

REMARKS ON DIMENSIONAL REDUCTION OF MULTIDIMENSIONAL COSMOLOGICAL MODELS

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Multidimensional cosmological models with factorizable geometry and their dimensional reduction to effective four-dimensional theories are analyzed on sensitivity to different scalings. It is shown that a non-correct gauging of the effective four-dimensional gravitational constant within the dimensional reduction results in a non-correct rescaling of the cosmological constant and the gravexciton/radion masses. The relationship between the effective gravitational constants of theories with different dimensions is discussed for setups where the lower dimensional theory results via dimensional reduction from the higher dimensional one and where the compactified space components vary dynamically.

1. Introduction

One of the basic features of general relativity and of string theory/M-theory is that gravity necessarily propagates in all dimensions as it inherently describes the dynamics of spacetime itself. Although this general feature will hold for all dimensions, its low-energy realization as Gauss's law will strongly depend on the concrete structure of spacetime, the number and size of extra-dimensional space components as well as on the dynamics of the space components and the chosen frame of the observer. For simplicity, let us consider a multidimensional model with warped product topology

$$M = M_{D_0} \times M_{D'} \quad (1)$$

consisting of an external ("our") D_0 -dimensional spacetime manifold M_{D_0} (with $D_0 = 4$) and a D' -dimensional compact space component $M_{D'}$ with characteristic

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size L . If r is the distance between two massive bodies in a static or nearly static background metric, then depending on this distance the masses will attract each other according to Newton's law in D dimensions for $r \lesssim L$ and in D_0 dimensions for $r \gtrsim L$, respectively

$$\begin{aligned} F_D(r) &= G_{N(D)} \frac{m_1 m_2}{r^{D-2}}, \\ F_{D_0}(r) &= G_{N(D_0)} \frac{m_1 m_2}{r^{D_0-2}}. \end{aligned} \quad (2)$$

The relationship between the fundamental Newton constant $G_{N(D)} \equiv \kappa_D^2 / (2S_{D-1})$ in D dimensions and the effective Newton constant $G_{N(D_0)} \equiv \kappa_{D_0}^2 / (2S_{D_0-1})$ in the lower dimensional subspace M_{D_0} can be obtained from Gauss' law in D dimensions^{1,2} and reads

$$G_{N(D_0)} = \frac{G_{N(D)} S_{D-1}}{S_{D_0-1} V_{D'}}. \quad (3)$$

Here, $S_d = 2\pi^{d/2} / \Gamma(d/2)$ denotes the surface area of the unit sphere in d dimensions, and $V_{D'} \sim L^{D'}$ is the volume of the extra-dimensional compact space. Because of $G_{N(4)} \equiv \kappa_0^2 / (8\pi) = M_{Pl(4)}^{-2}$ and $G_{N(D)} \equiv \kappa_D^2 / (2S_{D-1}) = M_{*(4+D')}^{-(2+D')}$, Eq. (3) dictates the relationship^{3a}

$$\kappa_0^2 = \kappa_D^2 / V_{D'} \implies M_{Pl(4)}^2 \sim V_{D'} M_{*(4+D')}^{2+D'} \quad (4)$$

between the Planck scale $M_{Pl(4)} = 1.22 \times 10^{19} \text{ GeV}$ and the fundamental mass scale $M_{*(4+D')}$.

Data from Cavendish-type experiments⁶ confirmed the effective four-dimensionality of our Universe to high precision at distances above the lower bound of 1 mm. The upper bound is set by gravity tests in our planetary system, whereas modifications of Gauss's law at galactic and inter-galactic scales are not ruled out⁷. The analysis of possible observational consequences of extra dimensions at distances over these well-tested scales requires not only a qualitatively correct procedure of dimensional reduction, but also a quantitatively correct one. The main purpose of this short remark is to demonstrate this issue, which specifically occurs for models in an Einstein frame formulation, with the help of a simple multidimensional cosmological toy model. We also briefly discuss the frame dependence of the effective Newton's law in the dimensionally reduced theory and its behavior under a slow dynamical evolution of the compactification scale of the internal space components.

