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AdS and stabilized extra dimensions in multi-dimensional gravitational models with nonlinear scalar curvature terms R^{-1} and R^4

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Abstract

We study multi-dimensional gravitational models with scalar curvature nonlinearities of types R^{-1} and R^4. It is assumed that the corresponding higher dimensional spacetime manifolds undergo a spontaneous compactification to manifolds with a warped product structure. Special attention has been paid to the stability of the extra-dimensional factor spaces. It is shown that for certain parameter regions the systems allow for a freezing stabilization of these spaces. In particular, we find for the R^{-1} model that configurations with stabilized extra dimensions do not provide a late-time acceleration (they are AdS), whereas the solution branch which allows for accelerated expansion (the dS branch) is incompatible with stabilized factor spaces. In the case of the R^4 model, we obtain that the stability region in parameter space depends on the total dimension $D = \dim(M)$ of the higher dimensional spacetime M. For $D > 8$ the stability region consists of a single (absolutely stable) sector which is shielded from a conformal singularity (and an antigravity sector beyond it) by a potential barrier of infinite height and width. This sector is smoothly connected with the stability region of a curvature-linear model. For $D < 8$ an additional (metastable) sector exists which is separated from the conformal singularity by a potential barrier of finite height and width so that systems in this sector are prone to collapse into the conformal singularity. This

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second sector is not smoothly connected with the first (absolutely stable) one. Several limiting cases and the possibility of inflation are discussed for the R^4 model.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Distance measurements of type Ia supernovas (SNe, Ia) [1] as well as cosmic microwave background (CMB) anisotropy measurements [2] performed during the past few years give strong evidence for the existence of dark energy—a smooth energy density with negative pressure which causes an accelerated expansion of the universe at the present time. This late-time acceleration stage should have started approximately 5 billion years ago and represents a second acceleration epoch after inflation which lasted for 10^{-35} s immediately after the big bang and ended 13.7 billion years ago.

The challenge to theoretical cosmology consists in finding a natural explanation of inflation and a late-time acceleration (dark energy) within the framework of string theory/M-theory or loop quantum gravity. Scenarios which address one or both of these issues are, e.g., string (pre-big bang) cosmology [3], a large number of brane-world scenarios (brane inflation [4], the ekpyrotic [5] and born-again universe [6] scenario, a late-time acceleration via *S*-branes [7]), as well as the string theory/M-theory scenarios with flux compactifications [8–12] and the recent set-ups of loop quantum cosmology [13].

A different starting point for explaining the late-time acceleration was taken in [14, 15], where it was shown that purely gravitational modifications of the (curvature-linear) Einstein–Hilbert action by including curvature-nonlinear terms of type $\bar{R}^{-1}$, $\bar{R}^{-n}$ could induce a positive effective cosmological constant, and with it an accelerated expansion. The corresponding phenomenological studies were performed in four spacetime dimensions by transforming the curvature-nonlinear theory into an equivalent curvature-linear theory with the nonlinearity degrees of freedom carried by an additional dynamical scalar field. (The used technique had been developed in earlier work since the 1980s [16–19].) A natural question which arises with regard to this phenomenological approach is whether it can naturally follow as a low-energy limit from some M-theory set-up or—looking from down-side up—whether it has a physically viable phenomenological extension to higher dimensions.

In the present paper, we take the latter (phenomenological) point of view and study higher dimensional extensions of purely gravitational non-Einsteinian models with scalar curvature nonlinearities of types $\bar{R}^{-1}$ and $\bar{R}^4$. Special emphasis will be laid on finding parameter regions (regions in moduli space) which ensure the existence of at least one minimum of the effective potential for the volume moduli of the internal spaces and which in this way allow for their stabilization. The latter fact is of special importance because the extra-dimensional space components should be static or nearly static at least from the time of primordial nucleosynthesis (otherwise the fundamental physical constants would vary, see, e.g., [20, 21][5], and observational bounds on the variation of the fine-structure constant could be violated [23]). During the last few years, problems of volume moduli stabilization have been studied, e.g., for models with large extra dimensions (Arkani-Hamed–Dimopoulos–Dvali (ADD) set-ups [24])

[5] First discussions of this subject date back to [22].

[25][6] as well as (more recently) for M-theory scenarios with flux compactifications [9–12, 27, 28] and for brane-gas systems [29].

Here we will mainly follow our earlier work on this subject [23, 30–34], where the stabilization of extra dimensions was studied for $(D_0 + D')$-dimensional bulk spacetimes with a product topology. The corresponding product manifolds are constructed from Einstein spaces M_i with scale (warp) factors which depend only on the coordinates of the external D_0-dimensional spacetime M_0 (the ansatz resembles a zero-mode approximation in the Kaluza–Klein formalism). As a consequence, the excitations of the scale factors (conformal excitations/excitations of the volume moduli) have the form of massive scalar fields (gravitational excitons/radions) living in the external spacetime. Stabilized volume moduli will correspond to positive eigenvalues of the mass matrix of these scalar fields, unstable configurations to tachyonic excitations.

The present work can be understood as a direct continuation of our earlier investigation on volume moduli stabilization in $\bar{R}^2$ models of purely geometrical type [33] as well as with magnetic (solitonic, Freund–Rubin type) form fields living in the extra dimensions (flux field compactifications) [34]. Its key results can be summarized as follows:

- A straightforward extension of a four-dimensional purely geometrical $\bar{R}^{-1}$ model to higher dimensions with subsequent dimensional reduction cannot simultaneously provide a late-time acceleration and a stabilization of the extra dimensions. A late-time acceleration is only possible for a solution branch which has a positive definite maximum of the effective potential, i.e. a positive definite effective cosmological constant, and not a negative definite minimum as would be required for a stabilization of the internal factor space components. This means that other, more sophisticated, extension scenarios would be needed to reach both goals simultaneously.
- In contrast to the $\bar{R}^2$ models of [33, 34], the $\bar{R}^4$ model shows a rich substructure of the stability region in parameter (moduli) space which crucially depends on the total dimension $D = D_0 + D'$ of the bulk spacetime. There exists one stability sector which is present for all dimensions $D \geqslant D_0 + 2$ and which smoothly tends to the stability sector of an $\bar{R}$ linear model when the $\bar{R}$ nonlinearity is switched off. This sector is shielded from the (probably unphysical) antigravity sector of the theory by a potential barrier of infinite height and width, and hence, it will be absolutely stable with regard to transitions of the system into the antigravity sector. Apart from this sector of absolute stability, there exists a second sector for total dimensions $D < 8$ which is separated from a conformal singularity and an antigravity sector beyond it by a potential barrier of finite height and width. Systems in this sector will only be metastable and prone to collapse into the conformal singularity and the antigravity sector. The metastable sector is separated from the stability sector of the $\bar{R}$ linear model by an essential singularity in $\bar{R}$ and the effective potential U_{eff} (when $\bar{R}$ and U_{eff} are considered as functions over a parameter subspace).
- In the analysed purely geometrical $\bar{R}$ nonlinear models, a stabilization of the internal factor spaces is necessarily connected with an AdS structure of the external spacetime. In order to uplift the AdS to a dS with a positive effective cosmological constant and the possibility of a late-time acceleration, additional matter fields should be included in the model (e.g., flux fields as in the $\bar{R}^2$ model of [34]).

The paper is structured as follows. In section 2, we present a brief technical outline of the transformation from a non-Einsteinian purely gravitational model with general scalar curvature nonlinearity of type $f(\bar{R})$ to an equivalent curvature-linear model with an additional nonlinearity carrying scalar field. Afterwards, we derive in section 3 criteria which ensure the

[6] See also [26].

existence of at least one minimum for the effective potential of the internal space scale factors (volume moduli). These criteria are then used in sections 4 and 5 to obtain the regions in parameter (moduli) space which allow for a freezing stabilization of the scale factors in models with scalar curvature nonlinearities of type $\bar{R}^{-1}$ and $\bar{R}^4$. The main results are summarized in section 6.

2. The general set-up

We consider a $D = (4 + D')$-dimensional nonlinear pure gravitational theory with the action functional

$$S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|\bar{g}|} f(\bar{R}), \tag{1}$$

where $f(\bar{R})$ is an arbitrary smooth function with mass dimension $\mathcal{O}(m^2)$ (m has the unit of mass) of a scalar curvature $\bar{R} = R[\bar{g}]$ constructed from the D-dimensional metric $\bar{g}_{ab}(a, b = 1, \ldots, D)$. D' is the number of extra dimensions and κ_D^2 denotes the D-dimensional gravitational constant which is connected with the fundamental mass scale $M_{*(4+D')}$ and the surface area $S_{D-1} = 2\pi^{(D-1)/2}/\Gamma[(D-1)/2]$ of a unit sphere in $D - 1$ dimensions by the relation [35–37]

$$\kappa_D^2 = 2S_{D-1}\big/ M_{*(4+D')}^{2+D'}. \tag{2}$$

The equation of motion for the theory (1) reads (see, e.g., [16–18])

$$f'\bar{R}_{ab} - \tfrac{1}{2} f \bar{g}_{ab} - \bar{\nabla}_a \bar{\nabla}_b f' + \bar{g}_{ab} \bar{\Box} f' = 0 \tag{3}$$

and has as trace

$$(D-1)\bar{\Box} f' = \frac{D}{2} f - f'\bar{R}. \tag{4}$$

We use the notation $\bar{\nabla}_a$ and $\bar{\Box}$ for the covariant derivative and the Laplacian with respect to the metric $\bar{g}_{ab}$, as well as the abbreviations $f' = df/d\bar{R}$, $\bar{R}_{ab} = R_{ab}[\bar{g}]$.

Before we endow the metric of the pure gravity theory (1) with an explicit structure, we recall that this $\bar{R}$ nonlinear theory is equivalent to a theory which is linear in another scalar curvature R but which contains an additional self-interacting scalar field. According to standard techniques [16–18], the corresponding R linear theory has the action functional,

$$S = \frac{1}{2\kappa_D^2} \int_M d^D x \sqrt{|g|} [R[g] - g^{ab} \phi_{,a} \phi_{,b} - 2U(\phi)], \tag{5}$$

where

$$f'(\bar{R}) = \frac{df}{d\bar{R}} := e^{A\phi} > 0, \qquad A := \sqrt{\frac{D-2}{D-1}}, \tag{6}$$

and where the self-interaction potential $U(\phi)$ of the scalar field ϕ is given by

$$U(\phi) = \frac{1}{2}(f')^{-D/(D-2)}[\bar{R} f' - f], \tag{7}$$

$$= \frac{1}{2} e^{-B\phi}[\bar{R}(\phi) e^{A\phi} - f(\bar{R}(\phi))], \qquad B := \frac{D}{\sqrt{(D-2)(D-1)}}. \tag{8}$$

The metrics $g_{ab}, \bar{g}_{ab}$ and the scalar curvatures $R, \bar{R}$ of the two theories (1) and (5) are conformally connected by the relations

$$g_{ab} = \Omega^2 \bar{g}_{ab} = [f'(\bar{R})]^{2/(D-2)} \bar{g}_{ab} \tag{9}$$

and

$$R = (f')^{2/(2-D)} \left\{ \bar{R} + \frac{D-1}{D-2}(f')^{-2}\bar{g}^{ab}\partial_a f'\partial_b f' - 2\frac{D-1}{D-2}(f')^{-1}\bar{\Box} f' \right\} \tag{10}$$

via the scalar field $\phi = \ln[f'(\bar{R})]/A$. This scalar field ϕ carries the nonlinearity degrees of freedom in $\bar{R}$ of the original theory, and for brevity we call it the nonlinearity field.

