

# MULTIDIMENSIONAL COSMOLOGY AND ASYMPTOTICAL ADS

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**Abstract.** A non-linear gravitational model with a multidimensional geometry and quadratic scalar curvature is considered. For certain parameter ranges, the extra dimensions are stabilized if the internal spaces have negative constant curvature. As a consequence, the 4-dimensional effective cosmological constant as well as the bulk cosmological constant become negative. The homogeneous and isotropic external space is asymptotically AdS<sub>4</sub>. The connection between the D-dimensional and the 4-dimensional fundamental mass scales sets an additional restriction on the parameters of the considered non-linear models.

## 1. Introduction

The multidimensionality of our Universe follows naturally from theories unifying different fundamental interactions with gravity, e.g. M/string theory (Polchinski, 1998). This idea has received a great deal of renewed attention over the last few years within the ‘brane-world’ description of the Universe. In the ADD brane model (Arkani-Hamed et al., 1998), the geometry is assumed to be factorizable as in the standard Kaluza-Klein model, i.e., the topology is the direct product of a non-warped manifold of the external space-time and warped manifolds of the internal spaces where the warp factors are functions of the external coordinates.

According to observations the internal space should be static or nearly static at least from the time of primordial nucleosynthesis (otherwise the fundamental physical constants would vary). This means that at the present evolutionary stage of the Universe the compactification scale of the internal space should either be stabilized and trapped at the minimum of some effective potential, or it should be slowly varying (similar to the slowly varying cosmological constant in the quintessence scenario (Wang et al., 2000)). In both cases, small fluctuations over stabilized or slowly varying compactification scales (conformal scales/geometrical moduli) are possible.

String theory suggests that the usual linear Einstein-Hilbert action should be extended with higher order non-linear curvature terms. Within this context we investigate the problem of large extra dimensions stabilization in such theories. We find that the stabilization of extra dimensions takes place only if additional internal spaces have a compact hyperbolic geometry and the effective 4-dimensional cosmological constant is negative. Additionally, we show that requiring the extra



dimensions to be dynamically stabilized is a sufficient condition for the bulk space-time to acquire a constant negative curvature. In the case that the external space  $M_0$  is homogeneous and isotropic this implies that  $M_0$  becomes asymptotically anti de Sitter (see Günther et al., 2002 for additional details).

### 2. General Theory

We consider a  $D = (4 + D')$  – dimensional non-linear pure gravitational theory with the action  $S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|\bar{g}|} f(\bar{R})$ , where  $f(\bar{R})$  is an arbitrary smooth function of a scalar curvature  $\bar{R} = R[\bar{g}]$  constructed from the D-dimensional metric  $\bar{g}_{AB}$  ( $a, b = 1, \dots, D$ ) and  $\kappa_D^2 = 8\pi/M_{Pl(4+D')}$  is the D-dimensional gravitational constant. The equation of motion for this theory reads  $f' \bar{R}_{ab} - \frac{1}{2} f \bar{g}_{ab} - \bar{\nabla}_a \bar{\nabla}_b f' + \bar{g}_{ab} \bar{\square} f' = 0$  where  $f' = df/d\bar{R}$ ,  $\bar{R}_{ab} = R_{ab}[\bar{g}]$ ,  $\bar{\nabla}_a$  is the covariant derivative with respect to the metric  $\bar{g}_{ab}$  and the corresponding Laplacian is denoted by  $\bar{\square}$ . For  $f'(\bar{R}) > 0$  the conformal transformation  $g_{ab} = \Omega^2 \bar{g}_{ab}$ , with  $\Omega = [f'(\bar{R})]^{1/(D-2)}$  reduces the non-linear theory to a linear one with an additional scalar field. We can further write

$$G_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} g^{mn} \phi_{,m} \phi_{,n} - \frac{1}{2} g_{ab} e^{\frac{-D}{\sqrt{(D-2)(D-1)}\phi}} (\bar{R} f' - f) \tag{2.1}$$

and

$$\bar{\square} \phi = \frac{1}{\sqrt{(D-2)(D-1)}} e^{\frac{-D}{\sqrt{(D-2)(D-1)}\phi}} \left( \frac{D}{2} f - f' \bar{R} \right), \tag{2.2}$$

where

$$f' = \frac{df}{d\bar{R}} := e^{A\phi} > 0, \quad A := \sqrt{\frac{D-2}{D-1}}. \tag{2.3}$$

Let us consider what will happen if, in some way, the scalar field  $\phi$  tends asymptotically to a constant:  $\phi \rightarrow \phi_0 \equiv \text{const}$ . As it follows, in this limit the non-linear theory leads to a linear one:  $f(\bar{R}) \sim c_1 \bar{R} + c_2$  with  $c_1 = f' = \exp(A\phi_0)$  and a cosmological constant  $-c_2/(2c_1)$ . Thus, in the limit  $\phi \rightarrow \phi_0$  the D-dimensional theory is asymptotically (A)dS with scalar curvature:  $\bar{R} \rightarrow -\frac{D}{D-2} \frac{c_2}{c_1}$ . Clearly, the linear theory reproduces this asymptotic (A)dS-limit for  $\phi \rightarrow \phi_0$  with the scalar curvature:

$$R \rightarrow 2 \frac{D}{D-2} U(\phi_0) = -\frac{D}{D-2} c_2 c_1^{-\frac{D}{D-2}}. \tag{2.4}$$