To start with, let us consider a cosmological model with factorizable geometry,

$$g = g^{(0)}(x) + \sum_{i=1}^n L_{Pl}^2 e^{2\beta^i(x)} g^{(i)}, \quad (5)$$

which is defined on the manifold (1) where, for generality, $M_{D'}$ is a direct product of n compact d_i -dimensional spaces: $M_{D'} = \prod_{i=1}^n M_i$, $\sum_{i=1}^n d_i = D'$. For simplicity,

^aIn brane-world models with non-factorizable geometry, the relation between κ_0 and κ_D differs from (4)^{4,5}.

we assume that the factors M_i are Einstein spaces: $R_{mn}[g^{(i)}] = \lambda^i g_{mn}^{(i)}$, $m, n = 1, \dots, d_i$ and $R[g^{(i)}] = \lambda^i d_i \equiv R_i$. The scale factors of the internal spaces depend on the coordinates of the four-dimensional external spacetime, $\beta^i = \beta^i(x)$.

Let $b_i \equiv L_{Pl} e^{\beta^i}$ and $b_{(0)i} \equiv L_{Pl} e^{\beta_0^i}$ denote the scales factors of the internal spaces M_i at arbitrary and at present time^b, respectively. Then the total volume of the internal spaces at the present time is given by

$$V_{D'} \equiv V_I \times v_0 \equiv \prod_{i=1}^n \int_{M_i} d^{d_i} y \sqrt{|g^{(i)}|} \times \left(\prod_{i=1}^n e^{d_i \beta_0^i} (L_{Pl})^{D'} \right) = V_I \times \prod_{i=1}^n b_{(0)i}^{d_i}. \quad (6)$$

The factor V_I is dimensionless and defined by geometry and topology of the internal spaces. In this section and the next one,

$$\tilde{\beta}^i = \beta^i - \beta_0^i \quad (7)$$

denote the deviations of the internal scale factors from their present day values.

For the demonstration of the scaling sensitivity it is sufficient to perform the dimensional reduction on the simplest multidimensional action

$$S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|g|} \{R[g] - 2\Lambda\} - \frac{1}{2} \int_M d^D x \sqrt{|g|} (g^{MN} \partial_M \Phi \partial_N \Phi + 2U(\Phi) + \dots) \quad (8)$$

for bulk matter in form of a minimally coupled scalar field Φ . The field Φ itself can be considered in its zero-mode approximation. This means that Φ and $U(\Phi)$ depend only on the coordinates of the external space, and the dimensional reduction of the model can be performed by a simple integration over the coordinates of the internal spaces.

In the next two sections we consider models with internal space scale factors which are stabilized in the minimum position of an effective potential and concentrate on different normalizations of the gravitational constant κ_0^2 and the mass scales via the volume $V_{D'}$. Subject of the third section will be the frame dependence of changes in the effective Newton's law of the dimensionally reduced theory when the scale factors are not yet stabilized. In the last section we briefly summarize our results.

^bA dynamical behavior of the extra dimensions results via dimensional reduction in a variation of the effective physical constants of the resulting four-dimensional theory (see e.g. section 4 below and relation (4), where an uncompensated changing in time of $V_{D'}$ would lead to the variation of κ_0). First discussions of this subject date back to Ref.⁸. The stabilization of internal spaces was discussed, e.g., in Refs.^{9,10}.

2. The four-dimensional effective model

As first step we perform a dimensional reduction of action (8), what results in the following four-dimensional effective theory:

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \prod_{i=1}^n e^{d_i \tilde{\beta}^i} \left\{ R[g^{(0)}] - G_{ij} g^{(0)\mu\nu} \partial_\mu \tilde{\beta}^i \partial_\nu \tilde{\beta}^j + \right. \\ \left. + \sum_{i=1}^n \tilde{R}_i e^{-2\tilde{\beta}^i} - 2\Lambda - g^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2\kappa_D^2 U(\Phi) - \dots \right\}. \quad (9)$$

Here the total volume of the internal spaces $V_{D'}$ is defined by Eq. (6), the gravitational constant κ_0^2 is given as $\kappa_0^2 = \kappa_D^2/V_{D'}$ in accordance with relation (3), and the notations $\tilde{R}_i := R_i e^{-2\tilde{\beta}^i} L_{Pl}^{-2}$, $G_{ij} = \delta_{ij} d_i - d_i d_j$ are used. In the ADD approach^{1,3} the electro-weak scale is assumed as the fundamental scale $M_{*(4+D')} \sim M_{EW}$ so that relation (4) leads to strong restrictions on the total volume $V_{D'}$ of the internal spaces.