Next, we assume that the D-dimensional bulk spacetime M undergoes a spontaneous compactification[7] to a warped product manifold

$$M = M_0 \times M_1 \times \cdots \times M_n \tag{11}$$

with metric

$$\bar{g} = \bar{g}_{ab}(X)\,\mathrm{d}X^a \otimes \mathrm{d}X^b = \bar{g}^{(0)} + \sum_{i=1}^{n} \mathrm{e}^{2\bar{\beta}^i(x)} g^{(i)}. \tag{12}$$

The coordinates on the $(D_0 = d_0 + 1)$-dimensional manifold M_0 (usually interpreted as our observable $(D_0 = 4)$-dimensional universe) are denoted by x and the corresponding metric by

$$\bar{g}^{(0)} = \bar{g}^{(0)}_{\mu\nu}(x)\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu. \tag{13}$$

For simplicity, we choose the internal factor manifolds M_i as d_i-dimensional Einstein spaces with metrics $g^{(i)} = g^{(i)}_{m_i n_i}(y_i)\,\mathrm{d}y_i^{m_i} \otimes \mathrm{d}y_i^{n_i}$, so that the relations

$$R_{m_i n_i}[g^{(i)}] = \lambda^i g^{(i)}_{m_i n_i}, \qquad m_i, n_i = 1, \ldots, d_i \tag{14}$$

and

$$R[g^{(i)}] = \lambda^i d_i \equiv R_i \tag{15}$$

hold. The specific metric ansatz (12) leads to a scalar curvature $\bar{R}$ which depends only on the coordinates x of the external space: $\bar{R}[\bar{g}] = \bar{R}(x)$. Correspondingly, also the nonlinearity field ϕ depends on x only: $\phi = \phi(x)$.

Passing from the $\bar{R}$ nonlinear theory (1) to the equivalent R linear theory (5) the metric (12) undergoes the conformal transformation $\bar{g} \mapsto g$ (see relation (9))

$$g = \Omega^2 \bar{g} = (\mathrm{e}^{A\phi})^{2/(D-2)}\bar{g} := g^{(0)} + \sum_{i=1}^{n} \mathrm{e}^{2\beta^i(x)} g^{(i)} \tag{16}$$

with

$$g^{(0)}_{\mu\nu} := (\mathrm{e}^{A\phi})^{2/(D-2)}\bar{g}^{(0)}_{\mu\nu}, \qquad \beta^i := \bar{\beta}^i + \frac{A}{D-2}\phi. \tag{17}$$

3. Freezing stabilization

The main subject of our subsequent considerations will be the stabilization of the internal space components. A strong argument in favour of stabilized or almost stabilized internal space scale factors $\bar{\beta}^i(x)$, at the present evolution stage of the universe, is given by the intimate relation between variations of these scale factors and those of the fine-structure constant α [23]. The strong restrictions on α variations in the currently observable part of the universe [39] imply a correspondingly strong restriction on these scale factor variations [23]. For this reason, we will concentrate below on the derivation of criteria which will ensure a freezing stabilization of the scale factors. Extending earlier discussions of models with $\bar{R}^2$ scalar curvature nonlinearities [33, 34] we will investigate here models of the nonlinearity types $\bar{R}^{-1}$ and $\bar{R}^4$.

[7] For a discussion of possible decompactification scenarios we refer to the recent work [38].

In [32] it was shown that for models with a warped product structure (12) of the bulk spacetime M and a minimally coupled scalar field living on this spacetime, the stabilization of the internal space components requires a simultaneous freezing of the scalar field. Here we expect a similar situation with a simultaneous freezing stabilization of the scale factors $\beta^i(x)$ and the nonlinearity field $\phi(x)$. According to (17), this will also imply a stabilization of the scale factors $\bar{\beta}^i(x)$ of the original nonlinear model.

In general, the model will allow for several stable scale factor configurations (minima in the landscape over the space of volume moduli). We choose one of them[8], denote the corresponding scale factors as β_0^i, and work further with the deviations

$$\hat{\beta}^i(x) = \beta^i(x) - \beta_0^i \tag{18}$$

as the dynamical fields[9]. After the dimensional reduction of the action functional (5) we pass from the intermediate Brans–Dicke frame to the Einstein frame via a conformal transformation

$$g^{(0)}_{\mu\nu} = \hat{\Omega}^2 \hat{g}^{(0)}_{\mu\nu} = \left(\prod_{i=1}^n \mathrm{e}^{d_i\hat{\beta}^i}\right)^{-2/(D_0-2)} \hat{g}^{(0)}_{\mu\nu} \tag{19}$$

with respect to the scale factor deviations $\hat{\beta}^i(x)$ [33, 34, 36]. As a result, we arrive at the following action

$$S = \frac{1}{2\kappa^2_{D_0}} \int_{M_0} \mathrm{d}^{D_0}x \sqrt{|\hat{g}^{(0)}|}\{\hat{R}[\hat{g}^{(0)}] - \bar{G}_{ij}\hat{g}^{(0)\mu\nu}\partial_\mu\hat{\beta}^i\partial_\nu\hat{\beta}^j - \hat{g}^{(0)\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U_{\mathrm{eff}}\}, \tag{20}$$

which contains the scale factor offsets β_0^i through the total internal space volume

$$V_{D'} \equiv V_I \times v_0 \equiv \prod_{i=1}^n \int_{M_i} \mathrm{d}^{d_i}y\sqrt{|g^{(i)}|} \times \prod_{i=1}^n \mathrm{e}^{d_i\beta_0^i} \tag{21}$$

in the definition of the effective gravitational constant $\kappa^2_{D_0}$ of the dimensionally reduced theory

$$\kappa^2_{(D_0=4)} = \kappa^2_D \big/ V_{D'} = 8\pi \big/ M_4^2 \quad\Longrightarrow\quad M_4^2 = \frac{4\pi}{S_{D-1}} V_{D'} M_{*(4+D')}^{2+D'}. \tag{22}$$

Obviously, at the present evolution stage of the universe, the internal space components should have a total volume which would yield a four-dimensional mass scale of the order of the Planck mass $M_{(4)} = M_{Pl}$. The tensor components of the midisuperspace metric (the target space metric on $\mathbb{R}^n_T$) $\bar{G}_{ij}(i,j = 1,\ldots,n)$, its inverse metric $\bar{G}^{ij}$ and the effective potential are given as [40, 41]

$$\bar{G}_{ij} = d_i\delta_{ij} + \frac{1}{D_0-2}d_i d_j, \qquad \bar{G}^{ij} = \frac{\delta^{ij}}{d_i} + \frac{1}{2-D}. \tag{23}$$

The effective potential has the explicit form

$$U_{\mathrm{eff}}(\hat{\beta},\phi) = \left(\prod_{i=1}^n \mathrm{e}^{d_i\hat{\beta}^i}\right)^{-\frac{2}{D_0-2}} \left[-\frac{1}{2}\sum_{i=1}^n \hat{R}_i\,\mathrm{e}^{-2\hat{\beta}^i} + U(\phi)\right], \tag{24}$$

where we abbreviated

$$\hat{R}_i := R_i \exp\left(-2\beta_0^i\right). \tag{25}$$

[8] Although the toy model ansatz (1) is highly oversimplified and far from a realistic model, we can roughly think of the chosen minimum, e.g., as the one which we expect to correspond to the current evolution stage of our observable universe.

[9] For simplicity, we work here with stabilized scale factors β_0^i which we assume as homogeneous and constant. In general, one can split the scale factors $\beta^i(x)$, e.g., into a coherent scale factor background $\beta_0^i(x)$ and non-coherent scale factor fluctuations $\hat{\beta}^i(x) = \beta^i(x) - \beta_0^i(x)$ over this background [31].

For completeness, we note that the original metric $\bar{g}_{ab}$ of the $\bar{R}$ nonlinear model and the final Einstein frame metric $\hat{g}_{\mu\nu}$ of the dimensionally reduced model are connected by the relation

$$\bar{g}_{ab} = (f')^{-\frac{2}{D-2}} \left[\left(\prod_{i=1}^{n} \mathrm{e}^{d_i \hat{\beta}^i} \right)^{-\frac{2}{D_0-2}} \hat{g}^{(0)}_{\mu\nu} + \sum_{i=1}^{n} \mathrm{e}^{2\beta^i} g^{(i)} \right] \tag{26}$$

which up to the nonlinearity induced conformal factor $(f')^{-2/(D-2)}$ takes (for scale factors depending only on the time coordinate) a similar form as in the recently analysed cosmological S-brane models of [7, 42].

A freezing stabilization of the internal spaces will be achieved[10] if the effective potential has at least one minimum with respect to the fields $\hat{\beta}^i(x)$. Assuming, without loss of generality, that one of the minima is located at $\beta^i = \beta^i_0 \Rightarrow \hat{\beta}^i = 0$, we get the extremum condition:

$$\left.\frac{\partial U_{\mathrm{eff}}}{\partial \hat{\beta}^i}\right|_{\hat{\beta}=0} = 0 \quad \Longrightarrow \quad \hat{R}_i = \frac{d_i}{D_0-2} \left(-\sum_{j=1}^{n} \hat{R}_j + 2U(\phi) \right). \tag{27}$$

From its structure (a constant on the lhs and a dynamical function of $\phi(x)$ on the rhs) it follows that a stabilization of the internal space scale factors can only occur when the nonlinearity field $\phi(x)$ is stabilized as well. In our freezing scenario this will require a minimum with respect to ϕ:

$$\left.\frac{\partial U(\phi)}{\partial \phi}\right|_{\phi_0} = 0 \quad \Longleftrightarrow \quad \left.\frac{\partial U_{\mathrm{eff}}}{\partial \phi}\right|_{\phi_0} = 0. \tag{28}$$

We arrived at a stabilization problem, some of the general aspects of which have been analysed already in [30–34]. For brevity, we only summarize the corresponding essentials as they will be needed for more detailed discussions in the next sections:

(i) Equation (27) implies that the scalar curvatures $\hat{R}_i$ and with them the compactification scales $\mathrm{e}^{\beta^i_0}$ (see relation (25)) of the internal space components are finely tuned

$$\frac{\hat{R}_i}{d_i} = \frac{\hat{R}_j}{d_j}, \qquad i, j = 1, \ldots, n. \tag{29}$$

(ii) The masses of the normal mode excitations of the internal space scale factors (gravitational excitons/radions) and of the nonlinearity field ϕ near the minimum position are given as [32]

$$m_1^2 = \cdots = m_n^2 = -\frac{4}{D-2} U(\phi_0) = -2\frac{\hat{R}_i}{d_i} > 0, \tag{30}$$

$$m_\phi^2 := \left.\frac{\mathrm{d}^2 U(\phi)}{\mathrm{d}\phi^2}\right|_{\phi_0} > 0. \tag{31}$$

(iii) The value of the effective potential at the minimum plays the role of an effective 4D cosmological constant of the external (our) spacetime M_0:

$$\Lambda_{\mathrm{eff}} := \left. U_{\mathrm{eff}} \right|_{\substack{\hat{\beta}^i=0,\\ \phi=\phi_0}} = \frac{D_0-2}{D-2} U(\phi_0) = \frac{D_0-2}{2} \frac{\hat{R}_i}{d_i}. \tag{32}$$

[10] An alternative stabilization mechanism can consist, e.g., in the recently proposed dynamical stabilization in the vicinity of enhanced symmetry points [43]. In our present discussion we will not analyse such scenarios.

