$$

Coupled first-order equations

$$
\begin{aligned}
\tilde{j}_1 &= -n_e e\, c\, \frac{2}{N} \sum_{n=1}^{N} \exp(-i\psi_n) \\
\frac{d\tilde{E}_x}{dz} &= -\frac{\mu_0 c \widehat{K}}{4\gamma} \cdot \tilde{j}_1 \\
\frac{d\psi_n}{dz} &= 2k_u \eta_n\,, \qquad n = 1...N \\
\frac{d\eta_n}{dz} &= -\frac{e}{m_e c^2 \gamma_r} \mathrm{Re}\left\{ \left(\frac{\widehat{K}\tilde{E}_x}{2\gamma_r} - \frac{i\mu_0 c^2}{\omega_\ell} \cdot \tilde{j}_1 \right) \exp(i\psi_n) \right\}
\end{aligned}
$$

Coupled first-order equations describe time evolution of
1) modulated current density
2) light wave amplitude $\tilde{E}_x$
3) ponderomotive phase ψ_n of electron number n $\quad (n = 1 \ldots N)$
4) relative energy deviation $\eta_n = (\gamma_n - \gamma_r)/\gamma_r$

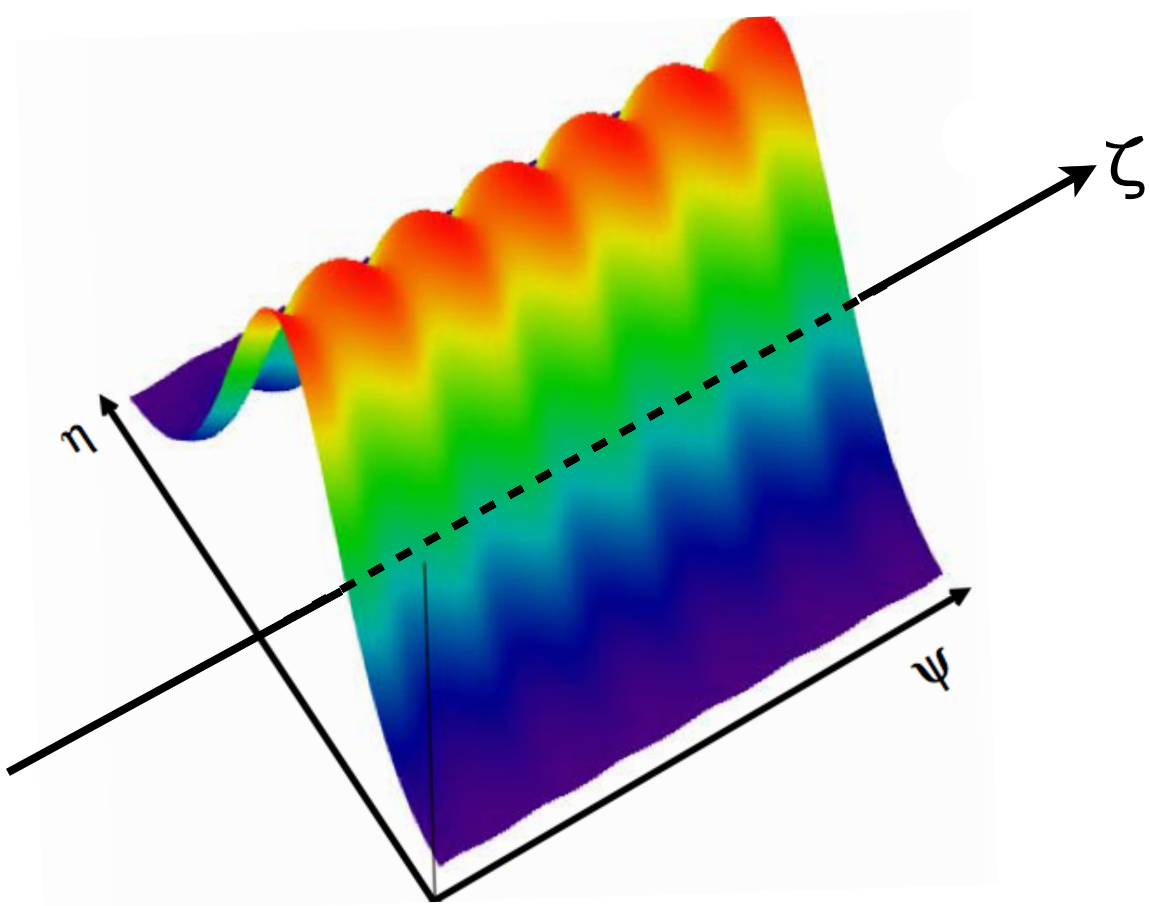
Many-body problem without analytical solution

The Third-Order Equation of the High-Gain FEL

Main physics of high-gain FEL is contained in the coupled first-order equations
Drawback: they can only be solved numerically. Goal: find differential equation containing only the electric field amplitude $\tilde{E}_x(z)$ of light wave.

For a "small" periodic density modulation the quantities ψ_n and η_n characterizing the particle dynamics in the bunch can be eliminated by defining a normalized particle distribution function

$$F(\psi, \eta, z) = \mathrm{Re}\left\{\tilde{F}(\psi, \eta, z)\right\} = F_0(\eta) + \mathrm{Re}\left\{\tilde{F}_1(\eta, z) \cdot e^{i\psi}\right\}$$



$F(\psi, \eta, z)$ obeys the Vlasov equation, a generalized continuity equation

$$\frac{dF}{dz} = \frac{\partial F}{\partial z} + \frac{\partial F}{\partial \psi}\frac{\partial \psi}{\partial z} + \frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial z} = 0$$

After many mathematical steps one finds the third-order equation

$$\boxed{\frac{\tilde{E}_x'''}{\Gamma^3} + 2i\,\frac{\eta}{\rho_{\mathrm{FEL}}}\,\frac{\tilde{E}_x''}{\Gamma^2} + \left(\frac{k_p^2}{\Gamma^2} - \left(\frac{\eta}{\rho_{\mathrm{FEL}}}\right)^2\right)\frac{\tilde{E}_x'}{\Gamma} - i\,\tilde{E}_x = 0\,.}$$

simplest form $\quad \tilde{E}_x''' - i\,\Gamma^3 \tilde{E}_x = 0$

$$\text{gain parameter} \quad \Gamma = \left[\frac{\mu_0 \widehat{K}^2 e^2 k_u n_e}{4\gamma_r^3 m_e}\right]^{1/3}$$

$$\text{space charge parameter} \quad k_p = \frac{\omega_p^*}{c}\sqrt{\frac{2\lambda_\ell}{\gamma_r \lambda_u}}\,, \qquad \omega_p^* = \sqrt{\frac{n_e e^2}{\gamma_r \varepsilon_0 m_e}}$$

$$\text{FEL parameter} \quad \rho_{\mathrm{FEL}} = \frac{\Gamma}{2k_u}$$

FEL bandwidth

Third-order differential equation is solved analytically by trial function

$$\tilde{E}_x(z) = A\exp(\alpha z)$$

Special case $\eta = 0$ and $k_p = 0$, i.e. energy on resonance and negligible space charge:

$$\alpha^3 = i\,\Gamma^3 \quad \Rightarrow \quad \alpha_1 = -i\Gamma\ , \quad \alpha_2 = (i+\sqrt{3})\Gamma/2\ , \quad \alpha_3 = (i-\sqrt{3})\Gamma/2$$

Second solution leads to exponential growth of $\tilde{E}_x(z)$. Power of light wave grows as

$$\exp(\sqrt{3}\Gamma z) \equiv \exp(z/L_{\mathrm{g0}})$$

Power gain length

$$L_{\mathrm{g0}} = \frac{1}{\sqrt{3}\Gamma} = \frac{1}{\sqrt{3}}\left[\frac{4\gamma_r^3 m_e}{\mu_0 \widehat{K}^2 e^2 k_u n_e}\right]^{1/3}$$

General analytic solution of the third-order equation

Third-order differential equation is solved by assuming a z dependence of the form $\exp(\alpha z)$. Cubic equation for exponent α has three solutions $\alpha_1, \alpha_2, \alpha_3$. Field amplitude is linear combination of the three eigenfunctions

$$\tilde{E}_x(z) = c_1 V_1(z) + c_2 V_2(z) + c_3 V_3(z) \qquad V_j(z) = \exp(\alpha_j z)$$

First and second derivative

$$\begin{aligned}
\tilde{E}'_x(z) &= c_1\alpha_1 V_1(z) + c_2\alpha_2 V_2(z) + c_3\alpha_3 V_3(z) \\
\tilde{E}''_x(z) &= c_1\alpha_1^2 V_1(z) + c_2\alpha_2^2 V_2(z) + c_3\alpha_3^2 V_3(z)
\end{aligned}$$