Action (9) of the four-dimensional effective model is written in Brans-Dicke frame, i.e., it has the form of a generalized Brans-Dicke theory. As next step, we remove the explicit dilatonic coupling term in (9) by conformal transformation

$$g_{\mu\nu}^{(0)} = \Omega^2 \tilde{g}_{\mu\nu}^{(0)} := \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i} \right)^{\frac{-2}{D_0-2}} \tilde{g}_{\mu\nu}^{(0)} \quad (10)$$

and obtain the effective action in the Einstein-frame

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}[\tilde{g}^{(0)}] - \tilde{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_\mu \tilde{\beta}^i \partial_\nu \tilde{\beta}^j - \right. \\ \left. - \tilde{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2U_{eff} \right\}, \quad (11)$$

where the effective potential U_{eff} reads

$$U_{eff}[\tilde{\beta}, \Phi] = \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i} \right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2} \sum_{i=1}^n \tilde{R}_i e^{-2\tilde{\beta}^i} + \Lambda + \kappa_D^2 U(\Phi) + \dots \right] \quad (12)$$

and the notation $\tilde{G}_{ij} = \delta_{ij} d_i + d_i d_j / (D_0 - 2)$ is used. The internal spaces are stabilized at present time if this potential has a minimum at $\tilde{\beta}^i = 0$. Small conformal excitations of the internal spaces above this minimum have the form of massive scalar fields (gravexcitons/radions) in our four-dimensional spacetime⁹. For models with the scalar field Φ as the only bulk field, the stabilization occurs for fine-tuned scalar curvatures of the internal spaces¹⁰:

$$\frac{\tilde{R}_k}{d_k} = \frac{\tilde{R}_i}{d_i}, \quad (i, k = 1, \dots, n). \quad (13)$$

For the four-dimensional effective cosmological constant of such models holds

$$\Lambda_{eff} := U_{eff} \Big|_{\substack{\tilde{\beta}^i=0, \\ \Phi=\Phi_0}} = \frac{D_0 - 2 \tilde{R}_k}{2} \frac{1}{d_k}, \quad (14)$$

whereas the gravexciton/radion masses are defined by the relations

$$m_i^2 = -\frac{4\Lambda_{eff}}{D_0 - 2} = -2 \frac{\tilde{R}_k}{d_k} > 0, \quad i = 1, \dots, n. \quad (15)$$

The important point is that the cosmological constant (14) and the gravexciton/radion masses (15) in the Einstein frame are defined at the present time and in a model with present-time effective gravitational constant $\kappa_0^2 = \kappa_D^2/V_{D'}$. In the next section we will show that these properties do not hold when the scaling in the dimensional reduction scheme is chosen differently.

3. Alternative approach

In this section, we consider a dimensional reduction scheme which slightly differs from that of the previous section and which was employed in many papers. In this scheme the stabilization positions of the functions β^i are not fixed from the very beginning, but instead they are found from the minimum condition of the effective potential. In other words, the scale factors β^i are not split into present-time stabilization positions β_0^i and small fluctuational components $\tilde{\beta}^i$ above them. The dimensional reduction of action (8) parallels that of the previous section with replacing $\tilde{\beta}^i$ by β^i . The result reads

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \prod_{i=1}^n e^{d_i \beta^i} \left\{ R[g^{(0)}] - G_{ij} g^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j + \right. \\ \left. + \sum_{i=1}^n R_i e^{-2\beta^i} - 2\Lambda - g^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2\kappa_D^2 U(\Phi) - \dots \right\}, \quad (16)$$

with an effective gravitational constant given by

$$\kappa_0^2 := \frac{\kappa_D^2}{V_I \times (L_{Pl})^{D'}} \implies M_{Pl(4)}^2 \sim V_I (L_{Pl})^{D'} M_{*(4+D')}^{2+D'}. \quad (17)$$

In this approach, $V_I \times (L_{Pl})^{D'}$ does not describe the total present-time volume of the internal spaces (with the exception of the stabilization position $\beta^i = 0$), and, according to the generalized Gauss's law ("gravity propagates in all dimensions") the constant κ_0 in Eq. (17) does not correspond to the present-time gravitational constant. Only in the case of $\beta^i = 0$, i.e. for Planckian stabilization scales $b_{(0)i} = L_{Pl}$, the constant κ_0^2 in Eq. (17) correctly corresponds to the four-dimensional Newtonian gravitational constant. Here, we assumed $V_I \sim \mathcal{O}(1)$, which usually holds for constant curvature spaces with normalization $R[g^{(i)}] = \pm d_i(d_i - 1)$. The fundamental mass scale is then of the order of the Planck scale, $M_{*(4+D')} \sim M_{(Pl)4}$.

