Class. Quantum Grav. **15** (1998) 2025–2035. Printed in the UK PII: S0264-9381(98)91943-8

Multidimensional perfect-fluid cosmology with stable compactified internal dimensions

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Received 20 February 1998

Abstract. Multidimensional cosmological models in the presence of a bare cosmological constant and a perfect fluid are investigated under dimensional reduction to ($D_0 = 4$)-dimensional effective models. Stable compactification of the internal spaces is achieved for a special class of perfect fluids. The external space behaves in accordance with the standard Friedmann model. Necessary restrictions on the parameters of the models are found to ensure dynamical behaviour of the external (our) universe in agreement with observations.

PACS numbers: 0450, 9880H

1. Introduction

The large-scale dynamics of the observable part of our present time universe is well described by the Friedmann model with four-dimensional Friedmann–Robertson–Walker (FRW) metric. However, it is possible that spacetime at short (Planck) distances might have a dimensionality of more than four and possess a rather complex topology [1]. String theory [2] and its recent generalizations—p-brane, M- and F-theory [3, 4] widely use this concept and give it a new foundation. From this viewpoint, it is natural to generalize the Friedmann model to multidimensional cosmological models (MCM) with topology [5],

$$M = \mathbb{R} \times M_0 \times M_1 \times \cdots \times M_n, \tag{1.1}$$

where for simplicity the M_i ($i = 0, \ldots, n$) can be assumed to be d_i-dimensional Einstein spaces. M_0 usually denotes the ($d_0 = 3$)-dimensional external space. One of the main problems in MCM consists in the dynamical process leading from a stage with all dimensions developing on the same scale to the actual stage of the universe, where we have only four external dimensions and all internal spaces have to be compactified and contracted to sufficiently small scales, so that they are apparently unobservable. To make the internal dimensions unobservable at an actual stage of the universe we have to demand their contraction to scales near to the Planck length $L_{Pl} \sim 10^{-33}$ cm. Obviously, such a compactification should be stable. Recently [6], we found a class of MCM possessing stable compactification of extra dimensions.

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0264-9381/98/072025+11$19.50

On the other hand, any realistic MCM should provide a dynamical behaviour of the external spacetime in accordance with the observable universe. The phenomenological approach with a perfect fluid as a matter source is widely used in usual four-dimensional cosmology. According to present-day observations, the dynamical behaviour of the universe after inflation is well described by the standard Friedmann model [7] in the presence of a perfect fluid. Thus it might be worthwhile to generalize this approach to the description of the post-inflationary stage in multidimensional cosmological models. It is desirable to get models where, on one hand, the internal spaces are stably compactified near Planck scales and, on the other hand, the external universe behaves in accordance with the standard Friedmann model.

Here we present a toy MCM which shows a principal possibility to reach this goal. This model is beyond the scope of MCM with stable compactification found in [6]. The main difference consists in an additional time-dependent term in the effective potential that provides the needed dynamical behaviour of the external spacetime. This term is induced by a special type of fine tuning of the parameters of a multicomponent perfect fluid. Although such a fine tuning is a strong restriction on the matter content of the model, many important cases of physical interest are described by this class of perfect fluid. We note that a similar class of perfect fluids was considered in [8], where MCMs were integrated in the case of an absent cosmological constant and Ricci-flat internal spaces. As a result particular solutions with static internal spaces had been obtained. According to section 4 of the present paper these solutions are not stable and a bare cosmological constant and internal spaces with non-vanishing curvature are necessary conditions for their stabilization. Here we show that with the help of suitably chosen parameters the model can be further improved to solve two problems simultaneously. First, the internal spaces undergo stable compactification. Second, the external space behaves in accordance with the standard Friedmann model.

The paper is organized as follows. In section 2, the general description of the considered model is given. In section 3, the effective potential is obtained under dimensional reduction to a D_0-dimensional (usually $D_0 = 4$) effective theory in the Einstein frame. The problem of stable compactification is investigated in section 4 for a toy model with suitably chosen parameters. Here, it is shown that the external universe behaves as the standard Friedmann model. Conclusions and references complete the paper.

2. General description of the model

We consider a multidimensional cosmological model on a manifold (1.1) in the presence of a perfect fluid and a bare cosmological constant Λ. The metric of the model is parametrized as

$$g = g_{MN}\, \mathrm{d}X^M \otimes \mathrm{d}X^N = -\exp\left[2\gamma(\tau)\right] \mathrm{d}\tau \otimes \mathrm{d}\tau + \sum_{i=0}^{n} \exp\left[2\beta^i(\tau)\right] g_{(i)}. \qquad (2.1)$$

Manifolds M_i with the metrics $g_{(i)}$ are Einstein spaces of dimension d_i, i.e.