Since $V_j(0) = 1$ the coefficients c_j can be computed by specifying the initial conditions for $\tilde{E}_x(z)$, $\tilde{E}'_x(z)$ and $\tilde{E}''_x(z)$ at the beginning of the undulator at $z = 0$. The initial values can be expressed in matrix form by

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}'_x(0) \\ \tilde{E}''_x(0) \end{pmatrix} = \boldsymbol{\mathcal{A}} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad \text{with} \quad \boldsymbol{\mathcal{A}} = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \end{pmatrix}$$

Coefficient vector is given by

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \boldsymbol{\mathcal{A}}^{-1} \cdot \begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix}$$

Consider now the simple case $\eta = 0$ and $k_p = 0$, i.e. beam energy on resonance and negligible space charge. Then the eigenvalues are

$$\alpha_1 = -i\Gamma\,, \quad \alpha_2 = (i + \sqrt{3})\Gamma/2\,, \quad \alpha_3 = (i - \sqrt{3})\Gamma/2$$

$$\boldsymbol{\mathcal{A}}^{-1} = \frac{1}{3} \cdot \begin{pmatrix} 1 & i\,/\Gamma & -\,1\,/\Gamma^2 \\ 1 & (\sqrt{3} - i)/(2\Gamma) & (-i\,\sqrt{3} + 1)/(2\Gamma^2) \\ 1 & (-\sqrt{3} - i)/(2\Gamma) & (i\,\sqrt{3} + 1)/(2\Gamma^2) \end{pmatrix}$$

Start FEL process by an incident plane light wave of wavelength λ_ℓ and amplitude E_0

$$E_x(z,t) = E_0 \cos(k_\ell z - \omega_\ell t) \quad \text{with} \quad k_\ell = \omega_\ell / c = 2\pi/\lambda_\ell$$

Initial condition is

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x^{'}(0) \\ \tilde{E}_x^{''}(0) \end{pmatrix} = \begin{pmatrix} E_0 \\ 0 \\ 0 \end{pmatrix}$$

All three coefficients have the same value, $c_j = E_0/3$

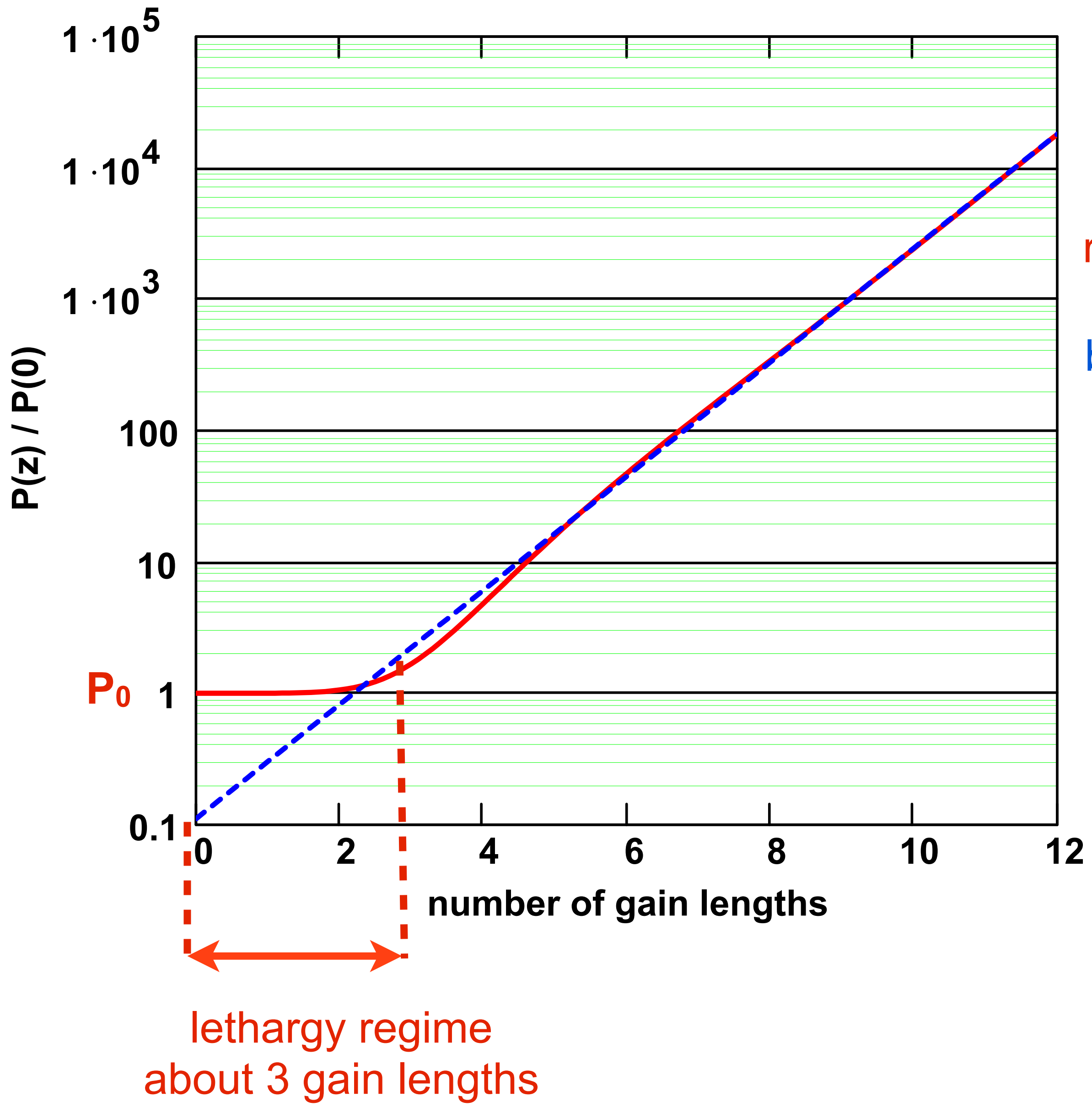
$$\Rightarrow \; \tilde{E}_x(z) = \frac{E_0}{3} \left[\exp(-i\Gamma z) + \exp((i+\sqrt{3})\Gamma z/2) + \exp((i-\sqrt{3})\Gamma z/2) \right]$$

First term oscillates along undulator axis, third term carries out a damped oscillation. Second term exhibits exponential growth and dominates at large z. FEL power grows asymptotically as

$$P(z) \cong \frac{P_0}{9} \exp(\sqrt{3}\Gamma z) = \frac{P_0}{9} \exp(z/L_{\mathrm{g0}}) \quad \text{for} \quad z \geq 3 L_{\mathrm{g0}}$$

P_0 power of incident seed light wave

FEL startup by seed laser radiation, incident power P_0



red curve: analytic solution of third-order equation

blue line: approximation $P(z) = (P_0/9)\exp(z/L_{g0})$

Applications of the High-Gain FEL Equations

FEL gain curve

Consider electron beam which is not on resonance but has still energy spread zero

$$\gamma \neq \gamma_r \quad \Rightarrow \quad \eta \neq 0 \qquad \sigma_\eta = 0$$

Lasing process seeded by incident plane wave
Gain $G(\eta, z)$ as a function of the relative energy deviation η and the position z in the undulator is

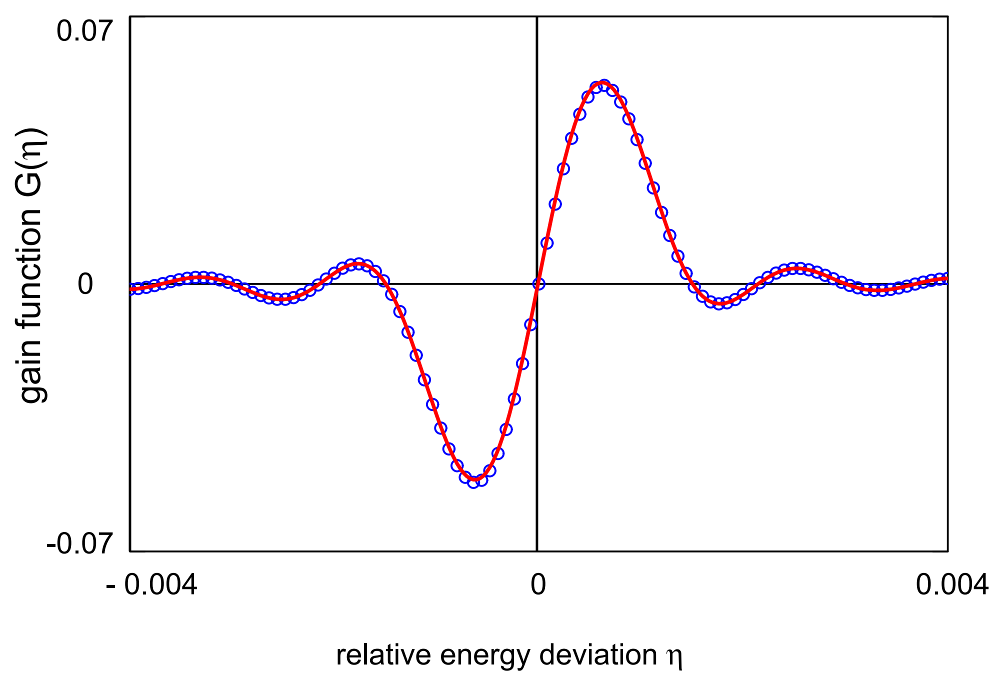
$$G(\eta, z) = \left(\frac{\tilde{E}_x(\eta, z)}{E_0} \right)^2 - 1$$