As next step, we demonstrate how the formal normalization of the gravitational constant in Eq. (17) results in a non-correct rescaling of the parameters of the four-dimensional effective theory (in Einstein frame) when $\beta_0^i \neq 0$. The conformal transformation to the Einstein frame reads:

$$g_{\mu\nu}^{(0)} = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{\frac{-2}{D_0-2}} \bar{g}_{\mu\nu}^{(0)}. \quad (18)$$

We observe that the four-dimensional Einstein frame metrics for the natural approach of the previous section (we denote it below by subscript "1", the corresponding external space metric is $\tilde{g}_{\mu\nu}^{(0)}$) and that of the formal approach of the present section (subscript "2", external space metric $\bar{g}_{\mu\nu}^{(0)}$) are connected as

$$\bar{g}_{\mu\nu}^{(0)} = \omega^2 \tilde{g}_{\mu\nu}^{(0)}, \quad \omega^2 = \left(v_0 / (L_{Pl})^{D'} \right)^{2/(D_0-2)} = \left(\prod_{i=1}^n e^{d_i \beta_0^i} \right)^{\frac{2}{D_0-2}}. \quad (19)$$

The Einstein frame form of action (16) is

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|\bar{g}^{(0)}|} \left\{ \bar{R}[\bar{g}^{(0)}] - \bar{G}_{ij} \bar{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - \bar{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2U_{eff} \right\} \quad (20)$$

with an effective potential given by

$$U_{eff}[\beta, \Phi] = \left(\prod_{i=1}^n e^{d_i \beta^i} \right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2} \sum_{i=1}^n R_i e^{-2\beta^i} + \Lambda + \kappa_D^2 U(\Phi) + \dots \right]. \quad (21)$$

The gravexciton/radion masses as well as the effective cosmological constants in the two approaches are connected by the following rescaling,

$$m_i^2|_2 = \left(\prod_{i=1}^n e^{d_i \beta_0^i} \right)^{\frac{-2}{D_0-2}} m_i^2|_1, \quad (22)$$

$$\Lambda_{eff}|_2 = \left(\prod_{i=1}^n e^{d_i \beta_0^i} \right)^{\frac{-2}{D_0-2}} \Lambda_{eff}|_1,$$

which results from the rescaled effective potential

$$U_{eff}|_2 = \left(\prod_{i=1}^n e^{d_i \beta_0^i} \right)^{\frac{-2}{D_0-2}} U_{eff}|_1 = \omega^{-2} U_{eff}|_1. \quad (23)$$

In its turn, the rescaling of the effective potential is caused by the formal and non-correct definition (17) of the four-dimensional gravitational constant κ_0 in the

second approach. The gravitational constants in the two approaches are connected by the relation

$$\kappa_0^2|_2 = \prod_{i=1}^n e^{d_i \beta_0^i} \kappa_0^2|_1 = \omega^{D_0-2} \kappa_0^2|_1, \quad (24)$$

which can also be obtained from the equation

$$\begin{aligned} \frac{1}{2\kappa_0^2|_2} \int d^{D_0} x \sqrt{|\bar{g}^{(0)}|} \bar{R}[\bar{g}^{(0)}] &= \frac{\omega^{D_0-2}}{2\kappa_0^2|_2} \int d^{D_0} x \sqrt{|\tilde{g}^{(0)}|} \tilde{R}[\tilde{g}^{(0)}] \\ &\equiv \frac{1}{2\kappa_0^2|_1} \int d^{D_0} x \sqrt{|\tilde{g}^{(0)}|} \tilde{R}[\tilde{g}^{(0)}]. \end{aligned} \quad (25)$$