$$R_{mn}\left[g^{(i)}\right] = \lambda^i g^{(i)}_{mn}, \qquad m, n = 1, \ldots, d_i \qquad (2.2)$$

and

$$R\left[g^{(i)}\right] = \lambda^i d_i \equiv R_i. \qquad (2.3)$$

In the case of constant curvature spaces parameters λ^i are normalized as $\lambda^i = k_i(d_i - 1)$ with $k_i = \pm 1, 0$. The scalar curvature corresponding to the metric (2.1) reads

$$R = \sum_{i=0}^{n} R_i \exp\left(-2\beta^i\right) + \exp\left(-2\gamma\right) \sum_{i=0}^{n} d_i \left[2\ddot{\beta}^i - 2\dot{\gamma}\dot{\beta}^i + \left(\dot{\beta}^i\right)^2 + \dot{\beta}^i \sum_{j=0}^{n} d_j \dot{\beta}^j\right]. \tag{2.4}$$

Matter fields we take into account in a phenomenological way as a m-component perfect fluid with energy–momentum tensor

$$T_N^M = \sum_{a=1}^{m} T^{(a)}{}_N^M, \tag{2.5}$$

$$T^{(a)}{}_N^M = \mathrm{diag}\left(-\rho^{(a)}(\tau), \underbrace{P_0^{(a)}(\tau), \ldots, P_0^{(a)}(\tau)}_{d_0 \text{ times}}, \ldots, \underbrace{P_n^{(a)}(\tau), \ldots, P_n^{(a)}(\tau)}_{d_n \text{ times}}\right) \tag{2.6}$$

and equations of state

$$P_i^{(a)} = \left(\alpha_i^{(a)} - 1\right)\rho^{(a)}, \qquad i = 0, \ldots, n, \qquad a = 1, \ldots, m. \tag{2.7}$$

It is easy to see that physical values of $\alpha_i^{(a)}$ according to $-\rho^{(a)} \leqslant P_i^{(a)} \leqslant \rho^{(a)}$ run the region $0 \leqslant \alpha_i^{(a)} \leqslant 2$. The conservation equations we impose on each component separately

$$T^{(a)}{}_{N;M}^M = 0. \tag{2.8}$$

Denoting by an overdot differentiation with respect to time τ, these equations read for the tensors (2.6)

$$\dot{\rho}^{(a)} + \sum_{i=0}^{n} d_i \dot{\beta}^i \left(\rho^{(a)} + P_i^{(a)}\right) = 0 \tag{2.9}$$

and have according to (2.7) the simple integrals

$$\rho^{(a)}(\tau) = A^{(a)} \prod_{i=0}^{n} a_i^{-d_i \alpha_i^{(a)}}, \tag{2.10}$$

where $a_i \equiv \mathrm{e}^{\beta^i}$ are scale factors of M_i and $A^{(a)}$ are constants of integration. It is not difficult to verify that the Einstein equations with the energy–momentum tensor (2.5)–(2.10) are equivalent to the Euler–Lagrange equations for the Lagrangian [9, 10]

$$L = \tfrac{1}{2}\mathrm{e}^{-\gamma+\gamma_0} G_{ij} \dot{\beta}^i \dot{\beta}^j - \mathrm{e}^{\gamma+\gamma_0}\left(-\tfrac{1}{2}\sum_{i=0}^{n} R_i \mathrm{e}^{-2\beta^i} + \kappa^2 \sum_{a=1}^{m} \rho^{(a)} + \Lambda\right). \tag{2.11}$$

Here we use the notation $\gamma_0 = \sum_0^n d_i \beta^i$, Λ is a cosmological constant and κ^2 is a $\left(D = \sum_0^n d_i + 1\right)$-dimensional gravitational constant. The components of the minisuperspace metric read [5]

$$G_{ij} = d_i \delta_{ij} - d_i d_j. \tag{2.12}$$

The Lagrangian (2.11) can be obtained by dimensional reduction of the action

$$S = \frac{1}{2\kappa^2} \int_M \mathrm{d}^D x \sqrt{|g|} \{R[g] - 2\Lambda\} - \int_M \mathrm{d}^D x \sqrt{|g|} \rho + S_{YGH} = \frac{\mu}{\kappa^2} \int \mathrm{d}\tau\, L. \tag{2.13}$$

S_{YGH} is the standard York–Gibbons–Hawking boundary term and $\mu = \prod_{i=0}^n V_i$, where V_i is the volume of M_i (with unit scale factors): $V_i = \mathrm{vol}(M_i) = \int_{M_i} \mathrm{d}^{d_i} y \sqrt{\left|g^{(i)}\right|}$.