(iv) Relation (32) implies

$$\operatorname{sign}\Lambda_{\rm eff} = \operatorname{sign} U(\phi_0) = \operatorname{sign} R_i. \tag{33}$$

Together with condition (30) this shows that in a pure geometrical model stable configurations can only exist for internal spaces with negative curvature[11] $R_i < 0$ ($i = 1, \ldots, n$). Additionally, the effective cosmological constant $\Lambda_{\rm eff}$ as well as the minimum of the potential $U(\phi)$ should be negative too:

$$\Lambda_{\rm eff} < 0, \qquad U(\phi_0) < 0. \tag{34}$$

(v) Equations (21), (22), (25) and (29)–(32) yield the following scaling behaviour of the minimum related model parameters under a change of one of the offset scale factors $\beta^i_{0(1)} \longrightarrow \beta^i_{0(2)} := \ln(\lambda)\beta^i_{0(1)}$ as it will be induced, e.g., by a change of the minimum value $U(\phi_{0(1)}) \longrightarrow U(\phi_{0(2)})$:

$$\mathrm{e}^{\beta^i_{0(2)}} = \lambda\, \mathrm{e}^{\beta^i_{0(1)}} \qquad \Longrightarrow \qquad \mathrm{e}^{\beta^k_{0(2)}} = \lambda\, \mathrm{e}^{\beta^k_{0(1)}} \quad \forall k, \tag{35}$$

$$\frac{m^2_{k(1)}}{m^2_{k(2)}} = \frac{\hat{R}_{k(1)}}{\hat{R}_{k(2)}} = \frac{\Lambda_{\rm eff(1)}}{\Lambda_{\rm eff(2)}} = \frac{U(\phi_{0(1)})}{U(\phi_{0(2)})} = \lambda^2, \tag{36}$$

$$\frac{\kappa^2_{(D_0=4),(1)}}{\kappa^2_{(D_0=4),(2)}} = \frac{V_{D'(2)}}{V_{D'(1)}} = \lambda^{D'} = \left[\frac{U(\phi_{0(1)})}{U(\phi_{0(2)})}\right]^{D'/2}. \tag{37}$$

(vi) For a system which is almost stabilized at a freezing point $\phi \approx \phi_0$, the function $f(\bar{R})$ can be split into a constant background and small deviations,

$$f(\bar{R}) \approx c_1(\bar{R} - \bar{R}_0) + f(\bar{R}_0) \equiv c_1\bar{R} + c_2, \tag{38}$$

where $c_1 := f'(\bar{R}_0) = \exp(A\phi_0)$, $\bar{R}_0 \equiv \bar{R}(\phi_0)$, and $-c_2/(2c_1)$ plays the role of a cosmological constant. In the case of homogeneous and isotropic spacetime manifolds, linear purely geometrical theories with a constant Λ term necessarily imply an anti-de Sitter/de Sitter geometry so that the manifolds are Einstein spaces. Substitution of $f(\bar{R}) \to c_1\bar{R} + c_2$ into equation (3) proves this fact directly

$$\bar{R}_{ab} \longrightarrow -\frac{1}{D-2}\frac{c_2}{c_1}\bar{g}_{ab} \qquad \Longrightarrow \qquad \bar{R} \longrightarrow -\frac{D}{D-2}\frac{c_2}{c_1}. \tag{39}$$

Plugging the potential $U(\phi)$ from equation (8) into the minimum conditions (28), (31) yields with the help of $\partial_\phi \bar{R} = Af'/f''$ the conditions

$$\left.\frac{\mathrm{d}U}{\mathrm{d}\phi}\right|_{\phi_0} = \left.\frac{A}{2(D-2)}(f')^{-D/(D-2)}h\right|_{\phi_0} = 0, \qquad h := Df - 2\bar{R}f', \qquad \Longrightarrow \qquad h(\phi_0) = 0, \tag{40}$$

$$\left.\frac{\mathrm{d}^2U}{\mathrm{d}\phi^2}\right|_{\phi_0} = \frac{1}{2}A\,\mathrm{e}^{(A-B)\phi_0}[\partial_\phi\bar{R} + (A-B)\bar{R}]_{\phi_0} = \left.\frac{1}{2(D-1)}(f')^{-2/(D-2)}\frac{1}{f''}\partial_{\bar{R}}h\right|_{\phi_0} > 0, \tag{41}$$

[11] Negative constant curvature spaces M_i are compact if they have a quotient structure, $M_i = H^{d_i}/\Gamma_i$, where H^{d_i} and Γ_i are hyperbolic spaces and their discrete isometry group, respectively.

where the last inequality can be reshaped into the suitable form

$$f''\partial_{\bar{R}}h|_{\phi_0} = f''[(D-2)f' - 2\bar{R}f'']_{\phi_0} > 0. \tag{42}$$

Furthermore, we find from equation (40)

$$U(\phi_0) = \frac{D-2}{2D}(f')^{-\frac{2}{D-2}}\bar{R}(\phi_0) \tag{43}$$

so that (34) leads to the additional restriction

$$\bar{R}(\phi_0) < 0 \tag{44}$$

at the extremum. Via relation (10) the stabilization point curvatures of the $\bar{R}$ nonlinear and the R linear models are connected as

$$R_0 \approx (f')^{2/(2-D)}\bar{R}_0. \tag{45}$$

Thus, as the extra-dimensional scale factors approach their stability position the bulk spacetime curvature asymptotically (dynamically) tends to a negative constant value (see equation (44)). Because the effective cosmological constant is also negative ($\Lambda_{\text{eff}} < 0$), the homogeneous and isotropic external ($D_0 = 4$)-dimensional spacetime is asymptotically AdS_{D_0}. Together with the compact hyperbolic internal spaces $M_i = H^{d_i}/\Gamma_i$ this results in a spontaneous compactification scenario

$$\text{AdS}_D \longrightarrow \text{AdS}_{D_0} \times H^{d_1}/\Gamma_1 \times \cdots \times H^{d_n}/\Gamma_n. \tag{46}$$

In the next sections we will analyse the conditions (29)–(41), (44) on their compatibility with particular scalar curvature nonlinearities $f(\bar{R})$.

4. The R^{-1} model

Recently, it has been shown in [14] that cosmological models with a nonlinear scalar curvature term of type $\bar{R}^{-1}$ can provide a possible explanation of the observed late-time acceleration of our universe within a pure gravity set-up. The equivalent linearized model contains an effective potential with a positive branch which can simulate a transient inflation-like behaviour in the sense of an effective dark energy. The corresponding considerations have been performed mainly in four dimensions[12]. Here we extend these analyses to higher dimensional models—assuming that the scalar curvature nonlinearity is of the same form in all dimensions. We start from a nonlinear coupling of the type

$$f(\bar{R}) = \bar{R} - \mu/\bar{R}, \qquad \mu > 0. \tag{47}$$

In front of the $\bar{R}^{-1}$ term, the minus sign is chosen, because otherwise the potential $U(\phi)$ will have no extremum.

With the help of definition (6), we express the scalar curvature $\bar{R}$ in terms of the nonlinearity field ϕ and obtain two real-valued solution branches

$$\bar{R}_\pm = \pm\sqrt{\mu}(\text{e}^{A\phi} - 1)^{-1/2} \quad \Longrightarrow \quad \phi > 0 \tag{48}$$

of the quadratic equation $f'(\bar{R}) = \text{e}^{A\phi}$. The corresponding potentials

$$U_\pm(\phi) = \pm\sqrt{\mu}\,\text{e}^{-B\phi}\sqrt{\text{e}^{A\phi} - 1} \tag{49}$$

have extrema for curvatures (see equation (40))

$$\bar{R}_{0,\pm} = \pm\sqrt{\mu}\sqrt{\frac{D+2}{D-2}} \qquad \text{e}^{A\phi_0} = \frac{2B}{2B-A} = \frac{2D}{D+2} > 1 \qquad \text{for} \quad D \geqslant 3 \tag{50}$$

[12] A discussion of pro and contra of a higher dimensional origin of $\bar{R}^{-1}$ terms can be found in [15].

and take for these curvatures the values

$$U_\pm(\phi_0) = \pm\sqrt{\mu}\sqrt{\frac{D-2}{D+2}}\,\mathrm{e}^{-B\phi_0} = \pm\sqrt{\mu}\sqrt{\frac{D-2}{D+2}}\left(\frac{2D}{D+2}\right)^{-D/(D-2)}. \tag{51}$$

The stability defining second derivatives (equation (41)) at the extrema (50),

$$\begin{aligned}\partial_\phi^2 U_\pm\big|_{\phi_0} &= \mp\sqrt{\mu}\frac{D}{D-1}\sqrt{\frac{D+2}{D-2}}\,\mathrm{e}^{B\phi_0}\\ &= \mp\sqrt{\mu}\frac{D}{D-1}\sqrt{\frac{D+2}{D-2}}\left(\frac{2D}{D+2}\right)^{-D/(D-2)},\end{aligned} \tag{52}$$

show that only the negative curvature branch $\bar{R}_-$ yields a minimum with stable internal space components. The positive branch has a maximum with $U_+(\phi_0) > 0$. According to (33) it can provide an effective dark energy contribution with $\Lambda_{\mathrm{eff}} > 0$, but due to its tachyonic behaviour with $\partial_\phi^2 U(\phi_0) < 0$ it cannot give stably frozen internal dimensions. This means that the simplest extension of the four-dimensional purely geometrical $\bar{R}^{-1}$ set-up of [14] to higher dimensions is incompatible with a freezing stabilization of the extra dimensions. A possible circumvention of this behaviour could consist in the existence of different nonlinearity types $f_i(\bar{R}_i)$ in different factor spaces M_i so that their dynamics can decouple one from the other. This could allow for a freezing of the scale factors of the internal spaces even in the case of a late-time acceleration with $\Lambda_{\mathrm{eff}} > 0$. Another circumvention could consist of a mechanism which prevents the dynamics of the internal spaces from causing strong variations of the fine-structure constant α. The question of whether one of these schemes could work within a physically realistic set-up remains to be clarified.

Finally, we note that in the minimum of the effective potential $U_{\mathrm{eff}}(\phi, \beta^i)$, which is provided by the negative curvature branch $\bar{R}(\phi)$, one finds excitation masses for the gravexcitons/radions and the nonlinearity field (see equations (30), (51) and (52)) of order

$$m_1 = \cdots = m_n \sim m_\phi \sim \mu^{1/4}. \tag{53}$$

For the four-dimensional effective cosmological constant Λ_{eff} defined in (32) one obtains in accordance with equation (51) $\Lambda_{\mathrm{eff}} \sim -\sqrt{\mu}$.

5. The R^4 model

In this section, we analyse a model with a curvature-quartic correction term of the type

$$f(\bar{R}) = \bar{R} + \gamma\bar{R}^4 - 2\Lambda_D. \tag{54}$$

This set-up contains no quadratic curvature terms and can be understood as a very rough approximate analogue of specific curvature corrected models[13] of M-theory (see, e.g., [19, 45, 46]). The investigation will be performed for an arbitrary number of dimensions, D.

We start by deriving the explicit form of the potential $U(\phi)$. For this purpose, we substitute $f(\bar{R})$ from (54) into relation (6),

$$f' = \mathrm{e}^{A\phi} = 1 + 4\gamma\bar{R}^3, \tag{55}$$

and resolve the latter equation for $\bar{R}$:

$$\bar{R} = (4\gamma)^{-1/3}(\mathrm{e}^{A\phi} - 1)^{1/3}, \qquad -\infty < \phi < \infty. \tag{56}$$

The potential $U(\phi)$ is then found from equation (8) as

$$U(\phi) = \tfrac{1}{2}\,\mathrm{e}^{-B\phi}\left[\tfrac{3}{4}(4\gamma)^{-1/3}(\mathrm{e}^{A\phi} - 1)^{4/3} + 2\Lambda_D\right]. \tag{57}$$

[13] The role of curvature-quartic corrections in M-theory inflation scenarios was recently discussed in [44].