(Field $\tilde{E}_x$ inside undulator depends implicitely on η through η dependence of the eigenvalues α_j)

remember: gain function G is defined as G=gain-1

Short undulator: low-gain limit

Take undulator magnet that is shorter than one gain length $L_{und} \leq L_{\rm g0}$



Red curve: high-gain theory

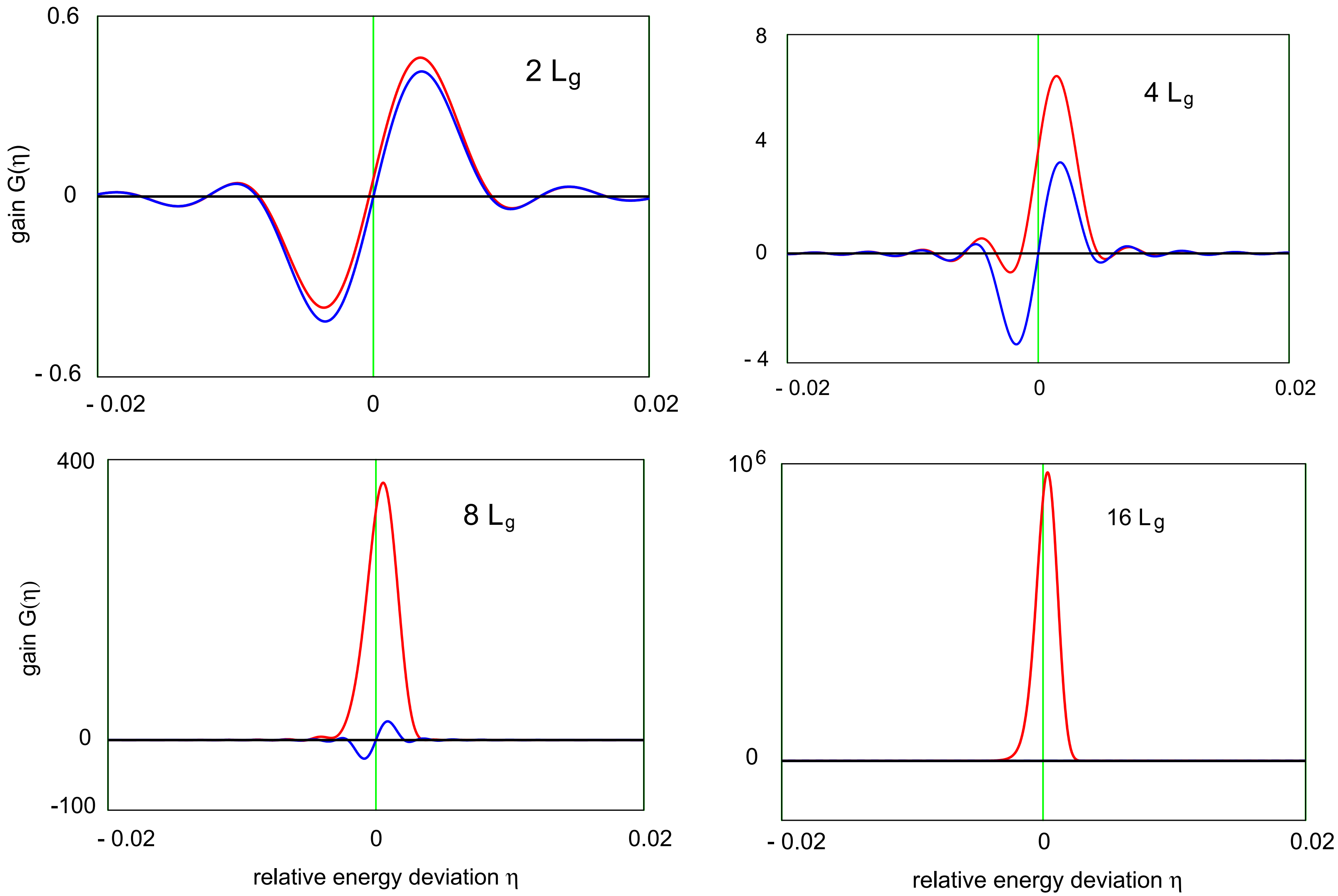
Blue circles: low-gain theory

Note: maximum gain is only 1.05 (G = gain -1= 0.05) $\Rightarrow$ low-gain regime.

The quantitative agreement proves that the low-gain FEL theory is the limiting case of the more general high-gain theory.

Long undulator: high-gain regime

Red: high-gain theory, blue: low-gain theory

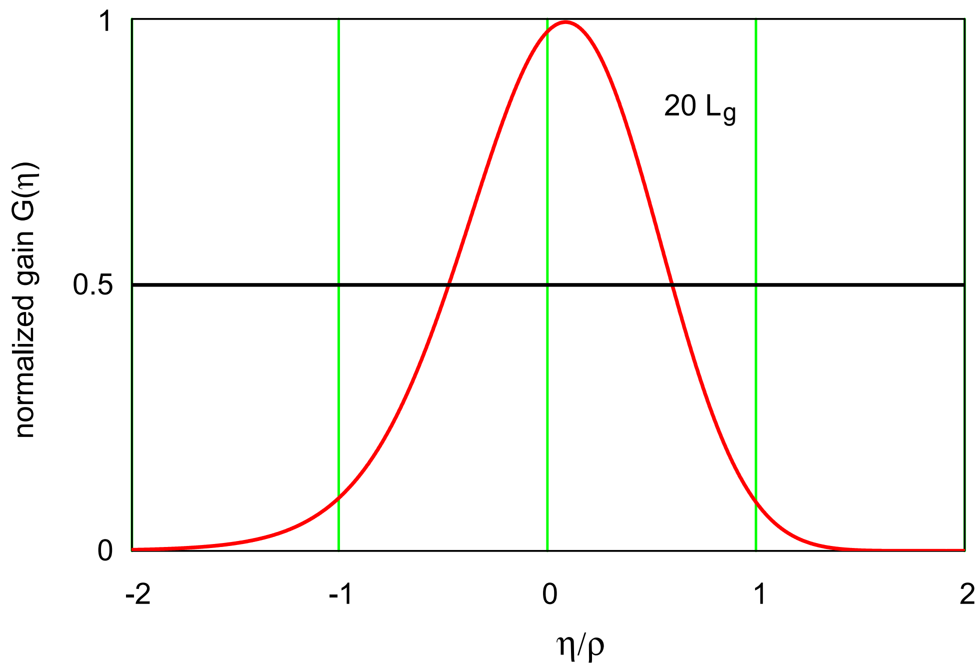


For $z \gg L_{g0}$: maximum amplification near $\eta = 0$ (on resonance)

Bandwidth of FEL

Analysis of third-order equation shows that FEL gain drops significantly when relative energy deviation exceeds the FEL ρ parameter

$$|\eta| > \rho_{\mathrm{FEL}} = \frac{1}{4\pi\sqrt{3}} \cdot \frac{\lambda_u}{L_{\mathrm{g0}}}$$



z dependent energy bandwidth

$$\Delta\eta(z) = 3\sqrt{\pi}\rho_{\mathrm{FEL}}\sqrt{\frac{L_{\mathrm{g0}}}{z}}$$