Hence, in this limit  $\bar{R}/R \rightarrow c_1^{\frac{D}{D-2}}$  and  $f' = c_1$ . We shall consider in the following the quadratic theory:  $f(\bar{R}) = \bar{R} + \alpha \bar{R}^2 - 2\Lambda_D$ . For this theory we obtain

$$1 + 2\alpha \bar{R} = e^{A\phi} \iff \bar{R} = \frac{1}{2\alpha} (e^{A\phi} - 1) \tag{2.5}$$

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and

$$U(\phi) = \frac{1}{2}e^{-B\phi} \left[ \frac{1}{4\alpha} (e^{A\phi} - 1)^2 + 2\Lambda_D \right]. \tag{2.6}$$

The condition  $f' > 0$  implies  $1 + 2\alpha\bar{R} > 0$ .

### 3. Dimensional Reduction

To investigate the stabilization of the extra dimensions, we have to specify the topology of the bulk space-time manifold and the metric on it. Let the  $D$ -dimensional space-time manifold (bulk)  $M$  have the form:  $M = M_0 \times M_1 \times \dots \times M_n$  with the metric on  $M$

$$\bar{g} = \bar{g}_{ab}(X)dX^a \otimes dX^b = \bar{g}^{(0)} + \sum_{i=1}^n e^{2\bar{\beta}^i(x)} g^{(i)}. \tag{3.1}$$

The coordinates on the  $(D_0 = d_0 + 1)$ -dimensional manifold  $M_0$  (usually assumed as our  $(D_0 = 4)$ -dimensional Universe) are denoted by  $x$  and the corresponding metric by  $\bar{g}^{(0)} = \bar{g}_{\mu\nu}^{(0)}(x)dx^\mu \otimes dx^\nu$ . Let the internal factor manifolds  $M_i$  be  $d_i$ -dimensional Einstein spaces with metric  $g^{(i)} = g_{m_i n_i}^{(i)}(y_i)dy_i^{m_i} \otimes dy_i^{n_i}$ , i.e.,  $R_{m_i n_i} [g^{(i)}] = \lambda^i g_{m_i n_i}^{(i)}$ ,  $m_i, n_i = 1, \dots, d_i$ . For the metric ansatz the scalar curvature  $\bar{R}$  depends only on  $x$ :  $\bar{R}[\bar{g}] = \bar{R}(x)$ . Thus  $\phi$  is also a function of  $x$ :  $\phi = \phi(x)$ . Without loss of generality we set the compactification scales of the internal spaces at present time at  $\beta^i = 0$  ( $i = 1, \dots, n$ ).  $D' = D - D_0 = \sum_{i=1}^n d_i$  is the number of the extra dimensions).

### 4. Stabilization of the Internal Space

Let us consider the case of one internal space for simplicity\*. A conformal transformation can yield, without loss of generality, the corresponding action (see Günther et al., 2002 for further information):

$$S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ R[\tilde{g}^{(0)}] - \tilde{g}^{(0)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \tilde{g}^{(0)\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2U_{eff}(\varphi, \phi) \right\}, \tag{4.1}$$

where  $\varphi := -\sqrt{\frac{d_1(D-2)}{D_0-2}} \beta^1$  and

$$U_{eff}(\varphi, \phi) = e^{2\varphi \sqrt{\frac{d_1}{(D-2)(D_0-2)}}} \left[ -\frac{1}{2} R_1 e^{2\varphi \sqrt{\frac{D_0-2}{d_1(D-2)}}} + U(\phi) \right]. \tag{4.2}$$

\* The only difference between a general model with  $n > 1$  internal spaces and the particular one with  $n = 1$  consists in an additional diagonalization of the geometrical moduli excitations.

