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Ion-trap analog of particle creation in cosmology

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We consider the transversal modes of ions in a linear radio-frequency trap where we control the time-dependent axial confinement to show that we can excite quanta of motion via a two-mode squeezing process. This effect is analogous to phenomena predicted to occur in the early universe, in general out of reach for experimental investigation. As a substantial advantage of this proposal in comparison to previous ones we propose to exploit the radial and axial modes simultaneously to permit experimental access of these effects based on state-of-the-art technology. In addition, we propose to create and explore entanglement between the two ions.

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I. INTRODUCTION

It is a fundamental prediction of quantum field theory that extreme conditions, such as nonadiabatic dynamics, can create pairs of particles out of the quantum vacuum. A prominent example is cosmological particle creation [1–3]. To provide an intuitive picture of such an effect, let us imagine two pendula coupled by a spring. The classical ground states with and without spring remain identical, however, the ground states of the quantum version differ in a fundamental way. Without the spring, we describe the system by a product of the individual ground states, while the two coupled pendula require entanglement of the nonseparable state (see also [4]). Now, if we remove the spring instantaneously such that the system has no time to react, we end up with two pendula which are not in their individual ground states, i.e., excited (see Fig. 1). The entanglement of the state corresponds to the correlation between the two pendula, e.g., if pendulum 1 was in the first excited state, then pendulum 2 has to match the excitation—while the total state of the system remains a pure state. This entanglement also implies that if we consider pendulum 1 only, by tracing over the degrees of freedom of pendulum 2, the effective state of pendulum 1 will be indistinguishable from a thermal (i.e., mixed) state.

In quantum field theory, this instantaneous or nonadiabatic removal of the spring is predicted to be caused by extreme circumstances, such as during the inflationary part of the expansion of the universe when wave packets get torn apart. This tearing-apart of waves is also the main mechanism responsible for Hawking radiation, i.e., black-hole evaporation [5,6]. Although Hawking radiation is created in a stationary background, following the time evolution of a wave packet at the horizon, we also see that it is torn apart—one part (pendulum) falling into the black hole and the other part escaping as Hawking radiation. In this case, the entanglement between the two "pendula" (one inside and the other outside the horizon) explains the thermal character of Hawking radiation. Here, we propose an experimentally realizable analog of this tearing-apart effect based on trapped ions. The radial modes of the two or more ions represent the two quantum pendula, while the spring is analogous to their Coulomb interaction within the axial trapping potential. We define the amplitude and the evolution in time of the latter by applying potentials to additional electrodes, controlling the axial motion of the ions and their mutual distance, respectively. Due to the unique control and accurate detection of the electronic and motional degrees of freedom, trapped ions are very good candidates for investigating these quantum effects; see also [7].

It might be illuminating to place our proposal into a broader context. Broadly speaking, it is an example of a quantum simulation or quantum simulator; see, e.g., [8–10]. More specifically, it can be regarded as belonging to the topic of analog gravity, which exploits analogies between gravitational phenomena (such as cosmological particle creation) and laboratory systems; see, e.g., [11–14]. For example, the analog of cosmological particle creation in Bose-Einstein condensates (see also [15] and [16]) has been discussed; see, e.g., [17] and [18]. In order to mention recent experimental progress, the measurement of the correlations emitted by an analog black-hole horizon in a Bose-Einstein condensate has been reported in [19]. The creation of correlated excitations in a setup which is more analogous to cosmological particle creation has been observed in [20].

In comparison to Bose-Einstein condensates, for example, trapped ions offer certain advantages. For ion traps, it is possible to detect Fock states down to the single-phonon level as well as squeezed states via quantum beating curves [21]. It is even possible to read out the motional quantum state via state tomography [22]. The obvious drawback is the limited number of ions. Further examples of the simulation of relativistic effects in ion traps can be found in [23–25].

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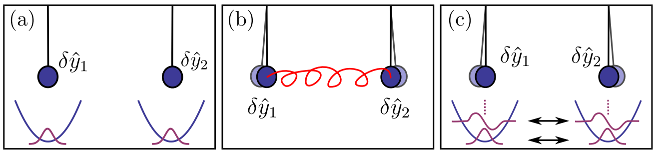

FIG. 1. Pictorial representation of the main mechanism. (a) Two uncoupled pendula have independent ground states. (b) For two pendula coupled by a string, the ground state is an entangled state. (c) After removing the coupling suddenly, the entangled state remains such that each pendulum separately is in an excited state. (b) represents the initial quantum vacuum state. The sudden removal of the spring then corresponds to the tearing-apart of waves (e.g., due to the cosmic expansion), which then leads to the creation of entangled pairs of particles.

II. EXCITATION OF PHONONS

We investigate a system of N ions of the identical species in a harmonic trapping potential characterized by a constant radial secular frequency ω_{rad}^2, provided by time-averaging the radio-frequency potential. In the axial direction, we specify the time-dependent confinement by $\omega_{\text{ax}}^2(t)$. This system is a generalization of the one-dimensional approach treated in [7], where only the motion along the axial direction has been investigated. A sketch of the setup for $N = 2$ is depicted in Fig. 2. Here we assume that the radial confinement is always stronger than the axial one, i.e., $\omega_{\text{rad}}^2 > \omega_{\text{ax}}^2(t)$. The classical equation of motion of the kth ion with coordinate $\mathbf{r}_k$ then reads

$$\ddot{\mathbf{r}}_k + \begin{pmatrix} \omega_{\text{ax}}^2(t) & 0 & 0 \\ 0 & \omega_{\text{rad}}^2 & 0 \\ 0 & 0 & \omega_{\text{rad}}^2 \end{pmatrix} \cdot \mathbf{r}_k = \gamma \sum_{l \neq k}^{N} \frac{\mathbf{r}_k - \mathbf{r}_l}{|\mathbf{r}_k - \mathbf{r}_l|^3}, \quad (1)$$

where the constant γ encodes the strength of the Coulomb repulsion between the ions. In the following, we focus on those solutions of (1) which start at $t = t_{\text{in}}$ in the equilibrium positions $\mathbf{r}_k(t_{\text{in}}) := \mathbf{r}_k^{\text{eq}} := (x_k^{\text{eq}}, 0, 0)^T$ when the trap is static initially. These solutions can be written as $\mathbf{r}_k(t) = b(t)\mathbf{r}_k^{\text{eq}}$,