3. The effective potential

Let us slightly generalize this model to the inhomogeneous case supposing that the scale factors $\beta^i = \beta^i(x)$ $(i = 0, \ldots, n)$ are functions of the coordinates x, where x are defined on the $D_0 = (1 + d_0)$-dimensional external spacetime manifold $\bar{M}_0 = \mathbb{R} \times M_0$ with the metric

$$\bar{g}^{(0)} = \bar{g}^{(0)}_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathrm{e}^{2\gamma}\, \mathrm{d}\tau^2 + \mathrm{e}^{2\beta^0(x)} g^{(0)}. \tag{3.1}$$

After conformal transformation of the external spacetime metric from the Brans–Dicke to the Einstein frame:

$$\begin{aligned} g = g_{MN}\, \mathrm{d}X^M \otimes \mathrm{d}X^N &= \bar{g}^{(0)} + \sum_{i=1}^{n} \exp\left[2\beta^i(x)\right] g^{(i)} \\ &= \Omega^2 \tilde{g}^{(0)} + \sum_{i=1}^{n} \exp\left[2\beta^i(x)\right] g^{(i)}, \end{aligned} \tag{3.2}$$

where

$$\Omega^2 = \left(\prod_{i=1}^{n} \mathrm{e}^{d_i \beta^i} \right)^{-2/(D_0-2)}, \tag{3.3}$$

the dimensionally reduced action (2.13) reads

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} \mathrm{d}^{D_0}x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] - \bar{G}_{ij}\, \tilde{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - 2U_{eff} \right\}, \tag{3.4}$$

where $\kappa_0^2 = \kappa^2 / V_I$ is the D_0-dimensional gravitational constant, $V_I = \prod_{i=1}^n V_i$, $\bar{G}_{ij}$ is the midisuperspace metric with the components

$$\bar{G}_{ij} = d_i \delta_{ij} + \frac{1}{D_0 - 2} d_i d_j, \qquad i, j = 1, \ldots, n \tag{3.5}$$

and the effective potential U_{eff} reads

$$U_{eff} = \left(\prod_{i=1}^{n} \mathrm{e}^{d_i \beta^i} \right)^{-2/(D_0-2)} \left[-\frac{1}{2} \sum_{i=1}^{n} R_i \mathrm{e}^{-2\beta^i} + \Lambda + \kappa^2 \sum_{a=1}^{m} \rho^{(a)} \right]. \tag{3.6}$$

The effective action (3.4) has the form of a usual four-dimensional (if $d_0 = 3$) theory and describes a self-gravitating σ-model with self-interaction. The internal scale factors play the role of scalar fields (dilatons in the starting Brans–Dicke frame) satisfying the wave equation

$$\bar{G}_{ij} \Box \beta^j \equiv \frac{1}{\sqrt{|\tilde{g}^{(0)}|}} \partial_\mu \left(\sqrt{|\tilde{g}^{(0)}|}\, \bar{G}_{ij}\, \tilde{g}^{(0)\mu\nu} \partial_\nu \beta^j \right) = \frac{\partial U_{eff}}{\partial \beta^i}. \tag{3.7}$$

In the Einstein frame the theory assumes the most natural form [11, 12] and beginning from this point the external spacetime metric $\tilde{g}^{(0)}$ is considered as the physical one. For this metric we adopt following ansatz:

$$\tilde{g}^{(0)} = \Omega^{-2} \bar{g}^{(0)} = \tilde{g}^{(0)}_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = -\mathrm{e}^{2\tilde{\gamma}}\, \mathrm{d}\tilde{\tau} \otimes \mathrm{d}\tilde{\tau} + \mathrm{e}^{2\tilde{\beta}^0(x)} g^{(0)}. \tag{3.8}$$

Thus external scale factors in the Brans–Dicke frame $a_0 = \mathrm{e}^{\beta^0} \equiv a$ and in the Einstein frame $\tilde{a}_0 = \mathrm{e}^{\tilde{\beta}^0} \equiv \tilde{a}$ are connected with each other by the relation

$$a = \left(\prod_{i=1}^{n} \mathrm{e}^{d_i \beta^i} \right)^{-1/(D_0-2)} \tilde{a}. \tag{3.9}$$

The energy densities $\rho^{(a)}$ of the perfect-fluid components are given by (2.10) and with the help of relation (3.9) can be rewritten as

$$\rho^{(a)} = \rho_0^{(a)} \prod_{i=1}^{n} a_i^{-\xi_i^{(a)}}, \tag{3.10}$$

where

$$\rho_0^{(a)} = A^{(a)} \frac{1}{\tilde{a}^{\alpha_0^{(a)} d_0}} \tag{3.11}$$

and

$$\xi_i^{(a)} = d_i \left(\alpha_i^{(a)} - \frac{\alpha_0^{(a)} d_0}{d_0 - 1} \right). \tag{3.12}$$