From its form with

$$U[\phi; \Lambda_D > 0, \gamma > 0] \geqslant 0 \quad \forall\phi \tag{58}$$

we immediately conclude that the minimum condition $U(\phi_0) < 0$ cannot be satisfied in the sector $(\Lambda_D > 0, \gamma > 0)$. In the remaining sectors, the potential will have a minimum if the extremum condition (40) and the minimum-ensuring inequality (42) are fulfilled simultaneously. In the present case, these conditions read

$$h[\Lambda_D, \gamma, \bar{R}] := [\gamma(D-8)\bar{R}^4 + (D-2)\bar{R} - 2D\Lambda_D]_{\phi_0} = 0 \tag{59}$$

and

$$f''\partial_{\bar{R}}h = 12\gamma\bar{R}^2[(D-2) + 4(D-8)\gamma\bar{R}^3]_{\phi_0} > 0, \tag{60}$$

respectively. From their structure, it follows that the dimension $D = 8$ will constitute an exceptional class of models (due to the cancellation of the highest order terms in (59), (60)). We will analyse this class of models in section 5.2.

5.1. Dimensions $D \neq 8$

Equation (59), $h[\Lambda_D, \gamma, \bar{R}] = 0$, is an algebraic equation in the variables $(\Lambda_D, \gamma, \bar{R})$ which defines a two-dimensional algebraic variety $\mathcal{V} \subset \mathcal{M}$ in the three-dimensional parameter space $\mathcal{M} = \mathbb{R}^3 \ni (\Lambda_D, \gamma, \bar{R})$. On this variety inequality (60) together with the restrictions (44) and (6)

$$\bar{R} < 0, \qquad f' = 1 + 4\gamma\bar{R}^3 > 0 \tag{61}$$

selects the parameter subset $\Upsilon \subset \mathcal{V}$ of stably compactified internal space configurations. Choosing Λ_D and γ as independent parameters, our main task will consist in obtaining the projection $\Theta_{(\Lambda_D,\gamma)} := \pi\Upsilon$ of the stability region $\Upsilon \subset \mathcal{V} \subset \mathcal{M}$ onto the (Λ_D, γ) plane. (By π we denote the projection itself.) Most of the information will be derived by finding restrictions on (Λ_D, γ) from the conditions which ensure the reality of $\bar{R}$ as a solution of the algebraic equation (59).

In order to obtain the solutions $\bar{R}$ of equation (59) explicitly, we follow standard techniques (see, e.g., [47] and appendix A) and consider first the associated cubic equation

$$u^3 + \frac{8D\Lambda_D}{\gamma(D-8)}u - \left[\frac{D-2}{\gamma(D-8)}\right]^2 = 0 \tag{62}$$

and its discriminant Q:

$$Q = r^2 + q^3, \qquad q := \frac{8D\Lambda_D}{3\gamma(D-8)}, \qquad r := \frac{1}{2}\left[\frac{D-2}{\gamma(D-8)}\right]^2. \tag{63}$$

Depending on the sign of Q, the cubic equation (62) has one real solution for $Q > 0$ or three real solutions for $Q \leqslant 0$, where in the case $Q = 0$ at least two of these solutions coincide. Denoting (one of) the real solution(s) by u_1, the four roots of the quartic equation (59) can then be obtained according to (A.11) and (A.12) as solutions of the two quadratic equations

$$\bar{R}^2 \pm \sqrt{u_1}\bar{R} + \tfrac{1}{2}\left(u_1 \pm \epsilon\sqrt{u_1^2 + 3q}\right) = 0 \tag{64}$$

with

$$\epsilon = -\mathrm{sign}\left(\frac{D-2}{\gamma(D-8)}\right). \tag{65}$$

Physically sensible solutions will correspond to the real roots of these equations.

Following this general scheme of analysis, we start from the discriminant Q which we rewrite for later convenience as

$$Q = r^2(1+z), \qquad z = z(\Lambda_D, \gamma) := q^3/r^2 = 4\gamma(8\Lambda_D/3)^3 w(D),$$
$$w(D) := \frac{D^3(D-8)}{(D-2)^4}. \tag{66}$$

In appendix B it is shown that the minimum-ensuring inequality (60) implies

$$z(\Lambda_D, \gamma) > -1 \tag{67}$$

so that Q is necessarily positive definite, $Q > 0$, and, hence, equation (62) has only one real-valued root [47]

$$u_1 = [r + Q^{1/2}]^{1/3} + [r - Q^{1/2}]^{1/3}$$
$$= r^{1/3} v_1(z) > 0, \tag{68}$$

where

$$v_1(z) := [1 + (1+z)^{1/2}]^{1/3} + [1 - (1+z)^{1/2}]^{1/3}. \tag{69}$$

It is now an easy task to explicitly analyse the pair of quadratic equations (64). They will only have real-valued roots, if at least one of the corresponding discriminants $\Delta_\pm$ is non-negative

$$\Delta_\pm = -u_1 \mp 2\epsilon\sqrt{u_1^2 + 3q} \geqslant 0. \tag{70}$$

(The subscripts $\pm$ in $\Delta_\pm$ correspond to the signs in equation (64).) Because of $u_1 > 0$ this holds only for

$$\Delta_{-\epsilon} = -u_1 + 2\sqrt{u_1^2 + 3q} \tag{71}$$

and under the additional reality-ensuring requirement

$$u_1^2 + 3q \geqslant 0. \tag{72}$$

Using the definitions (66) and (69) of z and $v_1(z)$, the inequalities $\Delta_{-\epsilon} \geqslant 0$ and (72) can be reshaped into the form

$$H_1(z) := v_1^6(z) + 64z \geqslant 0 \tag{73}$$

$$H_2(z) := v_1^6(z) + 27z \geqslant 0, \tag{74}$$

respectively.

From the graphics of the functions $H_{1,2}(z)$ depicted in figure 1, we read off that both inequalities (73) and (74) are satisfied for $z \geqslant -1$ and no additional restrictions are set by them on the allowed region of the parameter z.

Hence, we arrive at the result that the real-valued roots of the quartic equation (59) follow from equations (64) by identifying in them $\pm = -\epsilon$. The corresponding quadratic equation reads

$$\bar{R}^2 - \epsilon\sqrt{u_1}\bar{R} + \tfrac{1}{2}\left(u_1 - \sqrt{u_1^2 + 3q}\right) = 0 \tag{75}$$

and has solutions (we distinguish them again by subscripts $\pm$)

$$\bar{R}_{\epsilon,\pm} = \tfrac{1}{2}\left(\epsilon\sqrt{u_1} \pm \sqrt{2\sqrt{u_1^2 + 3q} - u_1}\right)$$
$$= r^{1/6} T_{\epsilon,\pm}(z), \tag{76}$$

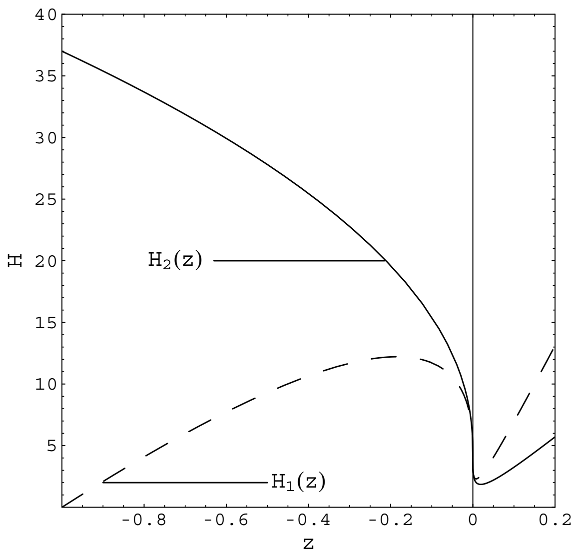

Figure 1. Auxiliary functions $H_1(z)$ and $H_2(z)$, which for $H_{1,2}(z) \geqslant 0$ ensure a non-negative discriminant $\Delta_{-\epsilon} \geqslant 0$ of the quadratic equations (64).

where

$$T_{\epsilon,\pm}(z) := \frac{1}{2}\left(\epsilon\sqrt{v_1} \pm \sqrt{2\sqrt{v_1^2 + 3z^{1/3}} - v_1}\right). \tag{77}$$

These roots are defined over the complete parameter region $z(\Lambda_D, \gamma) > -1$ so that further restrictions on (Λ_D, γ) can only follow from the additional requirements (61).

The first of these requirements, $\bar{R} < 0$, should be fulfilled for a successful freezing stabilization of the extra-dimensional factor spaces. From the structure of (76) and (77) we read off that $\bar{R}_{+,+}$ contradicts this bound, whereas for the remaining solutions it partially narrows the allowed parameter region as follows:

$$\begin{aligned} \bar{R}_{+,-}: &\quad 0 < z, \\ \bar{R}_{-,+}: &\quad -1 < z < 0, \\ \bar{R}_{-,-}: &\quad -1 < z. \end{aligned} \tag{78}$$

The second inequality, $f' = 1 + 4\gamma\bar{R}^3 > 0$, of (61) is analysed in appendix C and maps into the following parameter restrictions:

$$D < 8: \quad z < -w(D) = |w(D)|, \tag{79}$$

$$D > 8: \quad -w(D) < z. \tag{80}$$

So far, we have performed our analysis mainly in terms of the function $z(\Lambda_D, \gamma)$ and, hence, in terms of projections of the bounds (60) and $f' > 0$ on the (Λ_D, γ) plane. For completeness, we have to test whether all of the projected segments of $\mathcal{V}$ over the allowed region of the (Λ_D, γ) plane fulfil the additional bound[14] $\bar{R} < 0$, i.e., we should re-analyse (60) and $f' > 0$ in terms of $\bar{R}$. We start with (60). A simple case analysis gives

[14] The restrictions (78) are only necessary conditions for the existence of solutions $\bar{R} < 0$ of the quartic equation, and provide only partial information about (60) and $f' > 0$.