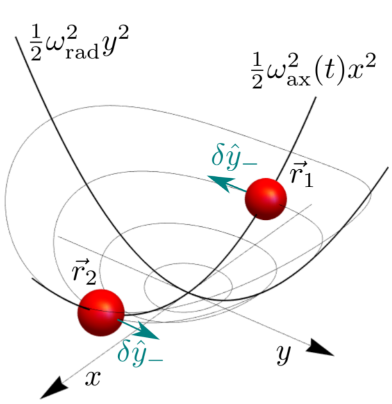

FIG. 2. Two ions with coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ in the potential energy landscape of a nonisotropic harmonic trap with time-dependent axial trap frequency $\omega_{\text{ax}}(t)$ and radial frequency ω_{rad}. The z direction is not shown. Classical motion along the axial x direction can amplify quantum fluctuations in the rocking mode $\delta\hat{y}_-$ along the y direction.

where the scale parameter $b(t)$ fulfills

$$\ddot{b}(t) + \omega_{\text{ax}}^2(t)b(t) = \frac{\left(\omega_{\text{ax}}^{\text{in}}\right)^2}{b^2(t)}, \quad (2)$$

with $\omega_{\text{ax}}^{\text{in}} = \omega_{\text{ax}}(t_{\text{in}})$. The boundary conditions are $b(t_{\text{in}}) = 1$ and $\dot{b}(t_{\text{in}}) = 0$. This means that the classical solution is fully determined as a time-dependent rescaling of the initial equilibrium positions (see, e.g., [26]).

However, the ions are quantum particles described by a wave function of a certain width, individual measurements of their positions have to deviate from and fluctuate around their classically predictable positions, revealing quantum fluctuations. Their position operator can be written as $\hat{\mathbf{r}}_k(t) = b(t)\mathbf{r}_k^{\text{eq}} + \delta\hat{\mathbf{r}}_k$, and in a semiclassical approximation, we assume that the deviations $\delta\hat{\mathbf{r}}_k$ remain small (because the mass of the ions is large, corresponding to a narrow width of their ground-state wave function).

Linearization and diagonalization of (1) then yields the Heisenberg equation of motion for the normal modes (phonons). While the axial phonons have been discussed in [7] we focus here on the radial phonons satisfying

$$\left(\frac{d^2}{dt^2} + \Omega_\kappa^2(t)\right)\delta\hat{y}_\kappa = 0. \quad (3)$$

Every radial normal mode $\delta\hat{y}_\kappa$ can be associated with one individual harmonic oscillator with time-dependent normal-mode frequency

$$\Omega_\kappa^2(t) = \omega_{\text{rad}}^2 - \frac{\omega_\kappa^2}{b^3(t)}, \quad (4)$$

where $\omega_\kappa^2 \geqslant 0$ is the κth eigenvalue of the matrix

$$M_{kl} = \delta_{kl} \sum_{j \neq k}^{N} \frac{\gamma}{\left|x_k^{\text{eq}} - x_j^{\text{eq}}\right|^3} - \frac{\gamma(1 - \delta_{kl})}{\left|x_k^{\text{eq}} - x_l^{\text{eq}}\right|^3}. \quad (5)$$

Especially, for the center-of-mass mode we have $\omega_0 = 0$ and for the rocking mode $\omega_1 = \omega_{\text{ax}}^{\text{in}}$.

In the following we show how the time dependence of the normal-mode frequencies $\Omega_\kappa(t)$ can lead to the excitation of phonons. At the initial instant t_{in} we express the position operator of each normal mode in terms of the harmonic oscillator ladder operators as ($\hbar = 1$)

$$\delta\hat{y}_\kappa(t_{\text{in}}) = \frac{1}{\sqrt{2\Omega_\kappa(t_{\text{in}})}}\hat{a}_\kappa^{\text{in}} + \text{H.c.} \quad (6)$$

For another given instant $t_{\text{out}} > t_{\text{in}}$, the operator evolves under the Heisenberg equation, (3), into

$$\delta\hat{y}_\kappa(t_{\text{out}}) = \frac{1}{\sqrt{2\Omega_\kappa(t_{\text{out}})}}\hat{a}_\kappa^{\text{out}} + \text{H.c.}, \quad (7)$$

where the final creation and annihilation operators $\hat{a}_\kappa^{\dagger\text{out}}$ and $\hat{a}_\kappa^{\text{out}}$ are linked to the initial ones via the Bogoliubov transformation

$$\hat{a}_\kappa^{\text{out}} = \alpha_\kappa^*\hat{a}_\kappa^{\text{in}} - \beta_\kappa^*\hat{a}_\kappa^{\dagger\text{in}}, \quad (8)$$

with the (complex) Bogoliubov coefficients α_κ and β_κ. For the initial ground state $|\Psi(t_{\text{in}})\rangle = |0\rangle$ in the κth radial mode, the mean number of created phonons is given by

$$\langle\hat{n}_\kappa^{\text{out}}\rangle = \langle\Psi(t_{\text{in}})|\hat{a}_\kappa^{\dagger\,\text{out}}\hat{a}_\kappa^{\text{out}}|\Psi(t_{\text{in}})\rangle = |\beta_\kappa|^2. \quad (9)$$

Hence, phonon creation takes place depending on the temporal evolution of $\Omega_\kappa(t)$ from $t_{\rm in}$ to $t_{\rm out}$, if $|\beta_\kappa| > 0$. Or in other words: The classical motion along the x axis induces the creation of phonons in the radial direction.