Normalized gain at $z = 20\ L_{\mathrm{g0}}$ as a function of η/ρ_{FEL}

Gain curve has a $FWHM \approx 1.0\,\rho_{FEL}$

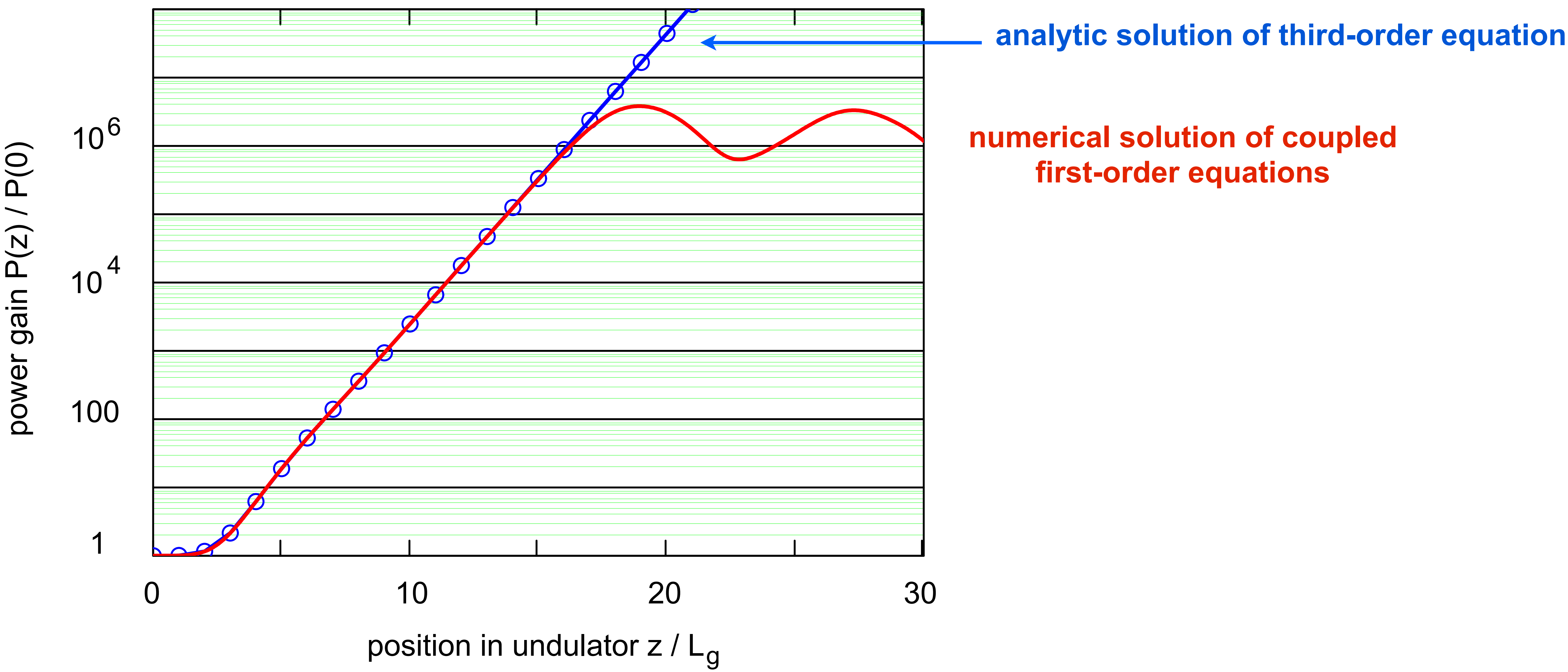
high-gain FEL acts as a narrow-band amplifier

typical bandwidth about 0.001

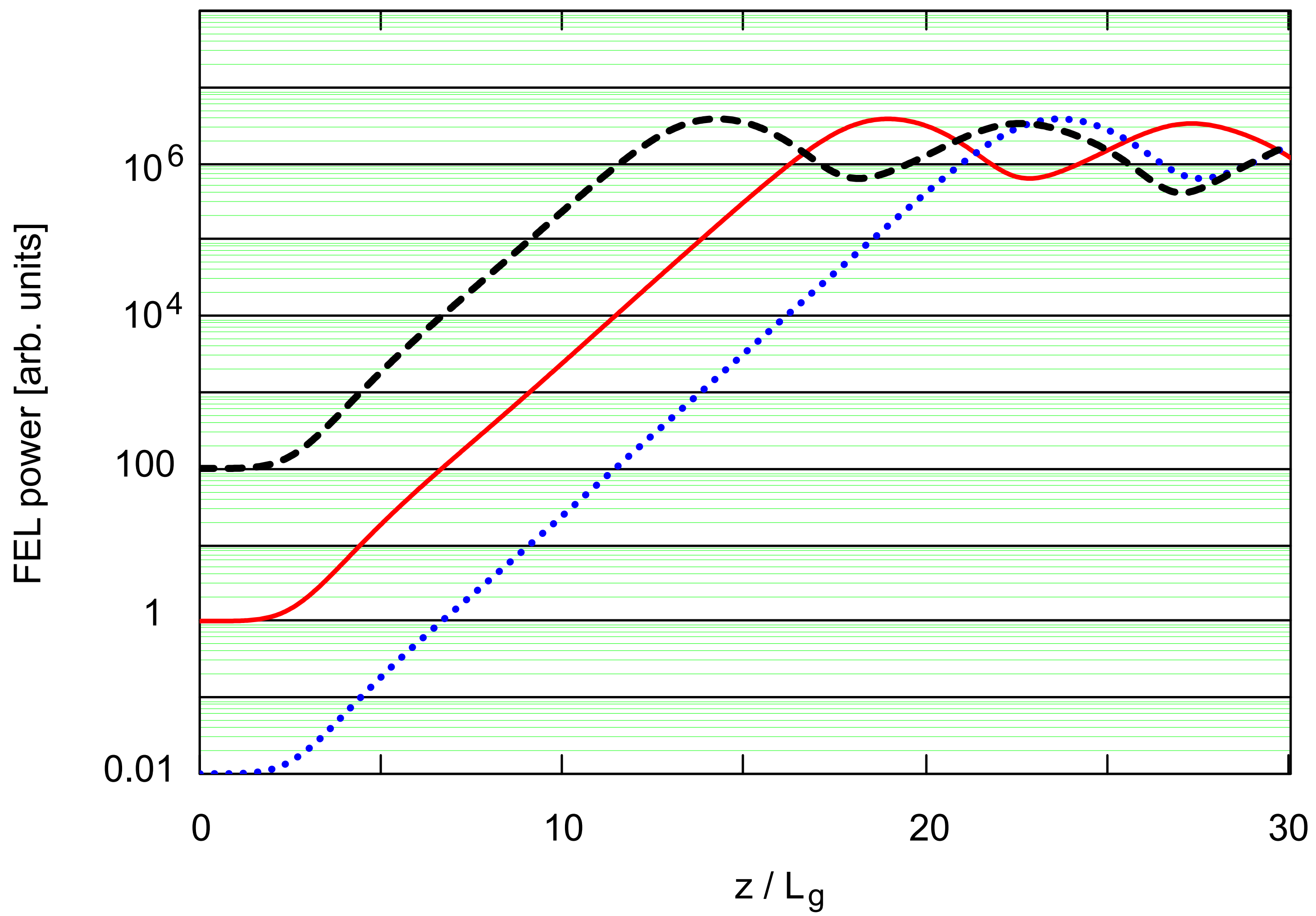
Numerical integration of the coupled FEL equations

Laser saturation

The numerical integration (by Runge-Kutta) of the coupled first-order differential equations can be used to study the regime of FEL saturation. The saturation is principally inaccessible with the analytic approach of the third-order equation which was derived under the assumption of a "small" periodic modulation of the beam current.



Comparison of different input powers of seed radiation.