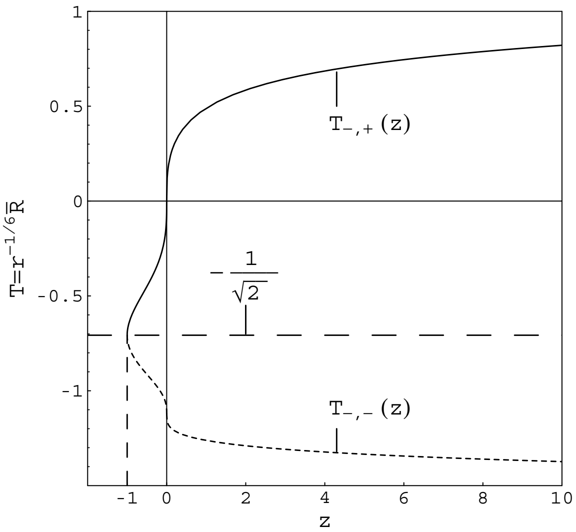

Figure 2. Rescaled scalar curvatures $T_{-,\pm}(z) := r^{-1/6}\bar{R}_{-,\pm}$.

$$\gamma > 0: \quad D > 8, \quad \epsilon = -: \quad \bar{R}^3 > -\left|\frac{1}{4\gamma}\frac{D-2}{D-8}\right|, \tag{81}$$

$$D < 8, \quad \epsilon = +: \quad \bar{R}^3 < \left|\frac{1}{4\gamma}\frac{D-2}{D-8}\right|, \tag{82}$$

$$\gamma < 0: \quad D > 8, \quad \epsilon = +: \quad \bar{R}^3 > \left|\frac{1}{4\gamma}\frac{D-2}{D-8}\right| > 0, \tag{83}$$

$$D < 8, \quad \epsilon = -: \quad \bar{R}^3 < -\left|\frac{1}{4\gamma}\frac{D-2}{D-8}\right| \tag{84}$$

and shows that configurations with $D > 8, \gamma < 0$ violate the bound $\bar{R} < 0$, whereas (82) is weaker than $\bar{R} < 0$. The remaining two inequalities (81) and (84) can be reshaped with the help of (77) and

$$\left|\frac{1}{4\gamma}\frac{D-2}{D-8}\right| = 2^{-3/2}r^{1/2} \tag{85}$$

(from equation (63)) as

$$T_{-,\pm} > -2^{-1/2}, \qquad T_{-,\pm} < -2^{-1/2}, \tag{86}$$

respectively. From the graphics of the functions $T_{-,\pm}(z)$ shown in figure 2 we read off that $T_{-,+}(z) > -2^{-1/2}$, $T_{-,-}(z) < -2^{-1/2}$ for $z > -1$. Hence, inequality (81) is fulfilled only by $\bar{R}_{-,+}$, whereas (84) selects $\bar{R}_{-,-}$. A comparison of the inequalities (81)–(84) with the condition $f' = 1 + 4\gamma\bar{R}^3 > 0$ and its implications

$$\gamma > 0: -(4\gamma)^{-1} < \bar{R}^3, \qquad \gamma < 0: \bar{R}^3 < |4\gamma|^{-1} \tag{87}$$

shows that the latter relations (87) are compatible with them and add no additional restrictions.

Combining the information about the inequalities (81)–(84) with (58) and relations (78)–(80) we arrive at the following parameter regions for configurations with a possible freezing stabilization of extra-dimensional (internal) factor spaces:

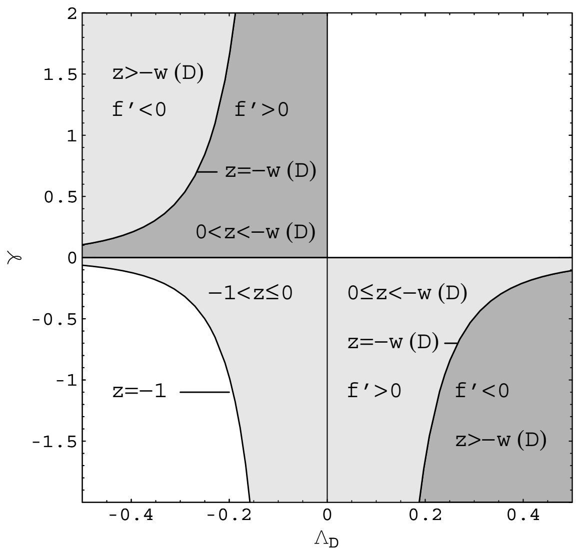

Figure 3. Projection $\Theta_{(\Lambda_D,\gamma)}$ of the stability region $\Upsilon \subset \mathcal{V} \subset \mathcal{M}$ of a subcritical model with $D < 8$ on the (Λ_D, γ) plane (shaded areas with $f' > 0$). The two curves $z(\Lambda_D, \gamma) = -w(D)$, given in equation (C.4), correspond to the conformal singularity $f' = 0$ where $\bar{R} \to -(4\gamma)^{-1/3}$ maps into $R \to -\infty$. They separate parameter regions with $U(\phi \to -\infty) \to +\infty$ (dark grey) from the regions with $U(\phi \to -\infty) \to -\infty$ (light grey). For $f' > 0$ the corresponding regions ensure the existence of an absolutely stable minimum (dark grey) and a metastable minimum (light grey, see the discussion in section 5.3). The $(f' < 0)$ regions have been included in the graphics for completeness.

$$D > 8: \quad \gamma > 0: \quad \bar{R}_{-,+}, \quad -w(D) < z < 0, \tag{88}$$

$$\gamma < 0: \quad \text{no stability}, \tag{89}$$

$$D < 8: \quad \gamma > 0: \quad \bar{R}_{+,-}, \quad 0 < z < |w(D)|, \tag{90}$$

$$\gamma < 0: \quad \bar{R}_{-,-}, \quad -1 < z < |w(D)|. \tag{91}$$

This result is shown schematically in figures 3 and 4. (The regions of formal extension to configurations with $f' < 0$ are included in the graphics for reasons of completeness. They are briefly discussed in section 5.3.)

5.2. The exceptional dimension $D = 8$

In this particular case, the $\bar{R}$ nonlinear terms in the minimum-ensuring conditions (59) and (60) cancel[15] and the bounds on the solution can be read off immediately:

$$\bar{R} = \tfrac{8}{3}\Lambda_D < 0, \qquad \gamma > 0. \tag{92}$$

($\bar{R} < 0$ follows from relation (43) and condition (44).) The restriction $f' = 1 + 4\gamma\bar{R}^3 > 0$ sets the same additional bound on the allowed parameter region

$$-1 < 4\gamma\left(\tfrac{8}{3}\Lambda_D\right)^3 < 0 \tag{93}$$

as (80) in the case of $D > 8$.

[15] For completeness, we note that in the set-up of the previous section 5.1 the formal limit $D \to 8$ (for D assumed as non-discrete and real valued) would correspond to the exceptional (singular) case $z \to 0$, $Q \to +\infty$.

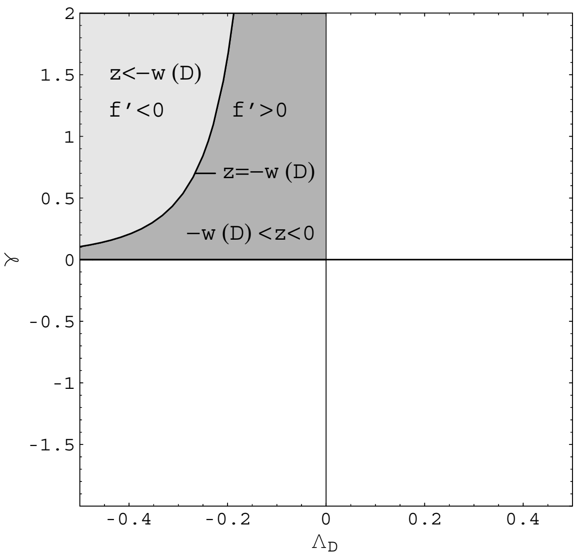

Figure 4. Projection $\Theta_{(\Lambda_D,\gamma)}$ of the stability region $\Upsilon \subset \mathcal{V} \subset \mathcal{M}$ of a model with $D \geqslant 8$ on the (Λ_D, γ) plane (shaded areas with $f' > 0$). The curve $z(\Lambda_D, \gamma) = -w(D)$, given in equation (C.4), corresponds to the conformal singularity $f' = 0$ where $\bar{R} \to -(4\gamma)^{-1/3}$ maps into $R \to -\infty$. It separates the parameter region with $U(\phi \to -\infty) \to +\infty$ (dark grey) from the region with $U(\phi \to -\infty) \to -\infty$ (light grey). For $f' > 0$ the corresponding region ensures the existence of an absolutely stable minimum (dark grey). The $(f' < 0)$ region is depicted for completeness. The lower half-plane $\gamma < 0$ corresponds to unstable configurations.

5.3. Stable and metastable configurations

In the previous two sections 5.1 and 5.2, it has been shown that models in $D \geqslant 8$ dimensions possess a stability region $\Theta_{(\Lambda_D,\gamma)} = \pi\Upsilon$, which is located in the upper (Λ_D, γ) half-plane. For models in $D < 8$ dimensions this region is larger and extends additionally into the lower (Λ_D, γ) plane (see relations (88)–(91) and figures 3 and 4). Now we will analyse the system over these stability regions in more detail.

We start with the asymptotic behaviour of the potential $U(\phi)$ in the limits $\phi \to \pm\infty$. In the high curvature limit[16] $\phi \to +\infty$ with dominant nonlinearity of type $f' = \mathrm{e}^{A\phi} = 1 + 4\gamma\bar{R}^3 \gg 1$ we find from (57)

$$U(\phi \to +\infty) \approx \tfrac{3}{8}(4\gamma)^{-1/3}\,\mathrm{e}^{(-B+4A/3)\phi} \tag{94}$$

and the sign of the coefficient in the exponent

$$\frac{4}{3}A - B = \frac{A}{3}\frac{D-8}{D-2} \tag{95}$$

[16] Here and in the subsequent considerations the high curvature limits are understood as formal limits (in the mathematical sense) within the framework of our simplified toy model. It is clear that in a model with scalar curvature nonlinearity of type $f(\bar{R}) = \bar{R} + \sum_{k=2}^{N} a_k \bar{R}^k$ gravitational self-interaction effects will dominate when $\sum_{k=2}^{N} a_k \bar{R}^{k-1} \gtrsim 1$. (For discussions of related subjects we refer to [48].) This means that a self-consistent treatment of the model would require techniques from (loop) quantum gravity or the high-energy sector of M-theory—which is out of the scope of the present paper.

leads for fixed γ to a qualitatively different behaviour for dimensions $D > 8$ and $D < 8$:

$$U(\phi \to +\infty) \to \frac{3}{8}(4\gamma)^{-1/3} \times \begin{cases} \infty & \text{for} \quad D > 8, \\ 1 & \text{for} \quad D = 8, \\ 0 & \text{for} \quad D < 8. \end{cases} \tag{96}$$

The existence of a critical dimension (in our case $D = 8$) is a rather general feature of gravitational theories with polynomial scalar curvature terms (see, e.g., [17, 49]). It can be easily demonstrated for a model with curvature nonlinearity of type

$$f(\bar{R}) = \sum_{k=0}^{N} a_k \bar{R}^k \tag{97}$$

for which the ansatz

$$\mathrm{e}^{A\phi} = f' = \sum_{k=0}^{N} k a_k \bar{R}^{k-1} \tag{98}$$

leads, similarly to (7), to a potential

$$U(\phi) = \frac{1}{2}(f')^{-D/(D-2)} \sum_{k=0}^{N} (k-1) a_k \bar{R}^k. \tag{99}$$

In the limit $\phi \to +\infty$ the curvature will behave like $\bar{R} \approx c\mathrm{e}^{h\phi}$ where h and c can be defined from the dominant term in (98):

$$\mathrm{e}^{A\phi} \approx N a_N \bar{R}^{N-1} \approx N a_N c^{N-1} \mathrm{e}^{(N-1)h\phi}. \tag{100}$$

Here the requirement $f' > 0$ allows for the following sign combinations of the coefficients a_N and the curvature asymptotics $\bar{R}(\phi \to \infty)$:

$$\begin{aligned} &N = 2l: \quad \operatorname{sign}[a_N] = \operatorname{sign}[\bar{R}(\phi \to \infty)] \\ &N = 2l+1: \quad a_N > 0, \quad \operatorname{sign}[\bar{R}(\phi \to \infty)] = \pm 1. \end{aligned} \tag{101}$$