The generators of the Bogoliubov transformation, (8), are squeezing operators. Therefore the corresponding evolution of the initial ground state $|\Psi(t_{\rm in})\rangle$ is given by

$$|\Psi(t_{\rm out})\rangle = \hat{\mathcal{S}}_{\xi_\kappa}|0\rangle = \exp\left\{\frac{1}{2}\sum_\kappa \xi_\kappa \left(\hat{a}_\kappa^{\dagger \rm in}\right)^2 - \text{H.c.}\right\}|0\rangle$$
$$= |0\rangle + \frac{1}{\sqrt{2}}\sum_\kappa \xi_\kappa|2_\kappa\rangle + O\left(\xi_\kappa^2\right), \quad (10)$$

where the squeezing parameter ξ_κ is linked to the Bogoliubov coefficients via $|\beta_\kappa| = \sinh|\xi_\kappa|$ and $\arg\xi_\kappa = -(\arg\alpha_\kappa + \arg\beta_\kappa)$. Formula (10) features the characteristics of a squeezing operation, the creation of particles (here phonons) in pairs.

III. EXCITATION MODELS FOR TWO IONS

In the following we focus on the case of $N = 2$ ions (see Fig. 2) and investigate the phonon creation induced by different axial motions of the ions: first, by a collision between the ions described by a scale function $b_{\rm col}(t)$ and, second, by an expansion of the ions corresponding to a scale function $b_{\rm ex}(t)$. The time dependence of the axial confinement necessary to generate a given scale function $b(t)$ can be deduced from (2) to

$$\omega_{\rm ax}(t) = \sqrt{\frac{\left(\omega_{\rm ax}^{\rm in}\right)^2}{b^3(t)} - \frac{\ddot{b}(t)}{b(t)}}. \quad (11)$$

We focus here on trajectories where $\omega_{\rm ax}(t) \in \mathbb{R}$. However, there exist also trajectories $b(t)$ that can only be realized for temporarily negative $\omega_{\rm ax}^2$, which means for temporarily repulsive trapping potentials.

The scale function is linked to the (classical) mutual distance of the ions via

$$\Delta x(t) = x_2(t) - x_1(t) = b(t)\Delta x^{\rm eq}. \quad (12)$$

In the radial direction we have the two phonon modes

$$\delta\hat{y}_\pm = \frac{1}{\sqrt{2}}(\delta\hat{y}_1 \pm \delta\hat{y}_2). \quad (13)$$

This is the center-of-mass mode $\delta\hat{y}_+$ with frequency $\Omega_+^2 = \omega_{\rm rad}^2$ and the rocking mode $\delta\hat{y}_-$ with frequency $\Omega_-^2(t) = \omega_{\rm rad}^2 - \left(\omega_{\rm ax}^{\rm in}\right)^2/b(t)^3$. With (3) the equation of motion for the rocking-mode phonons is

$$\left(\frac{d^2}{dt^2} + \Omega_-^2(t)\right)\delta\hat{y}_- = 0. \quad (14)$$

A. Collision model

We consider now a special scale function

$$b_{\rm col}(t) = \left(1 + \frac{\Delta\Omega_{\rm col}^2}{\left(\omega_{\rm ax}^{\rm in}\right)^2}\frac{1}{\cosh^2(\omega_{\rm col}t)}\right)^{-\frac{1}{3}} \quad (15)$$

that parametrizes a collision between the ions. Starting at $t_{\rm in} \to -\infty$ in the equilibrium position with $b_{\rm col}(t_{\rm in}) = 1$, the ions approach each other, reach for $t = 0$ a minimal axial distance $\Delta x_{\rm min}$ at the turning point, and, finally, return to their initial positions for $t_{\rm out} \to +\infty$ with $b(t_{\rm out}) = 1$. The parameter $\Delta\Omega_{\rm col}^2$ describes the change in the rocking-mode frequency Ω_-^2 from $t_{\rm in}$ to $t = 0$ and determines the minimal distance of the ions. The parameter $\omega_{\rm col}$ determines the characteristic time scale of the collision.

This process is analogous to the sequence in Figs. 1(a)–1(c). Initially the ions can be far apart, such that there is no Coulomb coupling and no entanglement [Fig. 1(a)]. Around $t = 0$ the ions come very close and interact strongly [Fig. 1(b)]. Figuratively, this collision corresponds to an installation and a removal of the spring. If this happens sufficiently rapidly, the ions remain entangled even at large separations [Fig. 1(c)]. Using $b_{\rm col}(t)$ from Eq. (15) the differential Eq. (14) can be solved in terms of hypergeometric functions whose asymptotic behavior is known for $t \to \pm\infty$. As shown in Appendix A this yields the Bogoliubov coefficient,

$$|\beta_-^{\rm col}|^2 = \left|\frac{\cosh\left(\frac{\pi}{2}\sqrt{\frac{4\Delta\Omega_{\rm col}^2}{\omega_{\rm col}^2} - 1}\right)}{\sinh\left(\frac{\pi\Omega_{\rm in}}{\omega_{\rm col}}\right)}\right|^2, \quad (16)$$

where $\Omega_{\rm in} = \Omega_-(t_{\rm in})$. Here we focus on a regime of moderate and slow collisions, where $\omega_{\rm col} \ll \Delta\Omega_{\rm col} < \Omega_{\rm in}$. Especially, this implies that the system never reaches critical points with $\Omega_- = 0$, where the classical radial motion becomes unstable and the linear chain features a phase transition into a two-dimensional zigzag structure [27]. Under these assumptions, Eq. (16) can be approximated as

$$|\beta_-^{\rm col}|^2 \approx \exp\left[-2\pi\frac{(\Omega_{\rm in} - \Delta\Omega_{\rm col})}{\omega_{\rm col}}\right]. \quad (17)$$

