On Krein space related perturbation theory for MHD α^2-dynamos

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The spectrum of the spherically symmetric α^2-dynamo is studied in the case of idealized boundary conditions. Starting from the exact analytical solutions of models with constant $\alpha-$profiles a perturbation theory and a Galerkin technique are developed in a Krein-space approach. With the help of these tools a very pronounced $\alpha-$resonance pattern is found in the deformations of the spectral mesh as well as in the unfolding of the diabolical points located at the nodes of this mesh. Non-oscillatory as well as oscillatory dynamo regimes are obtained. An estimation technique is developed for obtaining the critical $\alpha-$profiles at which the eigenvalues enter the right spectral half-plane with non-vanishing imaginary components (at which overcritical oscillatory dynamo regimes form).

1 Resonant deformation of the spectral mesh of the mean field MHD α^2-dynamo

The mean field α^2-dynamo of magnetohydrodynamics (MHD) [1] plays a similarly paradigmatic role in MHD dynamo theory like the harmonic oscillator in quantum mechanics. In its kinematic regime this dynamo is described by a *linear* induction equation for the magnetic field. For spherically symmetric $\alpha-$profiles $\alpha(r)$ the vector of the magnetic field can be decomposed into poloidal and toroidal components and expanded in spherical harmonics. After additional time separation, the induction equation reduces to a set of $l-$decoupled boundary eigenvalue problems [1, 2]

$$\mathfrak{A}_\alpha \mathfrak{u} = \lambda \mathfrak{u},\ \mathfrak{u}(r \searrow 0) = \mathfrak{u}(1) = 0 \tag{1}$$

for matrix differential operators

$$\mathfrak{A}_\alpha := \begin{pmatrix} -A_l & \alpha(r) \\ A_{l,\alpha} & -A_l \end{pmatrix}, \quad A_l := -\partial_r^2 + \frac{l(l+1)}{r^2}, \quad A_{l,\alpha} := -\partial_r \alpha(r) \partial_r + \alpha(r)\frac{l(l+1)}{r^2} = \alpha(r) A_l - \alpha'(r)\partial_r \,. \tag{2}$$

The boundary conditions in (1) are idealized ones and formally coincide with those for dynamos in a high conductivity limit of the dynamo maintaining fluid/plasma. We will restrict our subsequent considerations to this case and assume a domain

$$\mathcal{D}(\mathfrak{A}_\alpha) = \left\{ \mathfrak{u} \in \tilde{\mathcal{H}} = L_2(0,1) \oplus L_2(0,1) |\ \mathfrak{u}(r \searrow 0) = \mathfrak{u}(1) = 0 \right\} \tag{3}$$

in the Hilbert space $(\tilde{\mathcal{H}}, (.,.))$.

The $\alpha-$profile $\alpha(r)$ is a smooth real function $C^2(0,1) \ni \alpha(r) :\ (0,1) \ \to\ \mathbb{R}$. It plays the role of the potential in dynamo models. Due to the fundamental symmetry of its differential expression,

$$\mathfrak{A}_\alpha = J \mathfrak{A}_\alpha^\dagger J, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{4}$$

the operator $\mathfrak{A}_\alpha$ is a symmetric operator in a Krein space $(\mathcal{K}, [.,.])$ with indefinite inner product $[.,.] = (J.,.)$ and for the chosen domain (3) it is also selfadjoint in this space $[\mathfrak{A}_\alpha x, y] = [x, \mathfrak{A}_\alpha y],\ x, y \in \mathcal{K}$. For constant $\alpha-$profiles $\alpha(r) \equiv \alpha_0 =$const, $r \in [0,1)$, the spectrum and eigenvectors of the operator matrix $\mathfrak{A}_{\alpha_0}$ are

$$\lambda_n^\pm = \lambda_n^\pm(\alpha_0) = -\rho_n \pm \alpha_0 \sqrt{\rho_n} \in \mathbb{R}, \quad n \in \mathbb{Z}^+, \quad \mathfrak{u}_n^\pm = \begin{pmatrix} 1 \\ \pm\sqrt{\rho_n} \end{pmatrix} u_n \in \mathbb{R}^2 \otimes L_2(0,1)\,, \tag{5}$$