In the case of one internal space ($n = 1$) the action and the effective potential are, respectively,

$$S = \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} \mathrm{d}^{D_0} x \sqrt{|\tilde{g}^{(0)}|} \left\{ \tilde{R}\left[\tilde{g}^{(0)}\right] - \tilde{g}^{(0)\mu\nu} \partial_\mu \varphi \, \partial_\nu \varphi - 2U_{eff} \right\} \tag{3.13}$$

and

$$U_{eff} = \mathrm{e}^{2\varphi[d_1/(D-2)(D_0-2)]^{1/2}} \left[-\tfrac{1}{2} R_1 \mathrm{e}^{2\varphi[(D_0-2)/d_1(D-2)]^{1/2}} + \Lambda + \kappa^2 \rho(\tilde{a}, \varphi) \right], \tag{3.14}$$

where we redefined the dilaton field as

$$\varphi \equiv -\sqrt{\frac{d_1(D-2)}{D_0 - 2}} \beta^1. \tag{3.15}$$

Let us split the scalar fields $\beta^i(x)$ in equations (3.4) and (3.6) into a background component $\bar{\beta}^i(x)$ and a small perturbational (fluctuation) component $\eta^i(x)$

$$\beta^i(x) = \bar{\beta}^i(x) + \eta^i(x). \tag{3.16}$$

Assuming that such a splitting procedure is well defined we get the corresponding equations of motion from (3.7) as

$$\Box \bar{\beta}^i = \left[\bar{G}^{-1}\right]^{ij} b_j(\bar{\beta}) \tag{3.17}$$

and

$$\Box \eta^i = \left[\bar{G}^{-1}\right]^{ij} a_{jk}(\bar{\beta}) \eta^k, \tag{3.18}$$

where

$$a_{ij} := \frac{\partial^2 U_{eff}}{\partial \beta^i \partial \beta^j}, \qquad b_i := \frac{\partial U_{eff}}{\partial \beta^i}. \tag{3.19}$$

With the help of an appropriate background depending $SO(n)$-rotation $S = S(\bar{\beta})$ we can diagonalize matrix $\left[\bar{G}^{-1} A\right]^i_k \equiv \left[\bar{G}^{-1}\right]^{ij} a_{jk}(\bar{\beta})$ and rewrite (3.18) in terms of normal modes $\psi = S^{-1}\eta$:

$$\tilde{g}^{(0)\mu\nu} \mathrm{D}_\mu \mathrm{D}_\nu \psi = S^{-1} \bar{G}^{-1} A S \psi \stackrel{def}{=} M^2 \psi, \tag{3.20}$$

where M^2 is a background depending diagonal mass matrix

$$M^2 = \mathrm{diag}\left[m_1^2(\bar{\beta}), \ldots, m_n^2(\bar{\beta})\right]. \tag{3.21}$$

D_μ denotes a covariant derivative

$$\mathrm{D}_\mu := \partial_\mu + \Gamma_\mu + A_\mu, \qquad A_\mu := S^{-1} \partial_\mu S \tag{3.22}$$

with $\Gamma_\mu + A_\mu$ as a connection on the fibre bundle $E\big(\bar{M}_0, \mathbb{R}^{D_0} \oplus \mathbb{R}^n\big)$ consisting of the base manifold $\bar{M}_0$ and vector spaces $\mathbb{R}^{D_0}_x \oplus \mathbb{R}^n_x = T_x\bar{M}_0 \oplus \big\{\big(\eta^1(x), \ldots, \eta^n(x)\big)\big\}$ as fibres. So, the background components $\bar{\beta}^i(x)$ via the effective potential U_{eff} and its Hessian a_{ij} play the role of a medium for the fluctuational components $\psi^i(x)$. Propagating in $\bar{M}_0$ filled with this medium the excitational modes (gravitational excitons [6]) change their masses as well as the direction of their 'polarization' defined by the unit vector in the fibre space

$$\xi(x) := \frac{\psi(x)}{|\psi(x)|} \in S^{n-1} \subset \mathbb{R}^n, \tag{3.23}$$

where S^{n-1} denotes the $(n-1)$-dimensional sphere. For considerations on interactions of gravitational excitons with gauge fields and corresponding observable effects we refer to [13].

We note that in the general case, when $m_i^2(\bar{\beta}) \neq m_j^2(\bar{\beta})$, $i \neq j$, due to the lack of $SO(n)$ invariance of (3.20) the connection A_μ itself cannot be interpreted as a $SO(n)$ gauge connection in pure gauge. This is only possible for $M^2 = m^2_{exci} I_n$, with I_n the unit matrix. Then a transformation

$$\begin{aligned} U\colon \psi &\mapsto \tilde{\psi} = U\psi \\ A_\mu &\mapsto \tilde{A}_\mu = UA_\mu U^{-1} - (\partial_\mu U)U^{-1} \\ \mathrm{D}_\mu &\mapsto \tilde{\mathrm{D}}_\mu = \partial_\mu + \Gamma_\mu + \tilde{A}_\mu \\ \mathrm{D}_\mu\psi &\mapsto \tilde{\mathrm{D}}_\mu\tilde{\psi} = U\mathrm{D}_\mu\psi \end{aligned} \tag{3.24}$$

leaves (3.20) invariant due to $M^2 \mapsto \tilde{M}^2 = UM^2U^{-1} = M^2$, and U is indeed a gauge transformation.