That means that particle creation becomes important only if $\Omega_{\rm in} - \Delta\Omega_{\rm col}$ is chosen sufficiently small,

$$\Omega_{\rm in} - \Delta\Omega_{\rm col} = O(\omega_{\rm col}), \quad (18)$$

while it is exponentially suppressed for $\Omega_{\rm in} - \Delta\Omega_{\rm col} \gg \omega_{\rm col}$. In fact, this statement is valid for generic scale functions, as long as the collision fulfills the given assumptions.

Assuming that $\Omega_-^2(t)$ is sufficiently slowly varying (such that a WKB approximation can be applied) and can be approximated by a parabola near its minimum (i.e., the turning point at $t = 0$), we find that the mean number of phonons is mainly dominated by the relation between two parameters: the normal-mode frequency $\Omega_-^2(0)$ and its curvature $\frac{d^2}{dt^2}\Omega_-^2(0)$, both evaluated at the turning point $t = 0$; see Appendix B. As $\Omega_-^2(t)$ is linked to the ion trajectories via (4) and (1) it is equivalent to state that the mean number of phonons after the collision is mainly dominated by the two parameters

$$p_1 := \left(\frac{\Delta x^{\rm eq}}{\Delta x^{\rm min}}\right)^3 \quad (19)$$

and

$$p_2 := \left(\frac{\omega_{\rm ax}(t=0)}{\omega_{\rm ax}^{\rm in}}\right)^2, \quad (20)$$

where $\omega_{\rm ax}(t = 0)$ describes the axial confinement at the instance when the ions reach their turning point. For example,

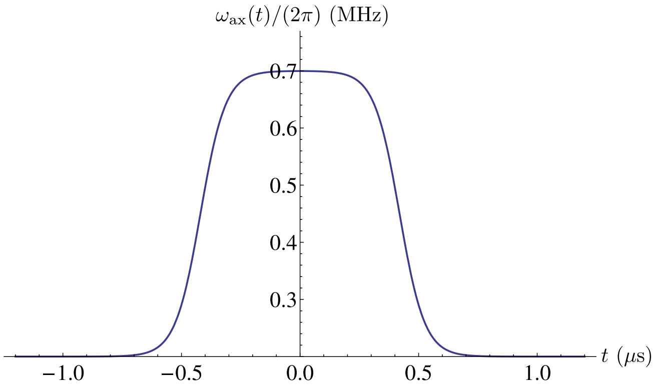

FIG. 3. Example of a time-dependent axial confinement characterized by $\omega_{\rm ax}(t)$ that leads to the classical motion illustrated in Fig. 4.

a model collision with trajectory $b_{\rm col}(t)$ defined in Eq. (15), yielding the two values p_1 and p_2, is obtained by choosing

$$\Delta\Omega^2_{\rm col}(p_1) = \left(\omega^{\rm in}_{\rm ax}\right)^2(p_1 - 1) \tag{21}$$

and

$$\omega^2_{\rm col}(p_1, p_2) = \left(\omega^{\rm in}_{\rm ax}\right)^2 \frac{3p_1(p_1 - p_2)}{2(p_1 - 1)}. \tag{22}$$

We can take advantage of this to obtain approximations for the Bogoliubov coefficients β_- of moderate and slow collisions with trajectories qualitatively similar to (15). Such a collision with given parameters p_1 and p_2 will lead to similar phonon excitations as model collisions, (15), having identical parameters. Therefore the Bogoliubov coefficient $|\beta_-|^2$ can be approximated as

$$|\beta_-|^2 \approx |\beta^{\rm col}_-(p_1, p_2)|^2, \tag{23}$$

where $\beta^{\rm col}_-(p_1, p_2)$ denotes $\beta^{\rm col}_-$ from (16) with the substitutions (21) and (22).

Let us exploit these results to propose a realistic implementation: a collision of two $^{25}{\rm Mg}^+$ ions trapped in a radial potential with frequency $\omega_{\rm rad} = 2\pi \times 3.5$ MHz and an initial axial potential with frequency $\omega^{\rm in}_{\rm ax} = 2\pi \times 0.2$ MHz. The initial equilibrium distance is $\Delta x^{\rm eq} \approx 19.1$ μm. As an example we consider the axial confinement presented in Fig. 3, where we increase $\omega_{\rm ax}(t)$ in approximately 0.5 μs from $\omega^{\rm in}_{\rm ax}$ to $\omega^{\rm max}_{\rm ax} = 2\pi \times 0.7$ MHz, keep it constant for around 0.5 μs and return, finally, to $\omega^{\rm in}_{\rm ax}$. In Fig. 4 the resulting ion trajectory is illustrated. The exact Bogoliubov coefficient can be evaluated either numerically to $|\beta_-|^2 \approx 0.18$ or approximately, based on (23), to $|\beta_-|^2 \approx 0.2$. A more extensive comparison between approximation (23) and the exact numerical results can be carried out by calculating the Bogoliubov coefficients for different final confinements $\omega^{\rm max}_{\rm ax}$ by both methods. The result is illustrated in Fig. 5 and indicates that we achieve a good agreement over several orders of magnitude.