The other combinations, $N = 2l : \operatorname{sign}[a_N] = -\operatorname{sign}[\bar{R}(\phi \to \infty)]$, $N = 2l+1 : a_N < 0, \operatorname{sign}[\bar{R}(\phi \to \infty)] = \pm 1$, would necessarily correspond to the $f' < 0$ sector, so that the complete consideration should be performed in terms of the extended conformal transformation technique of [18]. Such a consideration is out of the scope of the present paper and we restrict our attention to the cases (101). The coefficients h and c are then easily derived as $h = A/(N-1)$ and $c = \operatorname{sign}(a_N)|N a_N|^{-\frac{1}{N-1}}$. Plugging this into (99) one obtains

$$U(\phi \to +\infty) \approx \operatorname{sign}(a_N) \frac{(N-1)}{2N} |N a_N|^{-\frac{1}{N-1}} \mathrm{e}^{-\frac{D}{D-2}A\phi} \mathrm{e}^{\frac{N}{N-1}A\phi} \tag{102}$$

and that the exponent

$$\frac{D-2N}{(D-2)(N-1)} A \tag{103}$$

changes its sign at the critical dimension $D = 2N$:

$$U(\phi \to +\infty) \to \operatorname{sign}(a_N) \frac{(N-1)}{2N} |N a_N|^{-\frac{1}{N-1}} \times \begin{cases} \infty & \text{for} \quad D > 2N, \\ 1 & \text{for} \quad D = 2N, \\ 0 & \text{for} \quad D < 2N. \end{cases} \tag{104}$$

This critical dimension $D = 2N$ is independent of the concrete coefficient a_N and is only defined by the degree $N = \deg_{\bar{R}}(f)$ of the scalar curvature polynomial f. From the asymptotics (104) we read off that in the high curvature limit $\phi \to +\infty$, within our oversimplified classical framework, the potential $U(\phi)$ of the considered toy model shows asymptotical freedom for subcritical dimensions $D < 2N$, stable behaviour for $a_N > 0, D > 2N$ and a catastrophic instability for $a_N < 0, D > 2N$. We note that this general behaviour suggests a way to cure pathological (catastrophic) behaviour of polynomial $\bar{R}^{N_1}$ nonlinear theories in a fixed dimension $D > 2N_1$: by including higher order corrections up to order $N_2 > D/2$ the theory gets shifted into the non-pathological sector with asymptotical freedom. More generally, one is even led to the conjecture that the partially pathological behaviour of models in supercritical dimensions $D > 2N$ could be an artefact of a polynomial truncation of an (presently unknown) underlying non-polynomial $f(\bar{R})$ structure at high curvatures—which probably will find its resolution in a strong coupling regime of M-theory or in loop quantum gravity.

As the next step, we consider the opposite limit $\phi \to -\infty$ which corresponds to $f' \to 0$. From (7) we see that the potential $U(\phi)$ of a model with general polynomial $\bar{R}$ nonlinearity (97) behaves in this limit like

$$U(\phi \to -\infty) \approx -\tfrac{1}{2}\,\mathrm{e}^{-\frac{D}{D-2}A\phi} f(\bar{R}_{ci}), \tag{105}$$

where $\bar{R}_{ci}$ is one of the real roots of the polynomial $f'(\bar{R}) = 0$ given in (98). In other words, the potential $U(\phi)$ diverges like

$$U(\phi \to -\infty) \to -\mathrm{sign}[f(\bar{R}_{ci})] \times \infty \tag{106}$$

for any dimension D and any value $f(\bar{R}_{ci}) \neq 0$. This means that, for energetic reasons, the system will be repelled from configurations with $f' \approx 0$ by an infinitely high barrier $U(\phi \to -\infty) \to +\infty$ in the case $f(\bar{R}_{ci}) < 0$, and it will be catastrophically attracted (experience a collapse) to $f' \approx 0$ in the case $f(\bar{R}_{ci}) > 0$, $U(\phi \to -\infty) \to -\infty$.

Let us make this general consideration explicit for the $\bar{R}^4$ model (54). The polynomial $f' = 1 + 4\gamma\bar{R}^3$ has the single real-valued root $\bar{R}_{c2} = -(4\gamma)^{-1/3}$, which was used in (C.1)–(C.10) to map the inequality $f' > 0$ into the (Λ_D, γ) plane (result: the bounds $-w(D > 8) < z, z < -w(D < 8)$). Plugging this root into f we get

$$f(\bar{R}_{c2}) = -\left[\tfrac{3}{4}(4\gamma)^{-1/3} + 2\Lambda_D\right]. \tag{107}$$

From the condition $f(\bar{R}_{c2}) < 0$ for the existence of a repelling potential barrier with $U(\phi \to -\infty) \to +\infty$ limit one finds the following inequalities (regardless of the existence of a minimum)

$$\begin{aligned} \gamma > 0: &\quad \tfrac{8}{3}(4\gamma)^{1/3}\Lambda_D > -1, \\ \gamma < 0: &\quad \tfrac{8}{3}(4\gamma)^{1/3}\Lambda_D < -1. \end{aligned} \tag{108}$$

(Below we will show that the case $\gamma > 0$ holds for the $(f' > 0)$ sector, whereas $\gamma < 0$ will correspond to $f' < 0$.) Multiplication with $w(D)$ leads to the equivalent conditions

$$D > 8: \quad \gamma > 0: \quad -w(D) < z, \tag{109}$$

$$\gamma < 0: \quad z < -w(D), \tag{110}$$

$$D < 8: \quad \gamma > 0: \quad z < -w(D) = |w(D)|, \tag{111}$$

$$\gamma < 0: \quad |w(D)| = -w(D) < z. \tag{112}$$

We see that the asymptotical behaviour of the potential $U(\phi \to -\infty)$ is defined by similar parameter regions in z to those which control the existence of a minimum under the condition $f' > 0$. In both cases, the bound is connected with the critical value $f' = 0$ which corresponds to $z_c = -w(D)$. This is not a surprise because the two simultaneous conditions $f'(\bar{R}_{c2}) = 0$, $f(\bar{R}_{c2}) = 0$, which define the critical value z_c in the inequalities $f(\bar{R}_{c2}) < 0$,

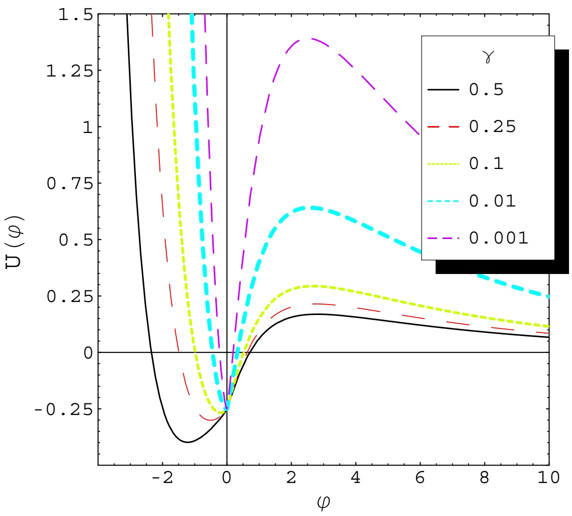

Figure 5. The typical form of a potential $U(\phi)$ with parameters (Λ_D, γ) from a subcritical $(D < 8)$ region of absolute stability $U(\phi \to -\infty) \to +\infty$, $U(\phi \to +\infty) \to +0$ in the physical sector $f' > 0$. Specifically, it is set $D = 6$, $\Lambda_D = -1/4$ for several values of γ (in Planck units).

$f(\bar{R}_{c2}) > 0$, fulfil the extremum condition $\partial_\phi U(\phi) = 0$, i.e. $h = Df - 2f'\bar{R} = 0$ in (40) and hence the quartic equation[17] (59).

Analogously to inequalities (108)–(112) we get from the condition $f(\bar{R}_{c2}) > 0$ for a catastrophically attracting potential, $U(\phi \to -\infty) \to -\infty$, that such an asymptotic holds over the sectors

$$D > 8: \quad \gamma > 0: \quad z < -w(D), \tag{113}$$

$$\gamma < 0: \quad -w(D) < z, \tag{114}$$

$$D < 8: \quad \gamma > 0: \quad |w(D)| = -w(D) < z, \tag{115}$$

$$\gamma < 0: \quad z < -w(D) = |w(D)|. \tag{116}$$

These sectors are complementary to those in (109)–(112). Here, we observe that $\gamma < 0$ for $f' > 0$ and $\gamma > 0$ for $f' < 0$. This is confirmed by a comparison of inequalities (109)–(116) with (79), (80) and the inequalities in footnote 23. Obviously, the regions $z < -w(D)$ for $D > 8$ and $z > -w(D) = |w(D)|$ for $D < 8$ correspond to a formal extension of the potential $U(\phi)$ into the $(f' < 0)$ sector. For completeness, these regions have been included in figures 3 and 4. The typical $U(\phi \to -\infty) \to \pm\infty$ behaviour of the potential is illustrated in figures 5 and 6.

From figure 6 we see that the minimum is separated by a barrier of finite height and width from the singularity at $\phi \to -\infty$. Hence, we find that the $\bar{R}^4$ models in the $(f' > 0)$ sector are absolutely stable for $\gamma > 0$ and metastable with a tendency to collapse into the singularity $f' \approx 0$ for $\gamma < 0$. We further see from the conformal relation (10) between the scalar curvature $\bar{R}$ of the nonlinear model and the curvature R of the equivalent linear model that $f' = 0$, and hence $z = |w(D)|$, corresponds to a conformal singularity: the finite curvature value $\bar{R}_{c2} = -(4\gamma)^{-1/3}$ of the nonlinear model is related to a curvature singularity $R \sim (f')^{-2/(D-2)}\bar{R}_{c2}$ in the associated linear model.

A detailed study of various limiting cases is given in appendix D. The corresponding main results can be summarized as follows:

[17] Obviously, it holds trivially $h = Df - 2f'\bar{R} = 0$ for $\bar{R} = \bar{R}_{c2}$.

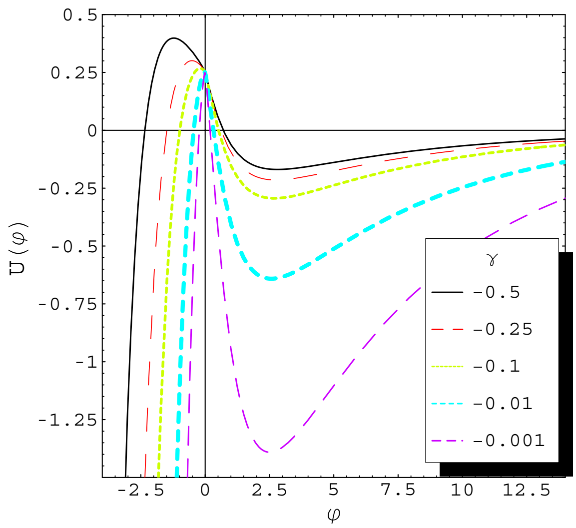

Figure 6. The typical form of a potential $U(\phi)$ of a subcritical $(D < 8)$ metastable system with $U(\phi \to -\infty) \to -\infty, U(\phi \to +\infty) \to -0$ in the physical sector $f' > 0$. It is set $D = 6, \Lambda_D = 1/4$ for several values of γ (in Planck units).

- The limit $(\Lambda_D \to -0, \gamma > 0)$ in the stable sector corresponds to a flat-space limit $\bar{R} \to -0$ which via (55) is associated with a freezing of the nonlinearity field ϕ: $f' \to 1$ at $\phi_0 \to 0$. In the metastable sector nothing special happens in the limit $(\Lambda_D \to 0, \gamma < 0)$.
- For $\Lambda_D \neq 0$ the limit $\gamma \to +0$ corresponds to a freezing of the nonlinearity field ϕ at $\phi_0 = 0$ and a smooth transition to a linear gravity model of Einstein–Hilbert type. In contrast, the limit $\gamma \to -0$ of the metastable $(D < 8)$ system results in an infinitely deep minimum $U(\phi_0, \gamma \to -0) \to -\infty$ at $\phi_0 = (1/A) \ln|3D/(D-8)|$ and a curvature singularity $\bar{R}_{-,-}(\gamma \to -0, D < 8) \approx -|(D-2)/[\gamma(D-8)]|^{1/3}$.
- The limit $z \to -1$ (in the metastable sector) corresponds to coalescing minimum and maximum of the potential $U(\phi)$ (with resulting inflection point at $z = -1$). For $z \leqslant -1$ the potential $U(\phi)$ has no minimum at all and the system is completely unstable.