Thus, if we restore in the second approach the correct four-dimensional gravitational constant, we obtain the same formulas as in the first approach:

$$\begin{aligned} S|_2 &= \frac{\omega^{D_0-2}}{\omega^{D_0-2} 2\kappa_0^2|_2} \int_{M_0} d^{D_0} x \sqrt{|\bar{g}^{(0)}|} \left\{ \bar{R}[\bar{g}^{(0)}] - \bar{G}_{ij} \bar{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - \right. \\ &\quad \left. - \bar{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2U_{eff}|_2 \right\} \\ &= \frac{1}{2\kappa_0^2|_1} \int_{M_0} d^{D_0} x \omega^{-D_0} \sqrt{|\bar{g}^{(0)}|} \left\{ \omega^2 \bar{R}[\bar{g}^{(0)}] - \omega^2 \bar{G}_{ij} \bar{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - \right. \\ &\quad \left. - \omega^2 \bar{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2\omega^2 U_{eff}|_2 \right\} \\ &= \frac{1}{2\kappa_0^2|_1} \int_{M_0} d^{D_0} x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}[\tilde{g}^{(0)}] - \tilde{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_\mu \tilde{\beta}^i \partial_\nu \tilde{\beta}^j - \right. \\ &\quad \left. - \tilde{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2U_{eff}|_1 \right\} \\ &\equiv S|_1. \end{aligned} \quad (26)$$

4. Frame dependence of Newton's gravitational force law and a possible dynamics of the gravitational "constant"

In the previous sections the main emphasis was laid on the correct scaling of the gravitational constant in present-time physical regimes when the sizes of the internal spaces are stabilized, and only small fluctuations over this stabilized scale factor background remain. In general, this stabilized scale factor regime will be the result and current end point of highly dynamical changes of the sizes of the internal spaces at early stages of the cosmological evolution before nucleosynthesis started^c. Taking into account that the effective gravitational constant κ_0^2 of our four-dimensional external spacetime is a derived constant which follows via dimensional reduction from a fundamental mass scale, we are naturally led to the question, whether κ_0^2

^cObservation data strongly restrict late-time variations of coupling constants, such as the fine structure constant α and gravitational constant κ_0 ¹².

also varies dynamically with the sizes of the internal space components and correspondingly depends on the external coordinates $\kappa_0 = \kappa_0(x)$, or whether it remains fixed by some mechanism. Below we will show that the answer of this question depends on the chosen metric frame of the effective four-dimensional spacetime.

We start the consideration by splitting the internal space scale factors

$$\beta^i(x) = \beta_0^i(x) + \tilde{\beta}^i(x) \quad (27)$$

into a slowly and coherently changing background component $\beta_0^i(x)$, which will define the averaged dynamics of the volume of the internal spaces, and small non-coherent particle-like excitations/fluctuations $\tilde{\beta}^i(x)$ over this background, which as previously will correspond to gravexcitons/radions. The dynamically averaged volume $V_{D'}(x)$ of the internal spaces is then defined in analogy with (6) as

$$\begin{aligned} V_{D'}(x) &\equiv V_I \times v_0(x) \equiv \prod_{i=1}^n \int_{M_i} d^{d_i} y \sqrt{|g^{(i)}|} \times \left(\prod_{i=1}^n e^{d_i \beta_0^i(x)} (L_{Pl})^{D'} \right) \\ &= V_I \times \prod_{i=1}^n b_{(0)i}^{d_i}(x). \end{aligned} \quad (28)$$

Subsequently, we present a sketchy non-relativistic analysis in terms of Newton's force law. We assume that the scale factor background changes slowly and smoothly enough, and the gravexciton/radion amplitudes are small enough to keep inside this approximation.

Next, we note that the effective gravitational force

$$F_{eff}(r) = \frac{G_{N(D)} S_{D-1} m_1 m_2}{S_{D_0-1} V_{D'} r^{D_0-2}} \quad (29)$$

between two masses m_1 and m_2 separated at a distance $r \gg L \sim V_{D'}^{1/D'}$ in the external spacetime M_{D_0} is the result of a dimensional reduction^d performed in the starting metric (5) of the total product space M_D . The distance r is correspondingly measured in the Brans-Dicke metric $g^{(0)}$ of the external spacetime M_{D_0} and $V_{D'}$ is the total volume of the internal space. This means that formally the constant volume $V_{D'}$ of a static internal space, for which the force law was derived, should be replaced by the total volume $V_{D',T}$ which is now defined by the non-truncated internal space scale factors $\beta^i(x)$:

$$\begin{aligned} V_{D'} \mapsto V_{D',T}(x) &= V_I \times L_{Pl}^{D'} \prod_{i=1}^n e^{d_i \beta^i(x)} \\ &= V_{D'}(x) \times \prod_{i=1}^n e^{d_i \tilde{\beta}^i(x)}. \end{aligned} \quad (30)$$