Furthermore, this realistic result permits us to predict that the mean phonon numbers created in the radial mode can be five times larger than the residual thermal excitation of $n_{\rm th} \approx 0.05$, achievable by current cooling techniques. In addition, the characteristic phonon distribution of the squeezed state

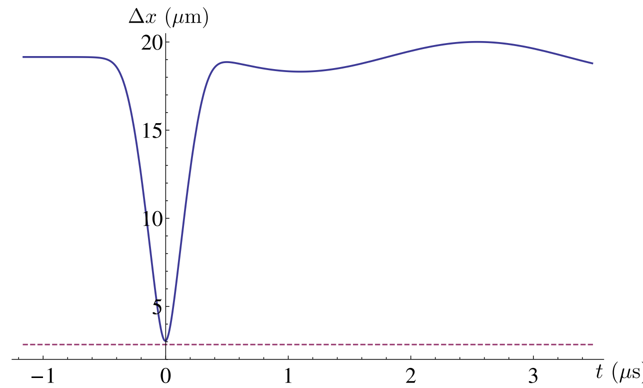

FIG. 4. Numerically calculated trajectory Δx for two ions confined in the axial potential with frequency $\omega_{\rm ax}(t)$ presented in Fig. 3. After the collision the ions oscillate relative to their initial equilibrium positions. The dashed red line shows the critical distance at which $\Omega_- = 0$, where the ion chain becomes instable.

allows us to clearly distinguish the pairwise created phonons from the thermal background. Therefore we conclude that effects analogous to cosmological particle creation should be observable in already state-of-the-art ion traps [28,29].

So far we have only treated collisions of two ions. However, collisions of the form of (15) permit also exact analytical expressions for the Bogoliubov coefficients of higher normal modes. In the limit of slow and moderate collisions they can be approximated by

$$|\beta_\kappa|^2 \propto \exp\left[-2\pi \frac{\sqrt{\omega^2_{\rm rad} - \omega^2_\kappa - \Delta\Omega_{\rm col}\frac{\omega_\kappa}{\omega^{\rm in}_{\rm ax}}}}{\omega_{\rm col}}\right]. \tag{24}$$

Consequently, particle creation in the κth normal mode becomes important only if

$$\left(\sqrt{\omega^2_{\rm rad} - \omega^2_\kappa - \Delta\Omega_{\rm col}\frac{\omega_\kappa}{\omega^{\rm in}_{\rm ax}}}\right) = O(\omega_{\rm col}). \tag{25}$$

For an increasing $\Delta\Omega_{\rm col}$, considerable creation of pairs of phonons occurs therefore first in the mode with the highest ω_κ. For large N in a linear chain of ions, this mode is called the zigzag mode.

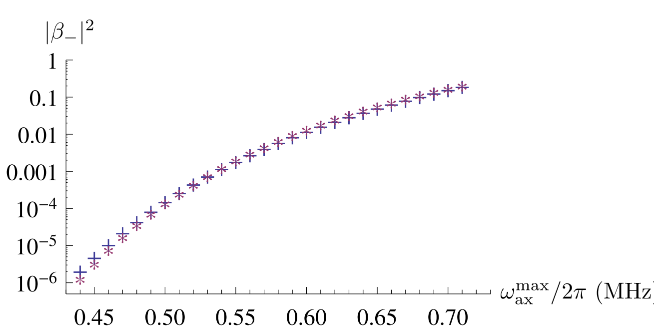

FIG. 5. Bogoliubov coefficient β_- obtained (crosses) numerically or (asterisks) based on Eq. (23) for collisions induced by an axial potential as presented in Fig. 3, where the peak confinement $\omega^{\rm max}_{\rm ax}$ is varied.

B. Expansion model

Another type of axial motion, which corresponds to an expansion of the mutual distance of the ions, is described by the scale function

$$b_{\rm ex}(t) = \left(1 - \frac{\Delta\Omega_{\rm ex}^2}{2(\omega_{\rm ax}^{\rm in})^2}(\tanh(\omega_{\rm ex}t) + 1)\right)^{-\frac{1}{3}}. \tag{26}$$

The parameter $\Delta\Omega_{\rm ex}^2$ describes the induced jump in the normal-mode frequency $\Omega_-^2(t)$, whereas $\omega_{\rm ex}$ determines how fast the expansion evolves. Inserting $b_{\rm ex}(t)$ into (14) yields a differential equation that is discussed in [3] as an example of cosmological particle creation. It can be solved in terms of hypergeometric functions whose asymptotic behavior is known for $t \to \pm\infty$. The Bogoliubov coefficient reads

$$|\beta_-^{\rm ex}|^2 = \frac{\sinh^2\left(\frac{\pi}{2}\frac{\Omega_{\rm out}-\Omega_{\rm in}}{\omega_{\rm ex}}\right)}{\sinh\left(\pi\frac{\Omega_{\rm in}}{\omega_{\rm ex}}\right)\sinh\left(\pi\frac{\Omega_{\rm out}}{\omega_{\rm ex}}\right)}, \tag{27}$$

where

$$\Omega_{\rm out} = \sqrt{\Omega_{\rm in}^2 + \Delta\Omega_{\rm ex}^2}. \tag{28}$$

For example, for very large $\omega_{\rm ex}$, this means that for a sudden quench, the Bogoliubov coefficient can be approximated to

$$\left|\beta_{\rm ex}^2\right| \approx \frac{(\Omega_{\rm out} - \Omega_{\rm in})^2}{4\Omega_{\rm in}\Omega_{\rm out}}. \tag{29}$$

However, in the case of moderate and slow expansions, i.e., $\omega_{\rm ex} \ll \Omega_{\rm in}$ the Bogoliubov coefficients become

$$|\beta_{\rm ex}|^2 \propto e^{-2\pi\Omega_{\rm in}/\omega_{\rm ex}}. \tag{30}$$

IV. ION-ION ENTANGLEMENT

After having discussed the excitation process of pairs of phonons in the last section, we now analyze the conditions for reaching entanglement between the ions and how robust this entanglement is against thermal disturbances. We discuss exclusively the case of $N = 2$ ions.