5.4. *Inflation in the* $(\gamma > 0)$ *sector*

For simplicity, we restrict our attention to the simplest models with only one internal factor space M_1. The requirement for non-vanishing negative curvature of this space (in order to ensure a late-time stabilization of the corresponding dimensions) holds only for total dimensions $D \geqslant 6$ (in the case of $D_0 = 4$). In terms of the normalized scale factor (the radion) of this internal space M_1

$$\varphi \equiv -\sqrt{\frac{d_1(D-2)}{D_0-2}}\hat{\beta}^1 \tag{117}$$

the effective potential (24) reads

$$U_{\text{eff}} = e^{2s\varphi}\left[U(\phi) - \frac{1}{2}\hat{R}_1 e^{2s_1\varphi}\right], \qquad s := \sqrt{\frac{d_1}{(D_0-2)(D-2)}}, \qquad s_1 = \frac{D_0-2}{d_1}s. \tag{118}$$

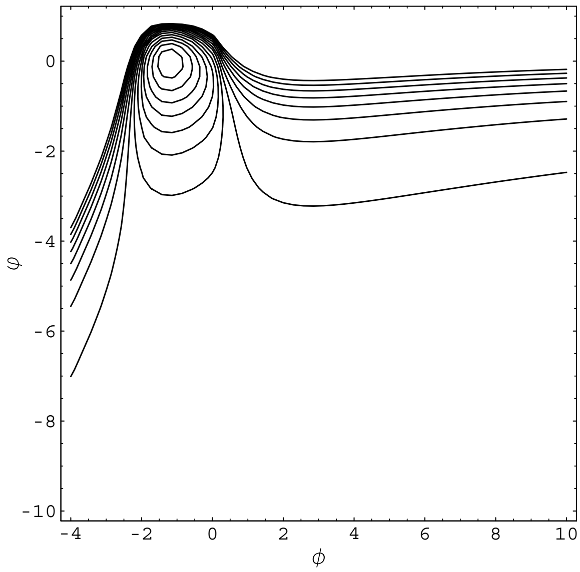

Figure 7. A contour plot of the effective potential $U_{\text{eff}}(\phi, \varphi)$ given in equation (118).

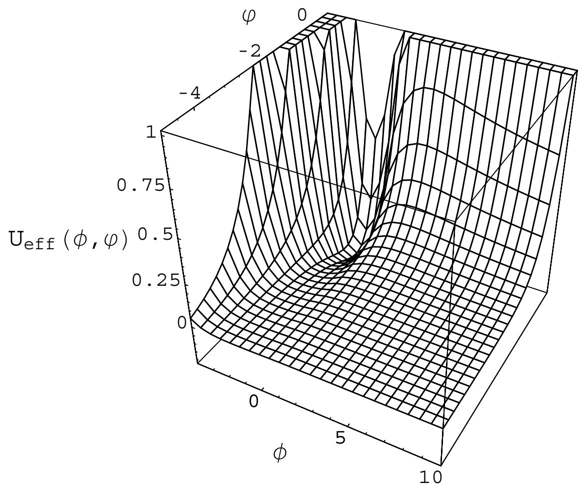

Figure 8. A 3D plot of the effective potential $U_{\text{eff}}(\phi, \varphi)$ given in equation (118).

In figures 7 and 8 its generic form is illustrated by a model with ($d_1 = 2$) extra dimensions and parameters $\gamma = 1/2$, $\Lambda_D = -1/4$. A general feature of an effective potential (118) with a minimum and a barrier asymptotic $U(\phi \to -\infty) \to +\infty$ is the necessary absence of a local maximum or a saddle point. This is easily seen from the extremum conditions

$$\partial_\varphi U_{\text{eff}}|_{\text{extr}} = 2s\, \mathrm{e}^{2s\varphi} \left[U(\phi) + \frac{D_0 - 2 - d_1}{2d_1} \hat{R}_1\, \mathrm{e}^{2s_1\varphi} \right]\Bigg|_{\text{extr}} = 0, \tag{119}$$

$$\partial_\phi U_{\text{eff}}|_{\text{extr}} = \mathrm{e}^{2s\varphi} \partial_\phi U(\phi)|_{\text{extr}} = 0 \tag{120}$$

and their implications. Condition (120) yields again the quartic equation (59). For $D > 8, \gamma > 0$ this equation has only a single solution $\bar{R}_{-,+}$ in the $(f' > 0)$ sector[18] so that the minimum is the only extremum. In the case $D < 8, \gamma > 0$ besides the negative curvature solution $\bar{R}_{+,-}$ of the minimum there exists a positive maximum $\bar{R}_{+,+} > 0$ (see equation (76)) so that at the corresponding extremum with regard to ϕ it holds $U(\phi_{\max}) > 0$ in accordance with equation (43). Hence, the condition (119) cannot be fulfilled due to $U(\phi_{\max}) > 0, \hat{R}_1 < 0, D_0 = 4, d_1 \geqslant 2 \Longrightarrow \partial_\varphi U_{\text{eff}} > 0$ and there will be no other extremum of the effective potential $U_{\text{eff}}(\phi, \varphi)$ apart from the already studied minimum at $(\phi = \phi_0, \varphi = 0)$.

This means that in the considered oversimplified model inflation of purely topological type [50] as, e.g., recently demonstrated for SUGRA inspired set-ups (racetrack inflation starting at a saddle point of the effective potential) in [51] is ruled out. In general, the too steep slopes of the exponential terms of the effective potential will spoil inflation, i.e., slow-roll behaviour seems unrealistic. We will demonstrate this with a region in the (ϕ, φ) plane where one still might hope to obtain a sufficiently gentle slope to induce the needed long-lasting accelerated expansion of the external space as well as an attraction to the global minimum (in order to ensure a late-time stabilization of the scalar fields). (See figures 7 and 8.) Such a region could be expected close to the maximum of the potential $U(\phi)$ where in rough approximation it holds

$$U_{\text{eff}} \approx \mathrm{e}^{2s\varphi} U(\phi). \tag{121}$$

The action functional (20) shows that the two scalar fields $\varphi_1 := \phi, \varphi_2 := \varphi$ live in a flat σ model target space. Hence, the estimate of the slow-roll parameters can be performed within a simplified version of the multi-field inflation scheme of [8, 11, 52]. Assuming in rough approximation that the external space has already flattened, the inflation parameters ϵ, η read

$$\epsilon = -\frac{\dot{H}}{H^2} \approx \frac{1}{2}\frac{|\partial U_{\text{eff}}|^2}{U_{\text{eff}}^2} \tag{122}$$

$$\eta = -\frac{\sum_{i=1}^2 \ddot{\varphi}_i \dot{\varphi}_i}{H|\dot{\varphi}|^2} \approx -\epsilon + \frac{\sum_{i,j=1}^2 \left(\partial_{ij}^2 U_{\text{eff}}\right)(\partial_i U_{\text{eff}})(\partial_j U_{\text{eff}})}{U_{\text{eff}}|\partial U_{\text{eff}}|^2}. \tag{123}$$

Here, $H = \dot{a}/a$ is as usual the Hubble parameter of the external space. Additionally, the following abbreviations have been introduced:

$$\partial_i := \frac{\partial}{\partial \varphi_i}, \qquad |\partial U_{\text{eff}}|^2 = \sum_{i=1}^2 (\partial_i U_{\text{eff}})^2, \qquad |\dot{\varphi}|^2 = \sum_{i=1}^2 \dot{\varphi}_i^2. \tag{124}$$

Inflation is possible for $\epsilon < 1$ and a sufficiently small slow-roll parameter $|\eta| \ll 1$. In the vicinity of the maximum in the ϕ-direction it holds $\partial_\phi U|_{\max} \approx 0$ so that ϵ, η are essentially defined by the slope in the φ-direction. Explicitly, one obtains in this region

$$\epsilon \approx 2s^2, \qquad \eta \approx 2s(1-s). \tag{125}$$

For a four-dimensional external space $D_0 = 4$ this yields from (118)

$$s^2 = \frac{d_1}{2(d_1+2)} < \frac{1}{2} \tag{126}$$

[18] From figure 2 and relation (88) one sees that the condition $f' > 0$ cuts the $T_{-,+}(z)$ branch at $z_{c2} = -w(D)$ (where $-1 < z_{c2} < 0$) and allows only for the piece $-w(D) < z < 0$ where $T_{-,+}(z) > T_{-,+}(z_{c2})$ holds. The remaining segment $-1 < z < -w(D)$ with $T_{-,+}(z) < T_{-,+}(z_{c2})$ and also the whole second solution $T_{-,-}(z)$ are completely located in the $(f' < 0)$ sector.

and the rough estimates

$$\begin{aligned} d_1 = 2: &\quad \epsilon \approx 0.5, &\quad \eta \approx 0.5, \\ d_1 = 3: &\quad \epsilon \approx 0.6, &\quad \eta \approx 0.49, \\ d_1 = 6: &\quad \epsilon \approx 0.75, &\quad \eta \approx 0.47. \end{aligned} \tag{127}$$

Because $|\eta| \ll 1$ is not satisfied, the considered toy model would produce an accelerated expansion ($\epsilon < 1$) which would be much too short for successful inflation. Whether a domain wall with the possibility of topological-type inflation could form between regions to the left and to the right (in the ϕ-direction) of the crest at the maximum of $U(\phi)$, remains an open question. Corresponding indications have been given in [19], but seem to require an additional detailed analysis—and probably an embedding of the toy model into a more general set-up with a richer structure.

6. Conclusion

In the present paper, we continued our investigation [33, 34] on multi-dimensional gravitational models with a non-Einsteinian form of the action. The corresponding action functional was assumed as a smooth function $f(\bar{R})$ of the scalar curvature $\bar{R}$ of a D-dimensional spacetime manifold with a warped product structure. The main subject of our considerations was the stabilization problem for the extra dimensions. As a technique, we used a reduction of the nonlinear gravitational model to a linear one with an additional self-interacting scalar field (nonlinearity scalar field ϕ). The factorized geometry allowed for a dimensional reduction of the considered model and a transition to the Einstein frame. As a result, we obtained an effective four-dimensional model with a nonlinearity scalar field and additional minimally coupled scalar fields which describe conformal excitations of the scale factors of the internal space (its zero-mode excitations).

In terms of these scalar fields we performed a detailed stability analysis for models with scalar curvature nonlinearities of type $f(\bar{R}) = \bar{R} - \mu/\bar{R}$, $\mu > 0$ and $f(\bar{R}) = \bar{R} + \gamma\bar{R}^4 - 2\Lambda_D$, where Λ_D plays the role of a D-dimensional bare (bulk) cosmological constant. As a stability condition, we assumed the existence of a minimum of the effective potential of the dimensionally reduced theory so that a late-time attractor of the system could be expected with freezing stabilization of the extra-dimensional scale factors and the nonlinearity field. It was shown in [33, 34], that for purely geometrical set-ups this is only possible for negative scalar curvatures ($\bar{R} < 0$), independently of the concrete form of the function $f(\bar{R})$.