It is clear that for large non-adiabatic and particle-like scale factor fluctuations $\tilde{\beta}^i(x)$ in (27) the force law approximation (29) will break down and should be

^dThe technical details are given in Ref.².

replaced by a field theoretic treatment based on a Green function technique as it was discussed, e.g., in Ref.¹. Subsequently, we will restrict our attention to two regimes for which the Newton's law (29) with the replacement (30) can be used as rough approximation of the gravitational attraction force: a long wave-length regime (I) without small short wave-length contributions so that a splitting (27) is not required^e [$\beta^i(x) = \beta_0^i(x)$]; and a regime where sufficiently small gravexciton/radion fluctuations $\tilde{\beta}^i(x)$ over a coherently varying long-wavelength background are present (regime II).

We will now analyze how the variations of $V_{D',T}(x)$ will affect the force law (29) in different frames of the external spacetime M_{D_0} . (We assume $D_0 = 4$.)

4.1. Brans-Dicke frame

In Newton's force law (29) the distance r is measured in the metric $g^{(0)}$ which coincides with the Brans-Dicke (BD) frame metric. This means that, in the BD frame, the only change in the force law (29) will be induced by the replacement of the total internal space volume according to (30):

$$F_{eff}(r) = \frac{G_{N(D)} S_{D-1}}{S_{D_0-1} V_{D',T}(x)} \frac{m_1 m_2}{r^{D_0-2}}. \quad (31)$$

As result the varying total internal space volume $V_{D',T}(x)$ leads to corresponding variations in the gravitational force between the masses m_1 and m_2 in the external space $M_{(D_0=4)}$. In both regimes we can define a varying effective gravitational "constant" as

$$\kappa_0^2(x) = \frac{\kappa_D^2}{V_{D',T}(x)} = \frac{\kappa_D^2}{V_I(LPl)^{D'}} \prod_{i=1}^n e^{-d_i \beta_0^i(x)}. \quad (32)$$

For absent scale factor fluctuations (regime I) $\beta_0^i(x)$ coincides with $\beta^i(x)$, whereas in regime II the small gravexciton/radion-like scale factor fluctuations $\tilde{\beta}^i(x)$ are split off from the slowly varying volume. In the latter regime (II) the action functional (9) reads

$$\begin{aligned} S = \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} & \frac{1}{2\kappa_0^2(x)} \prod_{i=1}^n e^{d_i \tilde{\beta}^i} \left\{ R[g^{(0)}] - G_{ij} g^{(0)\mu\nu} \partial_\mu \beta_0^i \partial_\nu \beta_0^j - \right. \\ & - 2G_{ij} g^{(0)\mu\nu} \partial_\mu \beta_0^i \partial_\nu \tilde{\beta}^j - G_{ij} g^{(0)\mu\nu} \partial_\mu \tilde{\beta}^i \partial_\nu \tilde{\beta}^j + \\ & \left. + \sum_{i=1}^n \tilde{R}_i e^{-2\tilde{\beta}^i} - 2\Lambda - g^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2\kappa_D^2 U(\Phi) - \dots \right\} \quad (33) \end{aligned}$$

and transition to regime I consists simply in the substitution $\tilde{\beta}^i(x) \rightarrow 0$, $\beta_0^i(x) \rightarrow \beta^i(x)$.

^eThis means that the total internal space volume $V_{D',T}(x)$ varies slowly enough that the gravitational force law will dominate over gravitational wave effects due to changes of the metric.

In action functional (33) the background scale factors $\beta_0^i(x)$ are via internal space volume $V_{D'}(x)$ and relation (32) responsible for the variations of the effective gravitational "constant" $\kappa_0^2(x)$, whereas the small scale factor fluctuations $\tilde{\beta}^i(x)$ can be considered as scalar particles (gravexcitons/radions) propagating over this background in the spacetime M_{D_0} .