Further, from (3.20) it is clear that a consideration of the excitational modes makes only sense if the characteristic spacetime scales $L_{\bar{\beta}}$ and L_ψ of the variations of the background fields $\bar{\beta}^i$ and the excitons ψ^i are of different order: $L_{\bar{\beta}} \gg L_\psi$. (Otherwise non-perturbative techniques should be applied.) Covering the external spacetime with domains Ω_c of intermedium characteristic length $L_c \approx |\Omega_c|^{1/(d_0+1)}$, $L_{\bar{\beta}} \gg L_c \gg L_\psi$ we can in a crude approximation replace the background fields $\bar{\beta}^i(x)$ in Ω_c by constants $\bar{\beta}^i_c$. According to (3.17), (3.20) and due to the regularity of the midisuperspace metric $\bar{G}_{ij}$ this implies an extremum condition on the effective potential in Ω_c

$$\left.\frac{\partial U_{eff}}{\partial \beta^i}\right|_{\bar{\beta}_c} = 0, \tag{3.25}$$

as well as a vanishing connection $A_\mu = 0$ and the constancy of matrix M^2. The only extremum that provides the constancy of $\bar{\beta}^i_c$ under perturbations ψ^i is a minimum and the exciton masses must be non-negative $m^2_{(c)i} := m_i^2(\bar{\beta}_c) \geqslant 0$ with at least one of them strictly positive. (The case of $m^2_{(c)i} = 0$, $m^2_{(c)j} > 0$ for some i, j corresponds to degenerate minima, as, for example, given in Sombrero-like potentials. The massless modes are similar to Goldstone bosons.)

Models with constant background fields on $\Omega_c = \bar{M}_0$ and with effective potentials U_{eff} depending only on the internal scale factors have been considered in [6, 14]. The corresponding action functional reads in this case

$$\begin{aligned} S = \frac{1}{2\kappa_0^2}\int_{\bar{M}_0} \mathrm{d}^{D_0}x \sqrt{\left|\tilde{g}^{(0)}\right|}\big\{\tilde{R}\big[\tilde{g}^{(0)}\big] - 2\Lambda_{(c)eff}\big\} \\ + \sum_{i=1}^{n} \tfrac{1}{2}\int_{\bar{M}_0} \mathrm{d}^{D_0}x \sqrt{\left|\tilde{g}^{(0)}\right|}\big\{-\tilde{g}^{(0)\mu\nu}\psi^i_{,\mu}\psi^i_{,\nu} - m^2_{(c)i}\psi^i\psi^i\big\}, \end{aligned} \tag{3.26}$$

where the effective cosmological constant $\Lambda_{(c)eff}$ is connected with the stable compactification position $a_{(c)i} = \exp \bar{\beta}_c^i$ by the relation $\Lambda_{(c)eff} \equiv U_{eff}(\vec{\beta}_c)$. From a physical point of view it is clear that the effective potential should satisfy following conditions:

(i) $a_{(c)i} \gtrsim L_{Pl}$,

(ii) $m_{(c)i} \leqslant M_{Pl}$,

(iii) $\Lambda_{(c)eff} \to 0$.

The first condition expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. The second condition means that the curvature of the effective potential should be less than the Planckian one. Of course, gravitational excitons can be excited at the present time if $m_i \ll M_{Pl}$. The third condition reflects the fact that the cosmological constant at the present time is very small: $|\Lambda| \leqslant 10^{-56}\ \mathrm{cm}^{-2} \approx 10^{-121}\Lambda_{Pl}$ where $\Lambda_{Pl} = L_{Pl}^{-2}$. Strictly speaking, in the case that the potential has several minima ($c > 1$) we can demand $a_{(c)i} \sim L_{Pl}$ and $\Lambda_{(c)eff} \to 0$ only for one of the minima to which the present state of the universe corresponds. For all other minima it may be $a_{(c)i} \gg L_{Pl}$ and $|\Lambda_{(c)eff}| \gg 0$.