We consider a system that is initially in a thermal state with sufficiently separated ions to consider them initially uncoupled, i.e., $\Omega_-(t_{\rm in}) = \omega_{\rm rad}$. In this case the operators $\hat{\delta}y_\pm$ and $\hat{\delta}y_{1/2}$ form two equivalent sets of normal modes and their corresponding initial creation and annihilation operators are linked via

$$\hat{a}_+^{\rm in} = \frac{1}{\sqrt{2}}\left(\hat{a}_1^{\rm in} + \hat{a}_2^{\rm in}\right) \tag{31}$$

and

$$\hat{a}_-^{\rm in} = \frac{1}{\sqrt{2}}\left(\hat{a}_1^{\rm in} - \hat{a}_2^{\rm in}\right). \tag{32}$$

Next the system becomes squeezed, for example, by an ion collision as discussed in Sec. III. Finally, the ions return to their initial positions, such that the coupling vanishes again.

First, we focus on a small squeezing parameter ξ and small thermal excitations within the radial mode

$$n_{\rm th} = \langle\hat{n}_1 + \hat{n}_2\rangle = 2\langle\hat{n}_1\rangle = 2\langle\hat{n}_2\rangle = 2\coth\left(\frac{\hbar\omega_{\rm rad}}{2k_BT}\right). \tag{33}$$

Here T is the (initial) temperature and k_B is the Boltzmann constant. We do not consider effects of thermal excitations in the axial modes because there is no coupling between axial and radial normal modes; see Eq. (3). The initial density operator can then be written as

$$\begin{aligned}\hat{\rho}_{\rm th}^{\rm in} &= (1 - n_{\rm th})|0\rangle_1|0\rangle_2\langle 0|_1\langle 0|_2 \\ &\quad + \frac{n_{\rm th}}{2}|1\rangle_1|0\rangle_2\langle 1|_1\langle 0|_2 + \frac{n_{\rm th}}{2}|0\rangle_1|1\rangle_2\langle 0|_1\langle 1|_2 \\ &\quad + O\left(n_{\rm th}^2\right).\end{aligned} \tag{34}$$

After the squeezing process described by the operator $\hat{\mathcal{S}}_{\xi_-}$ in Eq. (10), the final density operator reads

$$\hat{\rho}_{\rm th}^{\rm out} = \hat{\mathcal{S}}_{\xi_-}\hat{\rho}_{\rm th}^{\rm in}\hat{\mathcal{S}}_{\xi_-}^\dagger. \tag{35}$$

The partially transposed matrix of $\hat{\rho}_{\rm th}^{\rm out}$ possesses the eigenvalues $n_{\rm th} \pm |\xi_-|$ and consequently becomes negative definite for sufficiently large $|\xi_-|$. With the Peres-Horodecki criterion [30], which is a sufficient separability criterion for Gaussian states [31], it follows that the ions are entangled if and only if $|\xi_-| > n_{\rm th}$.

The former result was obtained by assuming small parameters $|\xi_-|$ and $n_{\rm th}$. However, for Gaussian states such as thermal states and squeezed thermal states (which we consider in our scenario), it is also possible to evaluate the Peres-Horodecki criterion for finite parameters ξ_- and $n_{\rm th}$. This was demonstrated in [32,33] and recently applied to analog gravity experiments in [34]. In Appendix C we adapt the formalism to our system and conclude that an initial thermal state with thermal excitations n_+ and n_- in the $\delta\hat{y}_\pm$ normal modes becomes entangled during a squeezing process in the $\delta\hat{y}_-$ mode if and only if the so-called symplectic eigenvalue

$$\lambda_-^{\rm PT} = \tfrac{1}{2}\sqrt{1 + 2n_-}\sqrt{1 + 2n_+}e^{-|\xi_-|} \tag{36}$$

satisfies $\lambda_-^{\rm PT} < 1/2$. As expected, in the limit of small squeezing parameters and small thermal excitations $2n_+ = 2n_- = n_{\rm th}$, this result coincides with the former entanglement criterion $|\xi_-| > n_{\rm th}$. To exemplify, we consider the collision between two ions which is depicted in Fig. 4. The associated Bogoliubov coefficient was determined in Sec. III as $|\beta_-|^2 \approx 0.18$, which is equivalent to a squeezing parameter of $|\xi_-| = \text{arsinh}(|\beta_-|) \approx 0.41$. Consequently, based on the symplectic eigenvalue in Eq. (36), such a squeezing is strong enough to entangle two ions in a thermal state with $n_{\rm th} < 0.5$. This upper bound is much larger than currently achievable, $n_{\rm th} \approx 0.05$. Furthermore, as outlined in Appendix C, the here considered squeezing is symmetric and one can infer from $\lambda_-^{\rm PT}$ also the strength of the entanglement by evaluating the entanglement of formation E_F [33]. For the particular example $n_{\rm th} \approx 0.05$ and $|\xi_-| \approx 0.41$ we obtain $E_F \approx 0.15$. This can be compared to a maximally entangled pair of qubits having $E_F = 1$ and a nonentangled state having $E_F = 0$ [35].

V. CONCLUSIONS

We have considered the radial modes of two or more ions in a trap which we accelerate in the axial direction. The shaping of the axial motion permits us to control the time-dependent coupling between the radial fluctuations and to create an characteristic excitation in these modes.