4.2. Hybrid frame

Removing the Brans-Dicke factor of the fluctuational components in action (33) by a conformal transformation of type (10)

$$g_{\mu\nu}^{(0)} = \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i} \right)^{\frac{-2}{D_0-2}} \hat{g}_{\mu\nu}^{(0)}, \quad (34)$$

we arrive at an action functional

$$\begin{aligned} S = \int_{M_0} d^{D_0} x \sqrt{|\hat{g}^{(0)}|} \frac{1}{2\kappa_0^2(x)} \left\{ \tilde{R} [\hat{g}^{(0)}] - G_{ij} \hat{g}^{(0)\mu\nu} \partial_\mu \beta_0^i \partial_\nu \beta_0^j - \right. \\ \left. - 2G_{ij} \hat{g}^{(0)\mu\nu} \partial_\mu \beta_0^i \partial_\nu \tilde{\beta}^j - \tilde{G}_{ij} \hat{g}^{(0)\mu\nu} \partial_\mu \tilde{\beta}^i \partial_\nu \tilde{\beta}^j - \right. \\ \left. - \hat{g}^{(0)\mu\nu} \kappa_D^2 \partial_\mu \Phi \partial_\nu \Phi - 2U_{eff} \right\}, \quad (35) \end{aligned}$$

which is in a hybrid Brans-Dicke-Einstein frame — Brans-Dicke frame with respect to the averaged scale factors $\beta_0^i(x)$, which define the effective gravitational constant $\kappa_0(x)$ via (32), and Einstein frame with respect to the gravexcitons/radions $\tilde{\beta}^i(x)$.

For a stabilization of the internal spaces with volume freezing we have

$$V_{D'}(x) \longrightarrow V_{D'}, \quad \beta_0^i(x) \longrightarrow \beta_0^i, \quad \kappa_0(x) \longrightarrow \kappa_0 \quad (36)$$

and we return to the simplified action functional (11) of section 2. We note, that similarly like possible present-time variations of the fine-structure "constant" α are strongly restricted by observational data^{12,13}, there exist strong observational restrictions on present-time variations of the effective gravitational constant κ_0 ^{12,14f}.

Let us now analyze how the starting Newton's gravitational force (29) should be changed for an observer in the hybrid frame. From the conformal transformation (34) follows the relation

$$r = \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i(x)} \right)^{\frac{-1}{D_0-2}} \hat{r} \quad (37)$$

^fFor completeness, we note that in string-theoretic setups an additional dynamically changing dilaton factor enters the definition (32) of the effective four-dimensional gravitational "constant" κ_0 . (See, e.g., Ref.¹².)

between the distances r and \hat{r} in BD and hybrid frame. On the other hand, the exact relation for the total internal space volume reads

$$V_{D',T}(x) = V_{D'}(x) \times \left(\prod_{i=1}^n e^{d_i \tilde{\beta}(x)^i} \right). \quad (38)$$

Plugging Eqs. (37), (38) into (31) we obtain the force law in the hybrid frame as

$$F_{eff}(r) = \frac{G_{N(D)} S_{D-1}}{S_{D_0-1} V_{D'}(x)} \frac{m_1 m_2}{\hat{r}^{D_0-2}} \implies \kappa_0^2(x) = \kappa_D^2 / V_{D'}(x), \quad (39)$$

i.e., the contributions of the small scale factor fluctuations $\tilde{\beta}^i(x)$ from total volume and the transformed distanced cancelled each other and we are left with a Newton's force law which depends only on the slowly varying part $V_{D'}(x)$ of the internal space volume. This means that for an observer in hybrid frame there is no need of averaging over small scale factor fluctuations because they are formally cancelling in the force law.

4.3. Einstein frame

Similar to the Einstein frame setup of section 2, we assume the internal space scale factors split into constant background components β_0^i and non-constant components $\tilde{\beta}^i(x)$

$$\beta^i(x) = \beta_0^i + \tilde{\beta}^i(x). \quad (40)$$

The total internal space volume is then defined as

$$V_{D',T}(x) = V_{D'} \times \left(\prod_{i=1}^n e^{d_i \tilde{\beta}(x)^i} \right), \quad (41)$$

with $V_{D'}$ given in Eq. (6). We note that, in general, the constant volume $V_{D'}$ can be interpreted as an arbitrarily fixed reference volume. For simplicity, we assume in our heuristic considerations that the non-constant scale factor components $\tilde{\beta}^i(x)$ are either slowly varying or, e.g., for stabilized scale factor backgrounds, sufficiently small to keep the description in terms of Newton's law physically sensible.