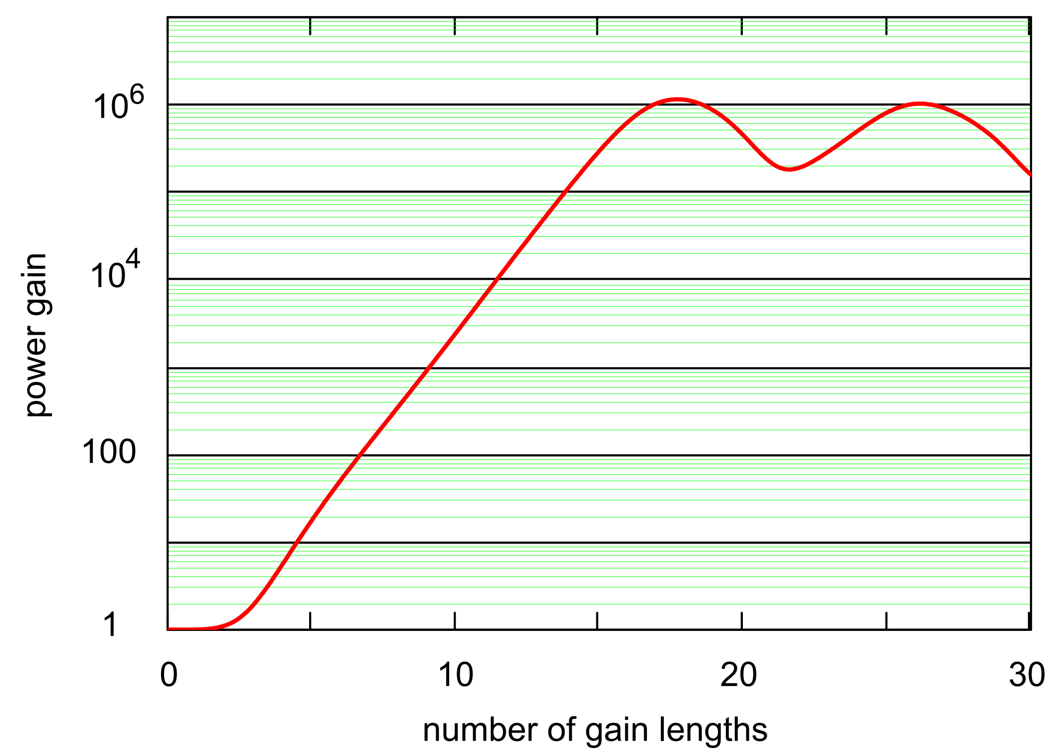


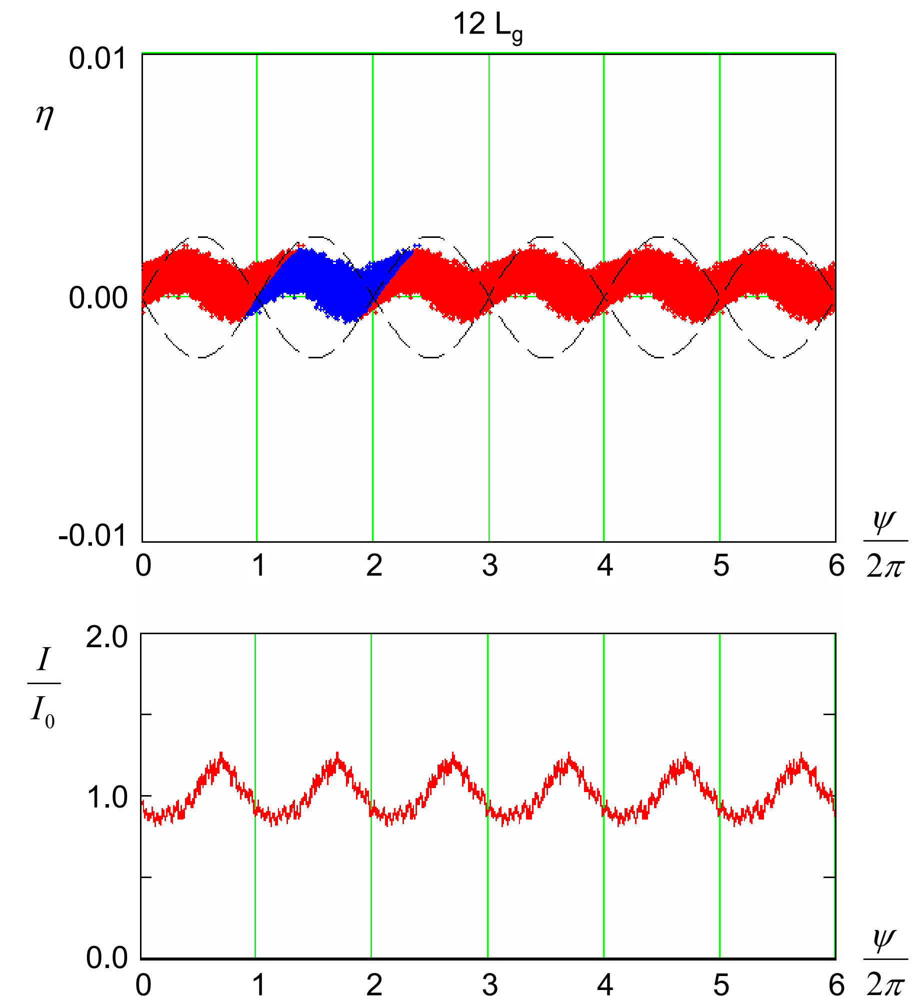
FEL power depends linearly on input power in exponential regime
However: saturation level is independent of input power

FEL power oscillates in the saturation regime $\Rightarrow$ energy is pumped back and forth between electron beam and light wave.

Simulation of microbunching

The coupled first-order differential equations permit to study microbunching
Use typical parameters of ultraviolet FEL FLASH



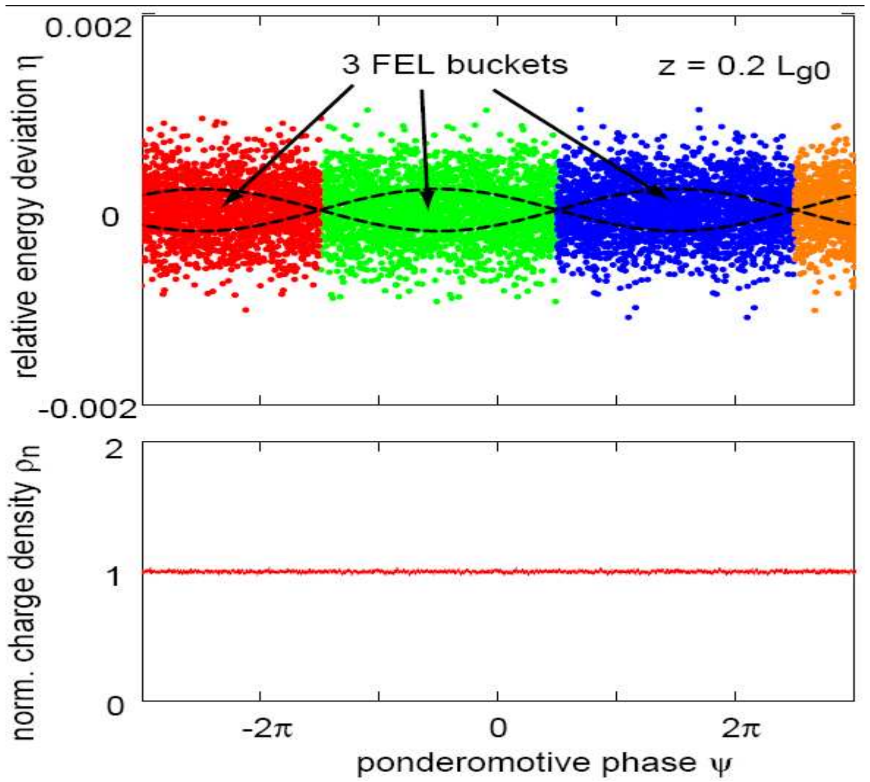


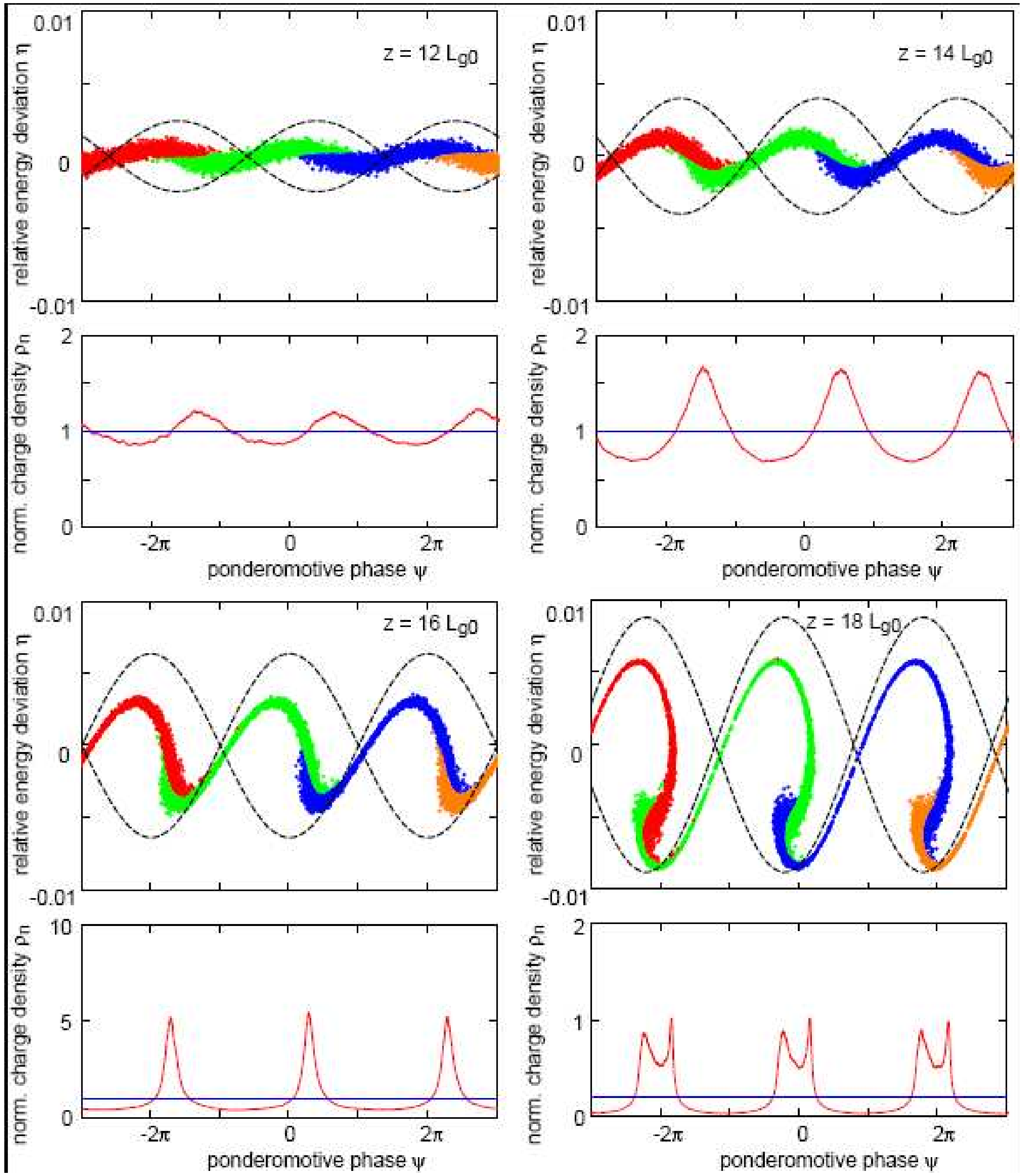
Numerical study of microbunching in a long undulator magnet