An advantage of this setup in comparison to previous proposals lies in exploiting axial and radial motion. This allows us to derive realistic parameters to enable the detection of phonon pair creation and entanglement. Furthermore, we predict for Gaussian states that the entanglement can be quantified experimentally by comparing correlations between pairs of ions to the thermal fluctuations of the individual ions. The process of phonon pair creation has been predicted to emerge in an analogous way in cosmological particle creation. It also displays similarities to black-hole evaporation, where the entanglement between the partners is often discussed in connection with the entropy of the black hole.

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APPENDIX A: BOGOLIUBOV COEFFICIENTS

The solutions of the rocking-mode differential equation (14), for the collision model (15), are the associated Legendre polynomials $P_\mu^\nu(z)$ with the substitutions

$$\begin{aligned} z &= \tanh(-\omega_{\rm col}t), \\ \nu &= i\frac{\Omega_{\rm in}}{\omega_{\rm col}}, \\ \mu &= \frac{1}{2}\left(i\sqrt{4\Delta\Omega_{\rm col}^2/\omega_{\rm col}^2 - 1} - 1\right). \end{aligned} \tag{A1}$$

Their asymptotic behavior is

$$P_\mu^\nu(\tanh(-\omega_{\rm col}t)) \overset{t\to-\infty}{\longrightarrow} \frac{e^{-i\Omega_{\rm in}t}}{\Gamma(1-\nu)} \tag{A2}$$

and

$$\begin{aligned} P_\mu^\nu(\tanh(-\omega_{\rm col}t)) \overset{t\to\infty}{\longrightarrow} &\frac{\Gamma(-\nu)}{\Gamma(-\nu-\mu)\Gamma(1+\mu-\nu)}e^{-i\Omega_{\rm in}t} \\ &- \frac{\sin(\pi\mu)\Gamma(\nu)}{\pi}e^{i\Omega_{\rm in}t}. \end{aligned} \tag{A3}$$

Hence, initial solutions with exponents $-i\Omega_{\rm in}t$ evolve into a mixture of solutions with positive and negative frequencies. The Bogoliubov coefficient $\beta_-^{\rm col}$ describes the conversion ratio of this process and is given by

$$|\beta_-^{\rm col}|^2 = \left|\frac{\sin(\pi\mu)\Gamma(\nu)\Gamma(1-\nu)}{\pi}\right|^2 = \left|\frac{\sin(\pi\mu)}{\sin(\pi\nu)}\right|^2. \tag{A4}$$

Back substitution yields, finally,

$$|\beta_-^{\rm col}|^2 = \left|\frac{\cosh\left(\frac{\pi}{2}\sqrt{\frac{4\Delta\Omega_{\rm col}^2}{\omega_{\rm col}^2} - 1}\right)}{\sinh\left(\frac{\pi\Omega_{\rm in}}{\omega_{\rm col}}\right)}\right|^2. \tag{A5}$$

APPENDIX B: WKB APPROXIMATION

We derive here the general exponential behavior of the Bogoliubov coefficients for slow and moderate collisions in a normal mode with frequency $\Omega^2(t)$. Moderate means that we stay away from the critical point, i.e.,

$$\Omega^2(t) > 0, \tag{B1}$$

while slow means that

$$\left|\frac{\dot{\Omega}(t)}{\Omega^2(t)}\right| \ll 1. \tag{B2}$$

For a typical collision $\Omega^2(t)$ reaches its minimum when the ions are closest and the scale function becomes minimal. Without loss of generality this happens at $t = 0$. As shown in [36], under these conditions a WKB approximation yields the exponential behavior of the Bogoliubov coefficient as

$$|\beta|^2 \propto \exp\left[-4\,\mathrm{Im}\left\{\int_0^{t_*}\Omega(t)dt\right\}\right], \tag{B3}$$

where t_* denotes the root of $\Omega(t)$ in the upper complex plane. Phonon creation happens more likely when the exponent is small. This can be achieved by working with low frequencies $\Omega(t)$ and small values for t_*.

Next, we calculate the exponent explicitly for collisions that are well described by a Taylor expansion

$$\Omega^2(t) \approx \Omega_{\rm min}^2 + \tfrac{1}{2}K^2t^2 \tag{B4}$$

in the region $|t| < |t_*|$, where

$$K^2 = \frac{d^2}{dt^2}\Omega^2(t)\Big|_{t=0} \tag{B5}$$

is the curvature. Their complex root is approximated by

$$t_* \approx i\sqrt{2}\frac{\Omega_{\rm min}}{K}. \tag{B6}$$

Finally, evaluating (B3) leads to

$$|\beta|^2 \propto \exp\left[-\sqrt{2}\pi\frac{\Omega_{\rm min}^2}{K}\right]. \tag{B7}$$

For the model collision with $b_{\rm col}(t)$ in (15), this yields the exponential behavior

$$|\beta_-^{\rm col}|^2 \propto \exp\left[-2\pi\frac{(\Omega_{\rm in} - \Delta\Omega_{\rm col})}{\omega_{\rm col}}\right], \tag{B8}$$

in agreement with (17).