and correspond to Krein space states of positive and negative type $[\mathfrak{u}_m^\pm, \mathfrak{u}_n^\pm] = \pm 2\sqrt{\rho_n}\delta_{mn},\ [\mathfrak{u}_m^\pm, \mathfrak{u}_n^\mp] = 0,\ \mathfrak{u}_n^\pm \in \mathcal{K}_\pm \subset \mathcal{K}$. The functions u_n in (5) are Riccati-Bessel functions

$$u_n(r) = N_n r^{1/2} J_{l+\frac{1}{2}}(\sqrt{\rho_n} r), \quad N_n := \frac{\sqrt{2}}{J_{l+\frac{3}{2}}(\sqrt{\rho_n})}, \quad (u_m, u_n) = \delta_{mn}, \quad \|u_n\| = 1\,. \tag{6}$$

Accordingly, the coefficients $\rho_n > 0$ in (5) are the squares of Bessel function zeros $J_{l+\frac{1}{2}}(\sqrt{\rho_n}) = 0,\ 0 < \sqrt{\rho_1} < \sqrt{\rho_2} < \cdots$. The branches $\lambda_n^\pm$ of the spectrum are real-valued linear functions of the parameter α_0 with slopes $\pm\sqrt{\rho_n}$ and form a mesh-like structure in the $(\alpha_0, \Re\lambda)-$plane, as depicted in Fig. 1a. The nodes of the spectral mesh in Fig. 1a are the intersection

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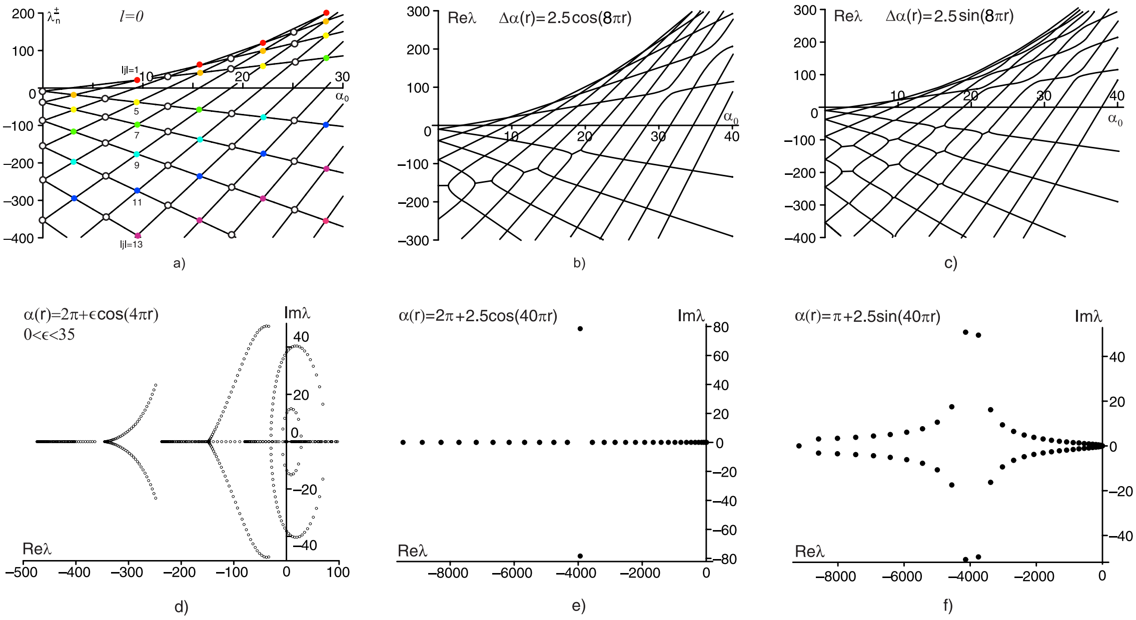