4. The model

A general analysis of the internal spaces stable compactification for MCM with the perfect fluid (2.10) is carried out in our paper [15]. In the present paper we investigate a particular class of effective potentials (3.6) with separating scale factor contributions from internal and external factor spaces

$$U_{eff} = \underbrace{\left(\prod_{i=1}^{n} \mathrm{e}^{d_i\beta^i}\right)^{-2/(D_0-2)} \left[-\tfrac{1}{2}\sum_{i=1}^{n} R_i \mathrm{e}^{-2\beta^i} + \Lambda\right]}_{U_{int}} + \underbrace{\kappa^2 \sum_{a=1}^{m} \rho_0^{(a)}}_{U_{ext}}. \tag{4.1}$$

We will show below, that such a separation, on the one hand, provides a stable compactification of the internal factor spaces due to a minimum of the first term $U_{int} = U_{int}\big(\beta^1, \ldots, \beta^n\big)$ as well as a dynamical behaviour of the external factor space due to $U_{ext} = U_{ext}\big(\tilde{\beta}^0\big)$. On the other hand, this separation crucially simplifies the calculations and allows an exact analysis. The price that we have to pay for the separation is a fine tuning of the parameters of the multicomponent perfect fluid

$$\begin{aligned} \alpha_0^{(a)} &= \frac{2}{d_0} + \frac{d_0 - 1}{d_0}\alpha^{(a)} \\ \alpha_i^{(a)} &= \alpha^{(a)}, \qquad i = 1, \ldots, n, \qquad a = 1, \ldots, m. \end{aligned} \tag{4.2}$$

Only in this case we have

$$\xi_i^{(a)} = -\frac{2d_i}{d_0 - 1} \tag{4.3}$$

yielding the compensation of the exponential prefactor for the perfect-fluid term in the effective potential (3.6). The corresponding components $\rho_0^{(a)}$ read, respectively,

$$\rho_0^{(a)} = A^{(a)} \frac{1}{\tilde{a}^{2+(d_0-1)\alpha^{(a)}}}. \tag{4.4}$$

Although the fine tuning (4.2) is a strong restriction, there exist some important particular models that belong to this class of multicomponent perfect fluids. For example, if $\alpha^{(a)} = 1$

the ath component of the perfect fluid describes radiation in the space M_0 and dust in the spaces $M_1, \ldots, M_n$. This kind of perfect fluid satisfies the condition $\sum_{i=0}^{n} d_i \alpha_i^{(a)} = D$ and is called super-radiation [16]. If $\alpha^{(a)} = 2$ we obtain the ultra-stiff matter in all M_i $(i = 0, \ldots, n)$ which is equivalent, for example, to a massless minimally coupled free scalar field. In the case where $\alpha^{(a)} = 0$ we get the equation of state $P_0^{(a)} = [(2-d_0)/d_0]\rho^{(a)}$ in the external space M_0 which describes a gas of cosmic strings if $d_0 = 3$: $P^{(a)} = -\frac{1}{3}\rho^{(a)}$ [17] and vacuum in the internal spaces $M_1, \ldots, M_n$. If $\alpha^{(a)} = \frac{1}{2}$ and $d_0 = 3$ we obtain dust in the external space M_0 and a matter with equation of state $P_i^{(a)} = -\frac{1}{2}\rho^{(a)}$ in the internal spaces M_i, $i = 1, \ldots, n$.

Let us first consider the conditions for the existence of a minimum of the potential $U_{int}\left(\beta^1, \ldots, \beta^n\right)$. According to [14] potentials U_{int} of type (4.1) have a single minimum if the bare cosmological constant and the curvature scalars of the internal spaces are negative $R_i, \Lambda < 0$. The scale factors $\left\{\beta_c^i\right\}_{i=1}^n$ at the minimum position of the effective potential are connected by a fine-tuning condition

$$\frac{R_i}{d_i} \mathrm{e}^{-2\beta_c^i} = \frac{2\Lambda}{D-2} \equiv \widetilde{C}, \qquad i = 1, \ldots, n \tag{4.5}$$

and the masses squared of the corresponding gravitational excitons are degenerate and given as

$$\begin{aligned} m_1^2 = \cdots = m_n^2 = m_{exci}^2 &= -\frac{4\Lambda}{D-2} \exp\left[-\frac{2}{d_0-1}\sum_{i=1}^{n} d_i \beta_c^i\right] \\ &= 2|\widetilde{C}|^{(D-2)/(d_0-1)} \prod_{i=1}^{n} \left|\frac{d_i}{R_i}\right|^{d_i/(d_0-1)}. \end{aligned} \tag{4.6}$$

Further, it was shown in [14] that the value of the potential U_{int} at the minimum is connected with the exciton mass by the relation

$$\Lambda_{int} := U_{int}\left(\beta_c^1, \ldots, \beta_c^n\right) = -\frac{d_0-1}{4} m_{exci}^2. \tag{4.7}$$

From equations (4.5) and (4.6) we see that exciton masses and minimum position $a_{(c)i} = \exp \bar{\beta}_c^i$ are constants that depend solely on the value of the bare cosmological constant Λ, the (constant) curvature scalars R_i and dimensions d_i of the internal factor spaces. This means that we have automatically $\Omega_c = \bar{M}_0$ from the very onset of the model. Hence the exciton approach in the present linear form breaks down only when the excitations become too strong so that higher-order terms must be included in the consideration or the phenomenological perfect-fluid approximation itself becomes inapplicable.