APPENDIX C: COVARIANCE MATRIX FORMALISM

To apply the entanglement criteria for Gaussian states developed in [32–34] to our system we define the phase-space vector with respect to the ion coordinates

$$\hat{\mathbf{R}}_{12} = \left(\delta\hat{y}_1 \quad \delta\dot{\hat{y}}_1 \quad \delta\hat{y}_2 \quad \delta\dot{\hat{y}}_2\right)^T. \tag{C1}$$

The corresponding covariance matrix reads

$$\sigma_{kl} := \tfrac{1}{2}\langle\hat{R}_k\hat{R}_l + \hat{R}_l\hat{R}_k\rangle. \tag{C2}$$

We also define the phase-space vector with respect to the normal coordinates

$$\hat{\mathbf{R}}_{\rm nor} = \left(\delta\hat{y}_+ \quad \delta\hat{p}_+ \quad \delta\hat{y}_- \quad \delta\hat{p}_-\right)^T = \boldsymbol{D}\cdot\hat{\mathbf{R}}_{12}, \tag{C3}$$

with the transformation matrix

$$\boldsymbol{D} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \tag{C4}$$

which satisfies $\boldsymbol{D} = \boldsymbol{D}^T = \boldsymbol{D}^{-1}$. The covariance matrices corresponding either to $\hat{\mathbf{R}}_{12}$ or to $\hat{\mathbf{R}}_{\rm nor}$ are linked via

$$\boldsymbol{\sigma}_{12} = \boldsymbol{D} \cdot \boldsymbol{\sigma}_{\rm nor} \cdot \boldsymbol{D}. \tag{C5}$$

We consider an initial thermal covariance matrix

$$\boldsymbol{\sigma}^{\rm in}_{\rm nor} = \frac{1}{2}\begin{pmatrix} 1+2n_+ & 0 & 0 & 0 \\ 0 & 1+2n_+ & 0 & 0 \\ 0 & 0 & 1+2n_- & 0 \\ 0 & 0 & 0 & 1+2n_- \end{pmatrix} \tag{C6}$$

with the thermal occupation numbers

$$n_\pm = \coth\left(\frac{\hbar\Omega_\pm}{2k_B T}\right). \tag{C7}$$

Its time evolution during a squeezing process is

$$\boldsymbol{\sigma}^{\rm out}_{\rm nor} = \boldsymbol{S} \cdot \boldsymbol{\sigma}^{\rm in}_{\rm nor} \cdot \boldsymbol{S}^T, \tag{C8}$$

where $\boldsymbol{S}$ is the symplectic matrix

$$\boldsymbol{S} = \begin{pmatrix} \mathrm{Re}\{\alpha_+\} & \mathrm{Im}\{\alpha_+\} & 0 & 0 \\ -\mathrm{Im}\{\alpha_+\} & \mathrm{Re}\{\alpha_+\} & 0 & 0 \\ 0 & 0 & \mathrm{Re}\{\alpha_- - \beta_-\} & \mathrm{Im}\{\alpha_- + \beta_-\} \\ 0 & 0 & -\mathrm{Im}\{\alpha_- - \beta_-\} & \mathrm{Re}\{\alpha_- + \beta_-\} \end{pmatrix} \tag{C9}$$

containing the Bogoliubov coefficients for the $\delta\hat{y}_-$ and the $\delta\hat{y}_+$ mode. Here we consider only squeezing of the $\delta\hat{y}_-$ mode, i.e., $|\alpha_+| = 1$. For the time evolution of $\boldsymbol{\sigma}_{12}$ this implies

$$\boldsymbol{\sigma}^{\rm out}_{12} = \boldsymbol{D} \cdot \boldsymbol{S} \cdot \boldsymbol{D} \cdot \boldsymbol{\sigma}^{\rm in}_{12} \cdot \boldsymbol{D} \cdot \boldsymbol{S}^T \cdot \boldsymbol{D}. \tag{C10}$$

As shown in [32], for Gaussian states the Peres-Horodecki criterion can be formulated as a criterion on the two symplectic eigenvalues $\lambda_\pm$ of the partial transposed convariance matrix

$$\left(\boldsymbol{\sigma}^{\rm out}_{12}\right)^{\rm PT} = \boldsymbol{T} \cdot \boldsymbol{\sigma}^{\rm out}_{12} \cdot \boldsymbol{T} \tag{C11}$$

with $\boldsymbol{T} = \mathrm{diag}(1, -1, 1, 1)$. The ions are entangled if one of the symplectic eigenvalues is smaller than $1/2$.

For our system we obtain the symplectic eigenvalues as the two positive eigenvalues of $i\boldsymbol{J} \cdot (\boldsymbol{\sigma}^{\rm out}_{12})^{\rm PT}$ to

$$\begin{aligned} \lambda^{\rm PT}_\pm &= \tfrac{1}{2}\sqrt{1+2n_-}\sqrt{1+2n_+}(|\alpha_-| \pm |\beta_-|) \\ &= \tfrac{1}{2}\sqrt{1+2n_-}\sqrt{1+2n_+}\, e^{\pm|\xi_-|}, \end{aligned} \tag{C12}$$

where

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{C13}$$

Therefore the ions are entangled if $\sqrt{1+2n_-}\sqrt{1+2n_+}\exp(-|\xi_-|) < 1$.

Furthermore, in the case of symmetric squeezing, the entanglement of formation E_F can be evaluated explicitly [33]. Squeezing is called symmetric when the two 2×2 matrices on the diagonal of $\boldsymbol{\sigma}^{\rm out}_{12}$ possess identical determinants, which is the case here. The entanglement of formation is then

$$E_F = \begin{cases} f(\lambda^{\rm PT}_-) & \text{if } 0 < \lambda^{\rm PT}_- < \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} \leqslant \lambda^{\rm PT}_-, \end{cases} \tag{C14}$$

with the function

$$f(x) = \frac{\left(\frac{1}{2}+x\right)^2}{2x}\ln\left(\frac{\left(\frac{1}{2}+x\right)^2}{2x}\right) - \frac{\left(\frac{1}{2}-x\right)^2}{2x}\ln\left(\frac{\left(\frac{1}{2}-x\right)^2}{2x}\right). \tag{C15}$$

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