Fig. 1 The spectral mesh of the operator matrix $\mathfrak{A}_\alpha$ for $l = 0$ (a); its resonant deformation due to harmonic perturbations of a constant α-profile (b), (c); the formation of overcritical oscillatory dynamo regimes for ϵ increasing from $\epsilon = 0$ ($\Im\lambda = 0$) to $\epsilon = 35$ ($\Re\lambda > 0$, $\Im\lambda \neq 0$ for some branches) (d); and the resonant unfolding of DPs in the complex plane (e), (f).

points of spectral branches $\lambda_n^\varepsilon = -\rho_n + \varepsilon\alpha_0\sqrt{\rho_n}$ and $\lambda_n^\delta = -\rho_n + \delta\alpha_0\sqrt{\rho_n}$ (where $\varepsilon = \pm$, $\delta = \pm$) and correspond to semi-simple double eigenvalues $\lambda_0^\nu = \varepsilon\delta\sqrt{\rho_n\rho_m}$ with pairwise linearly independent eigenvectors u_n^ε and u_m^δ of the operator $\mathfrak{A}_{\alpha_0^\nu}$ for $\alpha_0 = \alpha_0^\nu := \varepsilon\sqrt{\rho_n}+\delta\sqrt{\rho_m}$ (diabolical points (DPs)). Spectral branches of opposite Krein space types $\delta \neq \varepsilon$ intersect for both signs of α_0 at $\lambda_0^\nu < 0$. In contrast, intersections at $\lambda_0^\nu > 0$ are induced by spectral branches of positive type when $\alpha_0 > 0$, and of negative type when $\alpha_0 < 0$.

A perturbation of the α-profile of the form $\alpha(r) = \alpha_0^\nu + \Delta\alpha(r) = \alpha_0^\nu + \epsilon\varphi(r)$ yields a splitting of the double eigenvalue $\lambda = \lambda_0^\nu + \epsilon\lambda_1 + \ldots$, with leading coefficient λ_1 determined by the quadratic equation

$$\lambda_1^2 - \lambda_1\left(\varepsilon\frac{[\mathfrak{B}\mathrm{u}_n^\varepsilon, \mathrm{u}_n^\varepsilon]}{2\sqrt{\rho_n}} + \delta\frac{[\mathfrak{B}\mathrm{u}_m^\delta, \mathrm{u}_m^\delta]}{2\sqrt{\rho_m}}\right) + \varepsilon\delta\frac{[\mathfrak{B}\mathrm{u}_n^\varepsilon, \mathrm{u}_n^\varepsilon][\mathfrak{B}\mathrm{u}_m^\delta, \mathrm{u}_m^\delta] - [\mathfrak{B}\mathrm{u}_n^\varepsilon, \mathrm{u}_m^\delta]^2}{4\sqrt{\rho_n\rho_m}} = 0\,, \tag{7}$$

where

$$[\mathfrak{B}\mathrm{u}_m^\delta, \mathrm{u}_n^\varepsilon] = \int_0^1 \varphi\left[\left(\varepsilon\delta\sqrt{\rho_n\rho_m} + \frac{l(l+1)}{r^2}\right)u_m u_n + u_m' u_n'\right]dr. \tag{8}$$

One observes the typical Krein space behavior. Intersections of spectral branches corresponding to Krein-space states of the same type ($\varepsilon = \delta$) induce no real-to-complex transitions in the spectrum, whereas intersections of spectral branches of different types ($\varepsilon \neq \delta$) may in general be accompanied by real-to-complex transitions, see Fig. 1b,c. The spectral deformations show a very pronounced *resonance behavior* along parabolas indicated by white and coloured dots in Fig. 1a — leaving spectral regions away from these resonance parabolas almost unaffected [2]. For example, in the case $l = 0$ we find for *cosine* perturbations that the k-harmonics affect only the unfolding of diabolical points located strictly on the associated parabolas with index $j = 2k$, Fig. 1b,e. The effect of *sine* perturbations with mode number k is shown in Fig. 1c,f. As predicted by (7), we find a strongly pronounced unfolding of diabolical points located on the parabolas with $|j| = 2k \pm 1$. The DPs with $|j| = 2k \pm m$, $m > 1$ are less affected and the strength of the unfolding quickly decreases with increasing distance m to the resonant parabolas, see Fig. 1f. Details of an estimation technique for overcritical oscillatory dynamo regimes are given in [2].

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