Martin Dohlus, DESY

consider 3 slices

start with uniform distribution

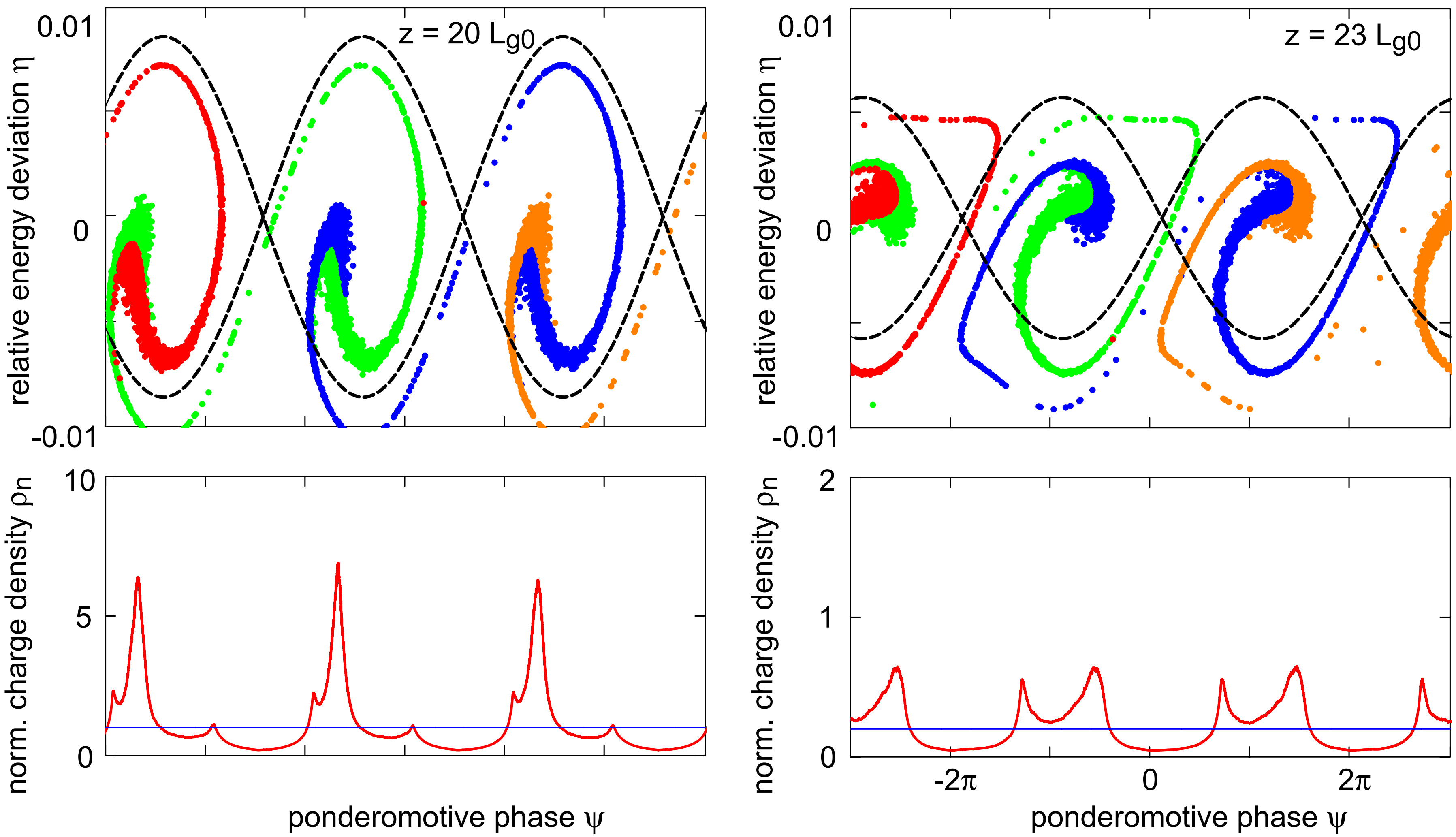




Note: the FEL buckets move, mainly in the lethargy regime

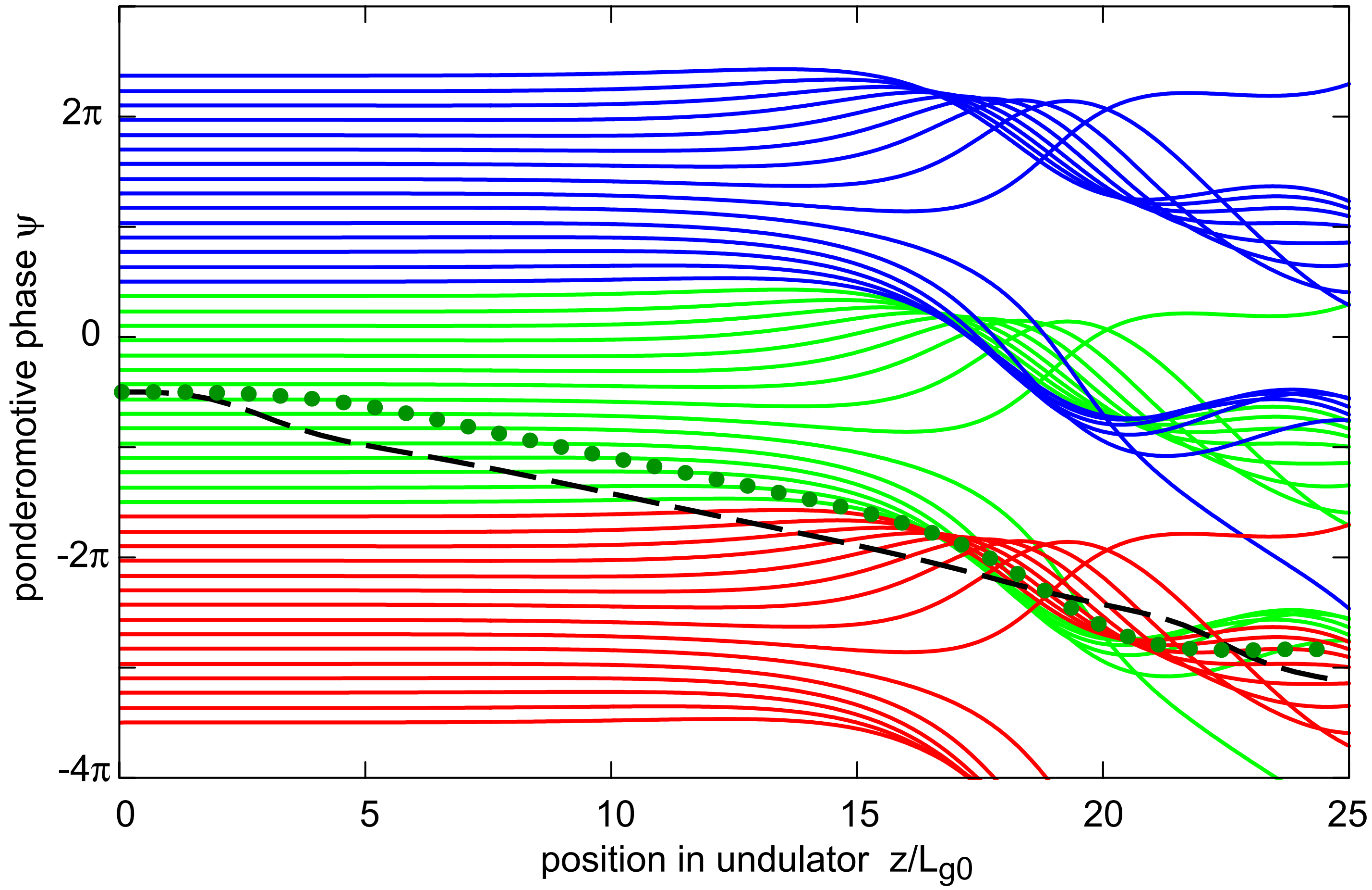
microbunches are formed in the right halves of the buckets