Let us now turn to the dynamical behaviour of the external factor space. For simplicity we consider the zero-order approximation, when all excitations are frozen, in the homogeneous case: $\tilde{\gamma} = \tilde{\gamma}(\tilde{\tau})$ and $\tilde{\beta} = \tilde{\beta}(\tilde{\tau})$. Then the action functional (3.26) with

$$U_{(c)eff} \equiv U_{eff}[\vec{\beta}_c, \tilde{\beta}(\tilde{\tau})] = U_{int}\left(\beta_c^1, \ldots, \beta_c^n\right) + U_{ext}[\tilde{\beta}(\tilde{\tau})] \equiv \Lambda_{int} + \bar{\rho}_0(\tilde{\tau}) \tag{4.8}$$

after dimensional reduction reads

$$\begin{aligned} S &= \frac{1}{2\kappa_0^2} \int_{\bar{M}_0} \mathrm{d}^{D_0} x \sqrt{|\tilde{g}^{(0)}|}\left\{\tilde{R}\left[\tilde{g}^{(0)}\right] - 2U_{(c)eff}\right\} \\ &= \frac{V_0}{2\kappa_0^2} \int \mathrm{d}\tilde{\tau} \left\{\mathrm{e}^{\tilde{\gamma}+d_0\tilde{\beta}} \mathrm{e}^{-2\tilde{\beta}} R\left[g^{(0)}\right] + d_0(1-d_0)\mathrm{e}^{-\tilde{\gamma}+d_0\tilde{\beta}} \left(\frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}\tilde{\tau}}\right)^2 - 2\mathrm{e}^{\tilde{\gamma}+d_0\tilde{\beta}}(\Lambda_{int} + \bar{\rho}_0)\right\} \\ &\quad + \frac{V_0}{2\kappa_0^2} d_0 \int \mathrm{d}\tilde{\tau} \frac{\mathrm{d}}{\mathrm{d}\tilde{\tau}} \left(\mathrm{e}^{-\tilde{\gamma}+d_0\tilde{\beta}} \frac{\mathrm{d}\tilde{\beta}}{\mathrm{d}\tilde{\tau}}\right), \end{aligned} \tag{4.9}$$

where usually $R\big[g^{(0)}\big] = kd_0(d_0-1)$, $k = \pm 1, 0$. The constraint equation $\partial L/\partial\tilde{\gamma} = 0$ in the synchronous time gauge $\tilde{\gamma} = 0$ yields

$$\left(\frac{1}{\tilde{a}}\frac{\mathrm{d}\tilde{a}}{\mathrm{d}\tilde{t}}\right)^2 = -\frac{k}{\tilde{a}^2} + \frac{2}{d_0(d_0-1)}(\Lambda_{int} + \bar{\rho}_0(\tilde{a})), \tag{4.10}$$

which results in

$$\begin{aligned}\tilde{t} + \text{constant} &= \int \frac{\mathrm{d}\tilde{a}}{\left[-k + (2\Lambda_{int}/d_0(d_0-1))\tilde{a}^2 + \left(2\kappa^2/d_0(d_0-1)\right)\sum_{a=1}^m A^{(a)}/\tilde{a}^{(d_0-1)\alpha^{(a)}}\right]^{1/2}},\\ &= \int \frac{\mathrm{d}\tilde{a}}{\left[-k + \frac{1}{3}\Lambda_{int}\tilde{a}^2 + \frac{1}{3}\kappa^2\sum_{a=1}^m A^{(a)}/\tilde{a}^{2\alpha^{(a)}}\right]^{1/2}},\end{aligned} \tag{4.11}$$

where in the last line we put $d_0 = 3$.

Thus, in the zero-order approximation we arrived at a Friedmann model in the presence of a negative cosmological constant Λ_{int} and a multicomponent perfect fluid. The perfect fluid has the form of a gas of cosmic strings for $\alpha^{(a)} = 0$, dust for $\alpha^{(a)} = \frac{1}{2}$ and radiation for $\alpha^{(a)} = 1$. As $0 \leqslant \alpha^{(a)} \leqslant 2$, the cosmological constant plays a role only for large $\tilde{a}$ and because of the negative sign of Λ_{int} the universe has a turning point at the maximum of $\tilde{a}$. To be consistent with present time observation we should take