Along the same scheme as for the hybrid frame, we obtain from the conformal transformation (10) a relation between the distance r measured in the BD frame and the corresponding distance r_E in the Einstein frame

$$r = \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i} \right)^{\frac{-1}{D_0-2}} r_E. \quad (42)$$

Substitution of Eqs. (41), (42) into (31) yields

$$F_{eff}(r) = \frac{G_{N(D)} S_{D-1}}{S_{D_0-1} V_{D'}} \frac{m_1 m_2}{r_E^{D_0-2}} \implies \kappa_0^2 = \kappa_D^2 / V_{D'}. \quad (43)$$

Again the contributions of the non-constant scale factor components $\tilde{\beta}^i(x)$ cancelled. We observe the interesting fact that in the Einstein frame the effective Newton's law in the dimensionally reduced theory does not change, irrespective of the dynamically changing volume $V_{D',T}(x)$ of the internal spaces, i.e., the changes in $V_{D',T}(x)$ are exactly compensated by inverse changes of the Einstein frame metric $\tilde{g}^{(0)}$ with respect to the original external space metric $g^{(0)}$. Hence, in the chosen oversimplified model an observer in Einstein frame will not be aware of the internal space dynamics in measurements based on non-relativistic approximations of the gravitational sector. This is also visible from the Einstein frame action (11), where the D_0 -dimensional gravity sector is given by a pure Einstein-Hilbert term with non-varying gravitational constant κ_0^2 . For given mass scales $M_{Pl} = \sqrt{8\pi}/\kappa_0$, $M_{*(4+D')} = (2S_{D-1}/\kappa_D^2)^{1/(2+D')}$ relation (43) fixes the internal space (reference) volume $V_{D'}$ completely.

Finally, we note that in the special case of a solely time dependent scale factor dynamics, the metric structure roughly parallels the M-theory inspired cosmological toy models of Ref.¹⁵ which live in warped product metrics of the type

$$g = \left(\prod_{i=1}^n e^{d_i \tilde{\beta}^i(t)} \right)^{\frac{-2}{D_0-2}} \tilde{g}_{\mu\nu}^{(0)} + \sum_{i=1}^n L_{Pl}^2 e^{2\beta^i(t)} g^{(i)}. \quad (44)$$

5. Conclusions

An essential part of any viable higher dimensional theory is a sensible scheme of dimensional reduction to an effective four-dimensional theory. On the one hand, the resulting effective theory should correctly describe our observable four-dimensional Universe (the external spacetime). On the other hand, it will contain explicit imprints and signatures of the extra-dimensional space components. The latter will give an opportunity to predict new observable phenomena, such as gravexcitons/radions which are geometrical moduli excitations of extra-dimensional spaces propagating as special types of particles in the observable Universe. A correct quantitative prediction of the derived physical parameters (such as the effective "fundamental" constants, the cosmological constant, the masses of gravexcitons/radions etc.) directly depends on the concrete scheme of dimensional reduction. In the present paper, we demonstrated this fact explicitly with the help of a multidimensional toy model with factorizable geometry as it is often used in KK and ADD approaches. For a model with stabilized internal spaces, we considered two different schemes of dimensional reduction with subsequent transformation to the Einstein frame. In the second approach, the present-time size of the internal spaces was not taken into account, and, correspondingly, it was left out of account that gravity should propagate in all dimensions. As a result, the effective four-dimensional gravitational constant was not correctly gauged what finally led to a non-correct rescaling of the parameters of the effective four-dimensional model (such as the effective cosmological constant and the gravexciton/radion masses).

Additionally, we discussed the relation between the chosen observer frame in the external spacetime M_{D_0} and the dependence of the effective gravitational coupling "constant" κ_0^2 in M_{D_0} on the dynamics of the compactified internal factor space. It was shown that in Brans-Dicke frame and a hybrid Brans-Dicke-Einstein frame κ_0^2 changes dynamically when the volume $V_{D'}(x)$ of the internal space changes, whereas in Einstein frame κ_0^2 can be hold fixed independently of the scale factor dynamics of the internal space. This constancy of κ_0^2 results from the special tuning between the internal space scale factors and the conformal factor of the external spacetime.

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