In order to obtain a stable compactification of the internal space, the potential  $U_{eff}(\varphi, \phi)$  should have a minimum with respect to  $\varphi$  and  $\phi$ . This is obvious with respect to the field  $\varphi$  because it is precisely the stabilization of this field that we aim to achieve. It is also clear that potential  $U_{eff}(\varphi, \phi)$  should have a minimum with respect to  $\phi$  because without stabilization of  $\phi$  the effective potential remains a dynamical function and the extremum condition  $\partial U_{eff}/\partial\phi|_{\varphi=0} = 0$  is not satisfied. Furthermore, the extrema of the potentials  $U_{eff}(\varphi, \phi)$  and  $U(\phi)$  with respect to the field  $\phi$  coincide with each other:

$$\frac{\partial U_{eff}}{\partial\phi} = e^{2\varphi\sqrt{\frac{d_1}{(D-2)(D_0-2)}}} \frac{\partial U(\phi)}{\partial\phi}. \quad (4.3)$$

Thus, the stabilization of the extra dimension takes place iff the field  $\phi$  goes to the minimum of the potential  $U(\phi)$ . In the general case  $\Lambda_D \neq 0$  it is easy to conclude that  $U|_{min} \geq 0$  for  $\Lambda_D \geq 0$  and  $U|_{min} < 0$  for  $\Lambda_D < 0$  (in the latter case  $-1 < 8\alpha\Lambda_D < 0$ ). Moreover, the extremum and minimum existence conditions imply that  $q = 8\alpha\Lambda_D > -1$  and  $\alpha > 0$ . It can then be shown (Günther et al., 2002) that the total potential  $U_{eff}(\varphi, \phi)$  also has a global minimum in the case when the potential  $U(\phi)$  has a negative minimum. Furthermore, the global minimum of  $U_{eff}$  is also negative as well as  $R_1$ , the scalar curvature of the internal space. The condition for the stabilization of the extra dimension thus leads asymptotically to a negative constant curvature bulk space-time. This takes place for  $\alpha > 0$  and  $-1 < 8\alpha\Lambda_D < 0$ . All other configurations are excluded. Moreover,  $\Lambda_{eff} \equiv U_{eff}|_{min}$  plays the role of a  $D_0$  - dimensional effective cosmological constant.

## 5. Conclusion and Discussion

We investigated a multidimensional gravitational model with a non-Einsteinian form of the action. In particular, we assumed that the action is an arbitrary smooth function of the scalar curvature  $f(R)$ . For such models, we concentrated our attention on the problem of extra dimension stabilization in the case of factorizable geometry. Conformal excitations described the internal space scale factors. A detailed stability analysis was carried out for a model with quadratic curvature term:  $f(R) = R + \alpha R^2 - 2\Lambda_D$ . It was shown that a stabilization is only possible for the parameter range  $-1 < 8\alpha\Lambda_D < 0$ .

This necessarily implies that the extra dimensions are stabilized if the compact internal spaces  $M_i$ ,  $i = 1, \dots, n$  have negative constant curvatures. The 4-dimensional cosmological constant (which corresponds to the minimum of the effective 4-dimensional potential) is also negative. As a consequence, the homogeneous and isotropic external ( $D_0 = 4$ )-dimensional space is asymptotically  $\text{AdS}_{D_0}$ .

From a cosmological perspective, it is of interest to consider the possibility of inflation for the 4-dimensional external space-time within our non-linear model.

For a linear multidimensional model with an arbitrary scalar field (inflaton) it can be shown that there is a possibility for inflation to occur if the scalar fields start to roll down from the region:

$$|U(\phi)| \geq |U(\phi)|_{min} \gg |R_1| e^{2\varphi \sqrt{\frac{D_0-2}{d_1(D-2)}}}, \tag{5.1}$$

where the effective potential reads

$$U_{eff} \approx e^{2\varphi \sqrt{\frac{d_1}{(D-2)(D_0-2)}}} U(\phi). \tag{5.2}$$

If  $e^{\sqrt{\frac{D-2}{D-1}}\varphi} \gg 1$  and hence  $U(\phi) \approx \frac{1}{8\alpha} e^{(2A-B)\phi} = \frac{1}{8\alpha} \exp \frac{D-4}{\sqrt{(D-2)(D-1)}}\phi$ , the slow-roll parameters  $\epsilon$  and  $\eta_{1,2}$  read  $\epsilon \approx \eta_1 \approx \eta_2 \approx \frac{2d_1}{(D-2)(D_0-2)} + \frac{(D-4)^2}{2(D-2)(D-1)}$ . For the dimensionality of our observable Universe  $D_0 = 4$ , these parameters are restricted to the range

$$\frac{3}{5} \leq \epsilon, \eta_1, \eta_2 \leq 1 \quad \text{for} \quad 6 \leq D \leq 10. \tag{5.3}$$

Thus, generally speaking, the slow-roll conditions for inflation are satisfied in this region. The scalar field  $\phi$  can act as inflaton and drive the inflation of the external space. It is clear that the estimates point only to the possibility for inflation to occur. For the considered model with negative effective cosmological constant inflation is not successfully completed if the reheating due to the decay of the  $\phi$ -field excitations and gravexcitons is not sufficient for a transition to the radiation dominated era. In any case, for scenarios with successful transition or without, the external space has a turning point at its maximal scale factor where the evolution changes from expansion to contraction (Felder et al., 2002). Obviously, for such models the negative effective cosmological constant forbids a late time acceleration of the Universe as indicated by recent observational data. This problem seems however to be cured if our model is generalized, e.g., by inclusion of additional form fields (Günther et al., 2002a).

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