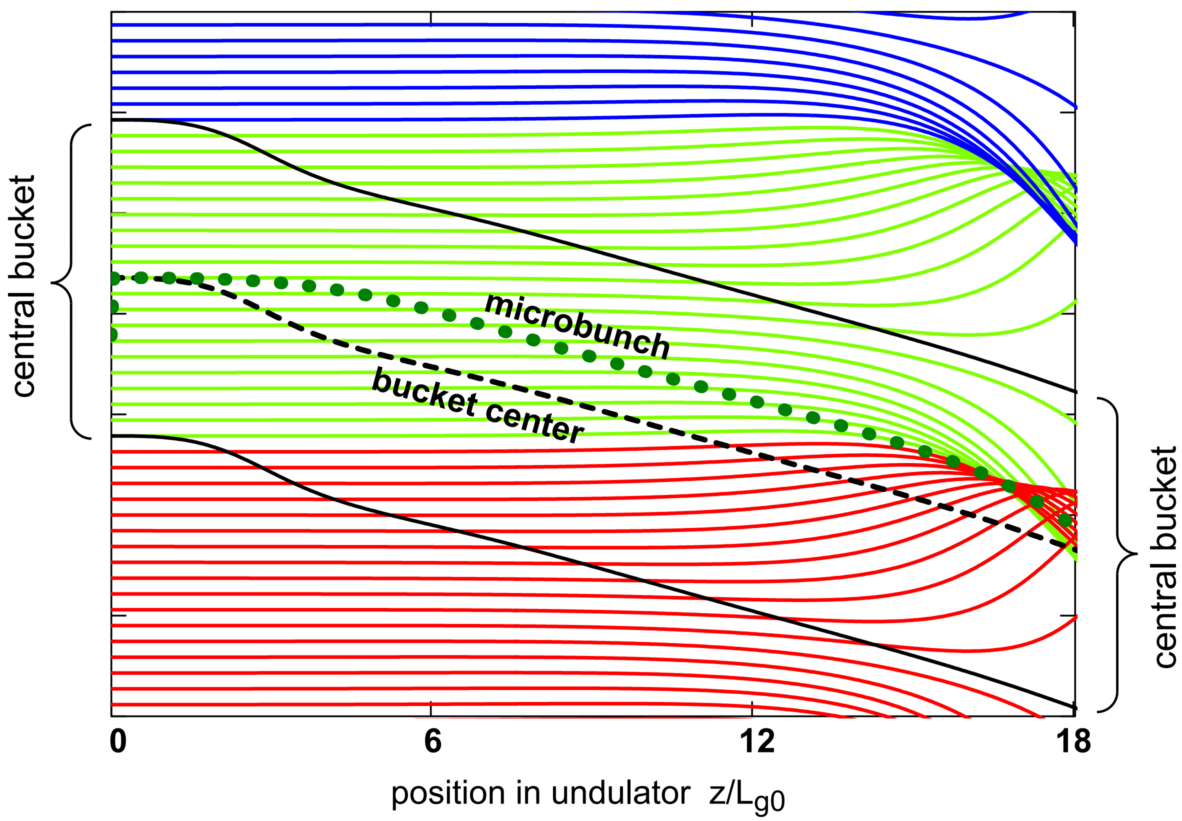
What happens if the undulator is too long?



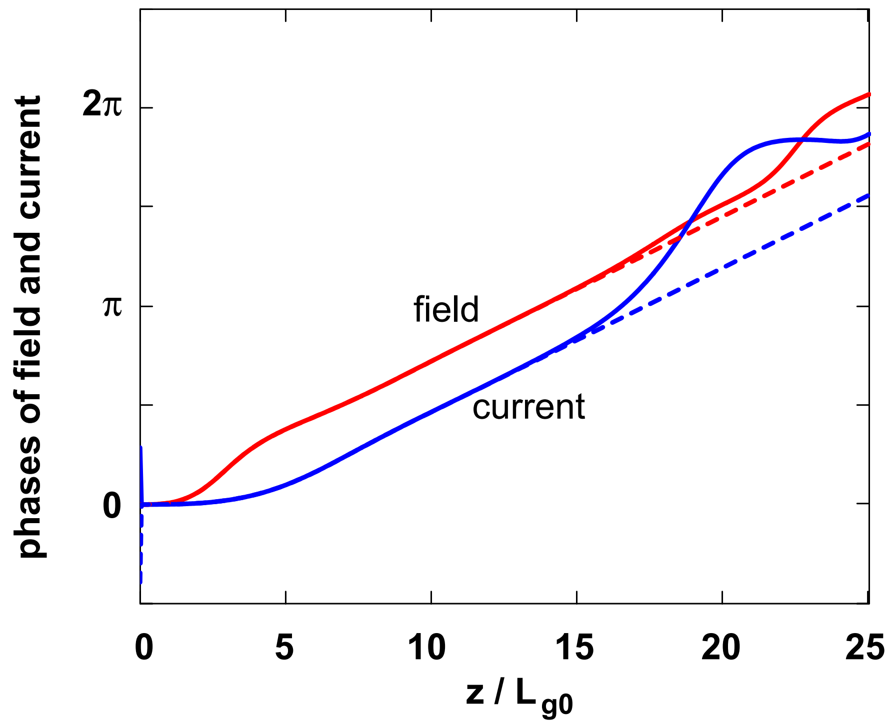
electrons move into left half of FEL buckets and take energy out of light wave