$$|\Lambda_{int}| \leqslant 10^{-121}\Lambda_{Pl}. \tag{4.12}$$

We note that due to (4.11) and in contrast with (3.26) the minimum value $U_{(c)eff}$ of the effective potential in (4.8) cannot be interpreted as a cosmological constant, even as a time-dependent one. Coming back to the gravitational excitons we see that according to (4.7) the upper bound (4.12) on the effective cosmological constant leads to ultra-light particles with mass $m_{exci} \leqslant 10^{-60}M_{Pl} \sim 10^{-32}$ eV. This is much less than the cosmic background radiation temperature at the present time $T_0 \sim 10^{-4}$ eV. It is clear that up to the present time such light particles behave as radiation and can be taken into account as an additional term $\rho_r = \kappa_0^2 A_r/3\tilde{a}^2$ in (4.11). It can be easily seen that we reconstruct the standard scenario if we consider the one-component ($m = 1$) case with $\alpha^{(1)} = \frac{1}{2}$, $\kappa^2 A^{(1)} \sim 10^{61}$ and $\kappa_0^2 A_r \sim 10^{117}$. Here we have at early stages a radiation-dominated universe and a dust-dominated universe at later stages of its evolution.

For completeness we note that via equations (4.6) and (4.7) the value of the effective cosmological constant has a crucial influence on the relation between the compactification scales of the internal factor spaces and their dimensions. In the case of only one internal negative curvature space $M_1 = H^{d_1}/\Gamma$ with $R_1 = -d_1(d_1-1)$ and compactification scale $a_{(c)1} = 10L_{Pl}$ we have, for example, the relation [14] $\Lambda_{int} = -(d_1-1)10^{-2(d_1+2)}L_{Pl}$, so that the bound (4.12) implies a dimension of this space of at least $d_1 = 59$. Taking instead of one internal space a set of two-dimensional hyperbolic g-tori $\left\{M_i = H^2/\Gamma\right\}_{i=1}^n$ [18] with compactification scale $a_{(c)i} = 10^2 L_{Pl}$ it is easy to check [14] that we need at least $n = 29$ such spaces to satisfy (4.12).

Of course, other values of the cosmological constant lead to other exciton masses and compactification–dimensionality relations. So, it is also possible to get models with much heavier gravitational excitons. For $\Lambda_{int} = -10^{-8}\Lambda_{Pl}$ we have, for example, $m = 10^{-4}M_{Pl}$ and the excitons are very heavy particles that should be considered as cold dark matter. If we take the one-component case $\alpha^{(1)} = 1$ we get at early times a radiation-dominated universe with a smooth transition to a cold dark-matter-dominated universe at later stages. But for this example it is necessary to introduce a mechanism that provides a reduction of the huge cosmological constant to the observable value $10^{-121}\Lambda_{Pl}$.

5. Conclusion

In the present paper we considered multidimensional cosmological models (MCM) with a bare cosmological constant and a perfect fluid as a matter source. It can be easily seen that there are only two classes of perfect fluids with stably compactified internal spaces. These kind of solutions are of utmost interest because an absence of a time variation of the fundamental constants in experiments [19, 20] shows that at the present time the extra dimensions, if they exist, should be static or nearly static.

The first class (see [6, 14]) consists of models with $\alpha_0^{(a)} = 0$. It leads to the vacuum equation of state in the external space M_0. All other $\alpha_i^{(a)}$ $(i = 1, \ldots, n)$ can take arbitrary values. This model can be used for a phemenological description of a multidimensional inflationary universe with smooth transition to a matter-dominated stage.

In the present paper we found a second class of perfect-fluid models with stable internal spaces. For these models the stability is induced by a fine tuning of the equation of state of the perfect fluid in the external and internal spaces (4.2). This class includes many important particular models and allows considerations of perfect fluids with different equations of state in the external space, among them also those that result in a Friedmann-like dynamics. Thus, this class of models can be applied for the description of the post-inflationary stage in multidimensional cosmology. For the models considered we found necessary restrictions on the parameters which, on one hand, ensure stable compactification of the internal spaces near the Planck length and, on the other hand, guarantee dynamical behaviour of the external (our) universe in accordance with the standard scenario for the Friedmann model.

This toy model gives a promising example of a multidimensional cosmological model which is not in contradiction with observations. Although, fine tuning is necessary to get an effective cosmological constant in accordance with the present-day observations. An interesting example of four-dimensional theories was presented in the recent papers [21, 22]. Here, the authors construct a metric-affine-measure theory to allow the integration measure in the action functional to be determined dynamically. With this approach they obtain an effective scalar field potential with zero minimum which needs no fine tuning of the original scalar field potential. One of the main differences between this theory and our effective four-dimensional model consists in the fact that in the latter case the scalar field is not an arbitrary matter field but has a pure geometrical nature (the scalar field is proportional to the logarithm of the internal scale factor) and its effective potential is determined uniquely by the form of the higher-dimensional action.

Acknowledgments

We thank S Shabanov and A Schakel for useful discussions during the preparation of this paper and the referee for drawing our attention to the papers [21, 22]. AZ thanks the Institute of Mathematics of the Potsdam University and the Institute of Theoretical Physics of the Berlin Free University for hospitality. The work was partially supported by DFG, grant 436 UKR 113.

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