Evolution of particle phases along undulator axis





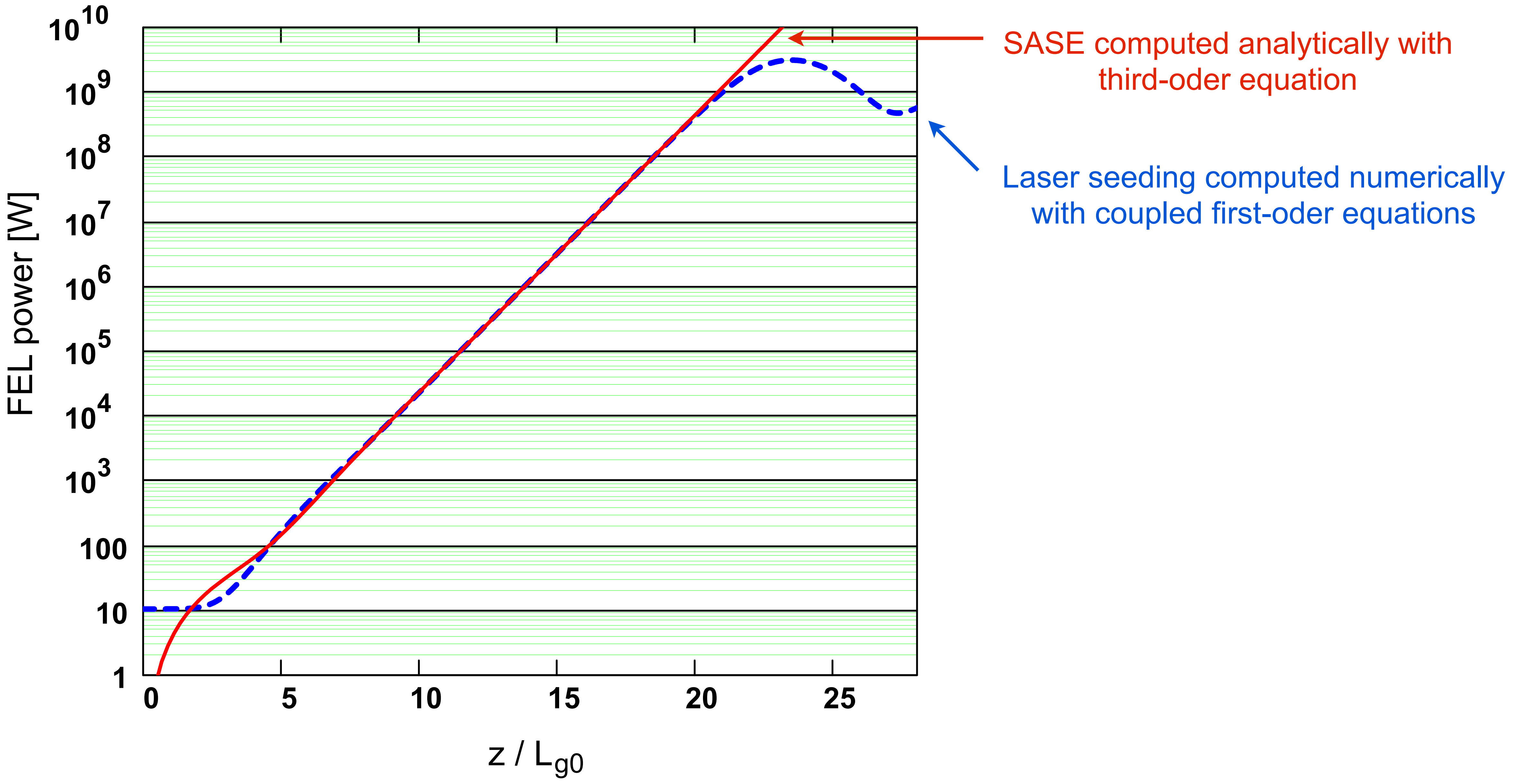
phases of field E_x and current j_1



Self Amplified Spontaneous Emission

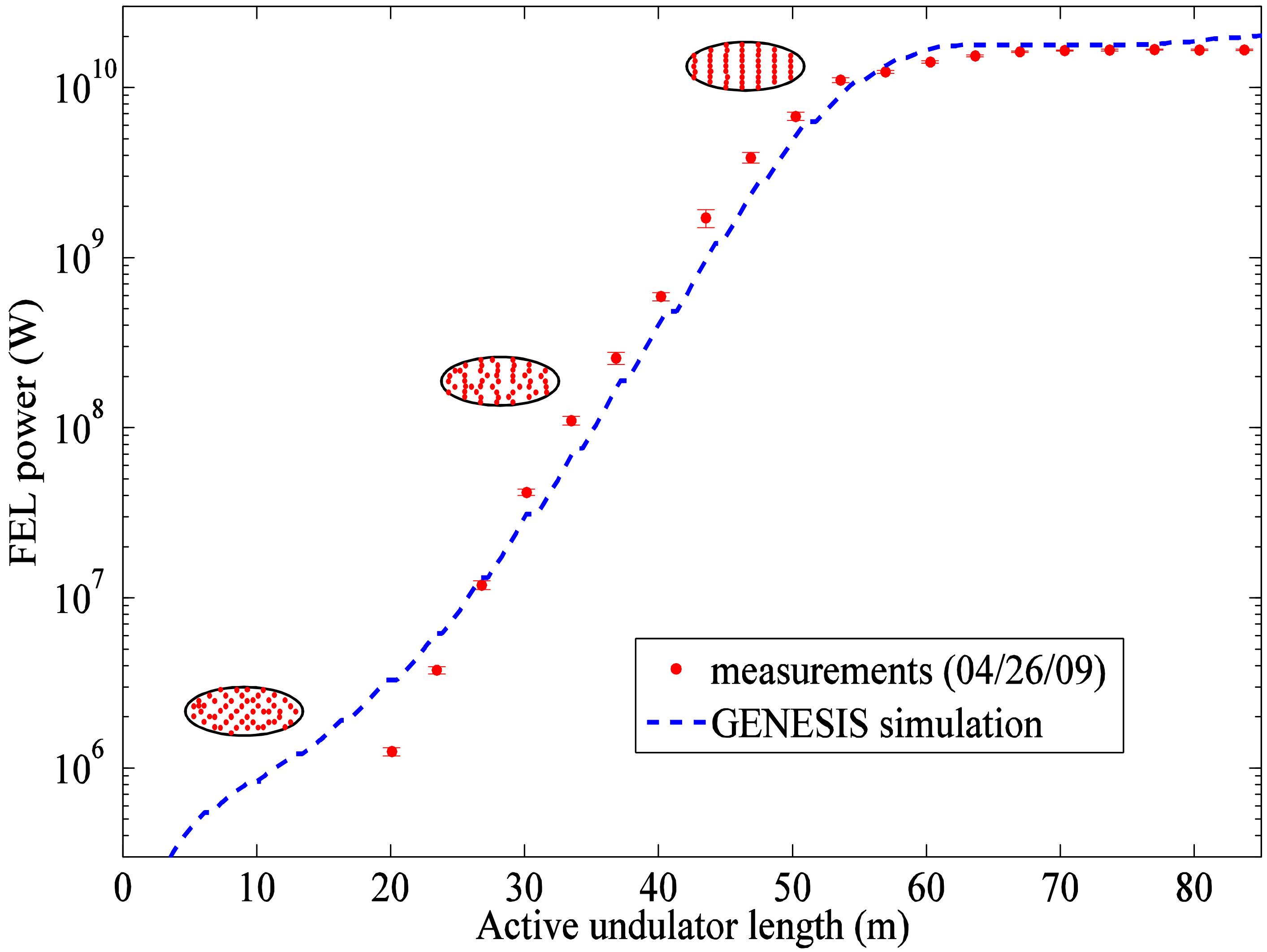
Modulated current density resulting from shot noise in electron beam

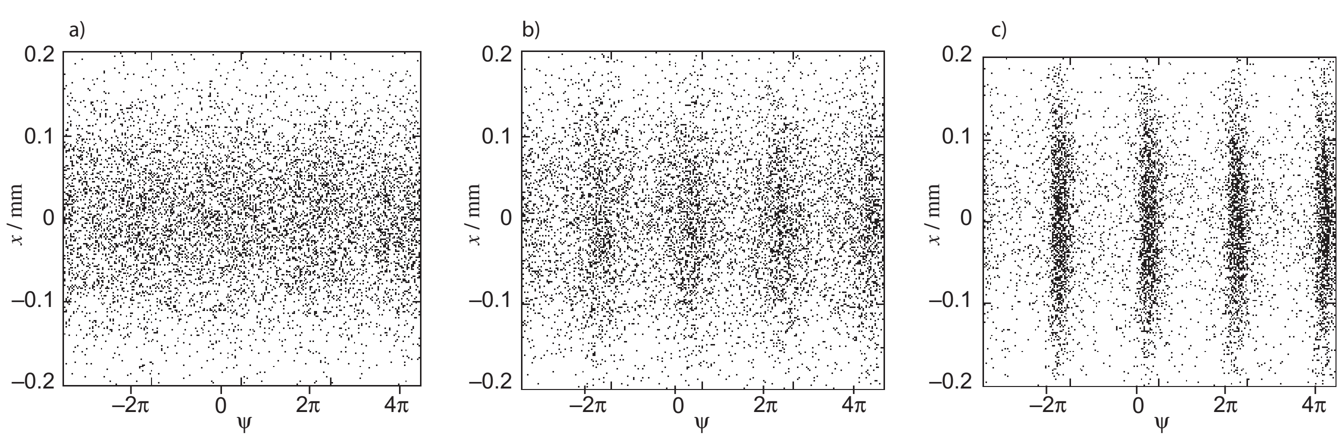
$$\tilde{j}_1 = \frac{\sqrt{2\,e\,|I_0|\,\Delta\omega}}{\sqrt{\pi}\,S_b} \qquad S_b \text{ beam cross section}$$



Measured power rise in LCLS at a wavelength of 1.5 Angström

Figure courtesy Zhirong Huang





Acknowledgements and references

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Furthermore, the following references have been extensively used:

Duke, P.J.: Synchrotron Radiation: Production and Properties. Oxford University Press, Oxford (2000)

Wiedemann, H.: Particle Accelerator Physics II, 2nd ed., Springer, Berlin (1999)

Clarke, J.A.: The Science and Technology of Undulators and Wigglers. Oxford University Press, Oxford (2004)

Wille, K.: The Physics of Particle Accelerators. An Introduction. Oxford University Press, Oxford (2001)

Murphy, J.B., Pellegrini, C.: Introduction to the physics of the free electron laser. In: Colson, W.B., Pellegrini, C., Renieri, A. (eds.) Laser Handbook vol. 6, Free Electron Lasers, p. 11. North Holland, Amsterdam (1990)

Colson, W.B.: Classical free electron laser theory. Laser Handbook vol. 6, p. 115. (1990)

Pellegrini, C., Reiche, S.: The development of X-ray free-electron lasers, IEEE J. Quantum Electron. **10**(6), 1393 (2004)

Pellegrini, C., Reiche, S.: Free-Electron Lasers. The Optics Encyclopedia, p. 1111 Wiley-VCH (2003)

Saldin, E.L., Schneidmiller, E.A., Yurkov, M.V.: The Physics of Free Electron Lasers. Springer, Berlin, Heidelberg (2000)

Huang, Z., Kim, K.-J. Review of x-ray free-electron laser theory. Phys. Rev. ST Accel. Beams **10**, 034801 (2007)