Four-dimensional purely gravitational models with $\bar{R}^{-1}$ curvature contributions have been proposed recently as a possible explanation of the observed late-time acceleration (dark energy) of the universe [14]. In section 4 of the present paper, we showed that higher dimensional models with the same $\bar{R}^{-1}$ scalar curvature nonlinearity reproduce (after the dimensional reduction) the two solution branches of the four-dimensional models. But due to their oversimplified structure these models cannot simultaneously provide a late-time acceleration of the external four-dimensional spacetime and a stabilization of the internal space. A late-time acceleration is only possible for one of the solution branches—for the one which yields a positive maximum of the potential $U(\phi)$ of the nonlinearity field. A stabilization of the internal spaces requires a negative minimum of $U(\phi)$ as it can be induced by the other solution branch. The question of whether this incompatibility could be resolved by different scalar curvature nonlinearities over the factor spaces (for each factor space M_i it might hold its own curvature nonlinearity $f_i(\bar{R}_i)$) is still open and deserves a separate analysis. We have left this issue to future investigations.

The considered $\bar{R}^4$ set-up was assumed as a highly simplified toy model analogue of the loop corrected gravity sector of M-theory [46]. The stability analysis of the higher dimensional model was reduced to a set of algebraic compatibility tests for the extremum condition (in the present case a quartic equation in the scalar curvature $\bar{R}$) and inequalities which ensure the existence of a minimum of the effective potential (non-tachyonic mass terms of the corresponding field excitations). For simplicity, we restricted the investigation to parameter regions $f' = 1 + 4\gamma\bar{R}^3 > 0$ which are smoothly connected with the curvature-linear model at $f' = 1$ without passing a conformal singularity. In the Brans–Dicke (Jordan) frame the latter requirement ensures an effective gravitational constant which is positive definite[19] and smoothly connected with that of a given BD frame model with a fixed (frozen) gravitational constant.

With the help of a projection technique in the $(\Lambda_D, \gamma, \bar{R})$ space $\mathcal{M}$ we identified regions which ensure the existence of a minimum of the effective potential, and hence fulfil a necessary condition for a successful freezing stabilization of the extra dimensions. The results can be summarized as follows. For systems with total spacetime dimensions $D \geqslant 6$ (in the case of a $(D_0 = 4)$-dimensional external spacetime) there exists a stable sector $\Theta_{1,(\Lambda_D,\gamma)} = \{\Lambda_D < 0 \cap \gamma > 0 \cap |z(\Lambda_D, \gamma)| < |w(D)|\}$ on the (Λ_D, γ) plane which in the limit $\gamma \to +0$ tends smoothly to the $\bar{R}$ linear sector. The corresponding transition is connected with a freezing of the nonlinearity field at the minimum of its potential $U(\phi)$, i.e., a diverging excitation mass, $m_\phi^2 \to +\infty$, due to a diverging Hessian of the potential $U(\phi)$. Models within $\Theta_{1,(\Lambda_D,\gamma)}$ are separated from the conformal singularity at $f' = 0$ (and the antigravity sector $f' < 0$ beyond it) by a potential barrier of infinite height and width and are, hence, absolutely stable with regard to transitions into the $(f' < 0)$ sector. Additionally, it was shown that the limit $\Lambda_D \to -0$ in the $\Theta_{1,(\Lambda_D,\gamma)}$ sector, is connected with a decompactification of the internal space components M_i, $i = 1, \ldots, n$ and a flattening $\bar{R} \to -0$ of the bulk spacetime M.

Apart from $\Theta_{1,(\Lambda_D,\gamma)}$ there exists a second stability sector $\Theta_{2,(\Lambda_D,\gamma)} = \{\gamma < 0 \cap -1 < z(\Lambda_D, \gamma) < -w(D)\}$ for dimensions $D < 8$. The potential $U(\phi)$ for such configurations is unbounded from below in the limit $f'(\phi \to -\infty) \to +0$ and has a minimum which is separated from the conformal singularity at $f' = 0$ (and the antigravity sector $f' < 0$ beyond it) by a potential wall of finite height and width. Configurations in this minimum would be metastable and prone to collapse into $f' = 0$. The $\Theta_{2,(\Lambda_D,\gamma)}$ sector is disconnected from $\Theta_{1,(\Lambda_D,\gamma)}$ by an essential singularity of $\bar{R}$ and $U(\phi_0)$ in the limit $\gamma \to -0$. In this limit the nonlinearity field ϕ freezes at $\phi_0(\gamma \to -0) \to (1/A)\ln(3D/|D - 8|)$ with $m_\phi^2 \to +\infty$, but simultaneously the scalar curvature diverges $\bar{R} \to -\infty$ and the potential deepens unboundedly $U(\phi_0) \to -\infty$. This behaviour is a strong indication for inconsistencies of $\Theta_{2,(\Lambda_D,\gamma)}$ configurations within the framework of the given limited set-up. The question of whether the $\Theta_{2,(\Lambda_D,\gamma)}$ sector of the considered oversimplified toy model will find a physically sensible interpretation within a still unknown extended curvature-nonlinear theory of gravity, a special UV limit of non-perturbative M-theory or within loop quantum gravity remains an open issue.

A further issue which was out of the scope of the present paper was the analysis of dynamical transitions between configurations which correspond to different solution branches $\bar{R}(\phi)$ of equation (6), $f'(\bar{R}) = \mathrm{e}^{A\phi}$. For the $\bar{R}^{-1}$ model of section 4, e.g., there exist two such branches $\bar{R}(\phi)$ which form a double cover[20] over given values of the parameter μ and the

[19] See footnote 23 for a few comments on the BD-antigravity sector with $f' < 0$.

[20] This is in contrast to the considered $\bar{R}^4$ model whose solutions $\bar{R}(\phi)$ are unambiguously defined by the sign of the nonlinearity parameter γ. The origin of this difference between the $\bar{R}^{-1}$ and the $\bar{R}^4$ model is in the number of real-valued solutions $\bar{R}_i(\phi)$ of equation (6) for given nonlinearity parameters μ or γ and ϕ. In the case of the $\bar{R}^{-1}$ model the corresponding quadratic equation has two real-valued solutions for given (μ, ϕ) whereas the cubic equation (55) has only one for fixed (γ, ϕ).

nonlinearity field ϕ. At early evolution stages of the universe, transitions between these two branches cannot be ruled out *a priori* and should be taken into account for a comprehensive description of the dynamics of the universe.

Finally, we note that the external spacetime in the considered pure geometrical models is necessarily AdS and the corresponding negative effective cosmological constant, $\Lambda_{\rm eff} < 0$, forbids a late-time acceleration. The situation can be cured by including additional matter fields. Examples are flux field stabilization scenarios [34] which provide certain moduli space sectors with a positive effective cosmological constant and external spacetimes of dS type.

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Appendix A. The quartic equation and its associated quadratic equation sets

In this appendix, we briefly re-derive the quadratic equation sets associated with the quartic equation (59). Following the Ferrari formalism as it is briefly described, e.g., in [47] we lay explicit emphasis on the sign rules[21] which are crucial for a correct derivation of the final solution set of the quartic equation.

The crux of the Ferrari formalism applied to a quartic equation of type (59),

$$x^4 + a_1 x + a_0 = 0, \tag{A.1}$$

consists in factoring it by transforming it into a difference of two quadratic terms

$$A^2 - B^2 = (A + B)(A - B) = 0 \tag{A.2}$$

so that solutions can be obtained from

$$A \pm B = 0. \tag{A.3}$$

Adding and subtracting a term $x^2 u + (u/2)^2$ in (A.1), with u an auxiliary function, one rewrites (A.1) as

$$\left(x^2 + \frac{u}{2}\right)^2 - u\left(x^2 - \frac{a_1}{u}x + \frac{\frac{1}{4}u^2 - a_0}{u}\right) = 0 \tag{A.4}$$

and requires the second term to be quadratic

$$\left(x^2 + \frac{u}{2}\right)^2 - u\left(x + \epsilon\sqrt{\frac{\frac{1}{4}u^2 - a_0}{u}}\right)^2 = 0. \tag{A.5}$$

Here, ϵ is a sign factor $\epsilon = \pm 1$ and we assume for definiteness

$$\sqrt{\frac{\frac{1}{4}u^2 - a_0}{u}} > 0 \qquad \text{for} \qquad \frac{\frac{1}{4}u^2 - a_0}{u} > 0. \tag{A.6}$$

[21] These sign rules are not displayed explicitly in [47].

The compatibility of equations (A.4) and (A.5) is ensured by the condition

$$-\frac{a_1}{u} = 2\epsilon\sqrt{\frac{\frac{1}{4}u^2 - a_0}{u}} \tag{A.7}$$

which on its turn is equivalent to the cubic equation

$$u^3 - 4a_0u - a_1^2 = 0. \tag{A.8}$$

We specify as in (59) and (63)

$$a_0 := -\frac{2D\Lambda_D}{\gamma(D-8)}, \qquad a_1 := \frac{D-2}{\gamma(D-8)}, \qquad -4a_0 = 3q, \qquad a_1^2 = 2r \tag{A.9}$$

and assume that u is a real-valued solution of equation (A.8). The analysis of section 5.1 shows that for stable configurations of the R^4 model it holds additionally $u > 0$. Using this condition as simplifying input information, we can rewrite the quartic equations (A.1) and (A.5) as

$$\left(x^2 + \frac{u}{2}\right)^2 - \left(\sqrt{u}x + \frac{\epsilon}{2}\sqrt{u^2+3q}\right)^2 = 0. \tag{A.10}$$

It factorizes according to (A.2) and (A.3) into a set of quadratic equations

$$x^2 \pm \sqrt{u}x + \tfrac{1}{2}\left(u \pm \epsilon\sqrt{u^2+3q}\right) = 0. \tag{A.11}$$

The sign factor ϵ follows from (A.6), (A.7) and (A.9) and $u > 0$ as

$$\epsilon = -\mathrm{sign}(a_1) = -\mathrm{sign}\left(\frac{D-2}{\gamma(D-8)}\right). \tag{A.12}$$

Appendix B. Sign analysis of the discriminant Q

The sign of Q can be obtained by mapping the minimum-ensuring inequality (60) into an equivalent inequality for z. For this purpose, we consider the critical surface $\Xi_{c1} \subset \mathcal{M}$ in the parameter space, where for $\gamma \neq 0$ the inequality (60) is replaced by an equality

$$\Xi_{c1} = \{(\Lambda_D, \gamma, \bar{R}) \in \mathcal{M} |\ \xi_{c1}[\Lambda_D, \gamma, \bar{R}] := (D-2) + 4(D-8)\gamma\bar{R}^3 = 0\}. \tag{B.1}$$

The intersection of this surface Ξ_{c1} with the algebraic variety $\mathcal{V} : h[\Lambda_D, \gamma, \bar{R}] = 0$ of the extremum condition will define a critical value z_{c1}. This value can be found explicitly by resolving (B.1) for $\bar{R}$, which gives

$$\bar{R}_{c1} := R|_{\Xi_{c1}} = -\left(\frac{D-2}{4(D-8)\gamma}\right)^{1/3}, \tag{B.2}$$

and plugging $\bar{R}_{c1}$ into the quartic equation (59). As a result, one obtains

$$z_{c1}(\Lambda_D, \gamma) = 4\gamma(8\Lambda_D/3)^3 w(D) = -1. \tag{B.3}$$

Now, small perturbations off the critical surface Ξ_{c1}, but along the variety $\mathcal{V}$, can be used to map the inequality (60) into its counterpart for z. Setting

$$\Lambda_D = \Lambda_{D,c1} + \delta\Lambda_D, \qquad \bar{R} = \bar{R}_{c1} + \delta\bar{R}, \qquad z = z_{c1} + \delta z \tag{B.4}$$

and keeping γ fixed, we get from the inequality (60)

$$72(D-8)\gamma^2\bar{R}_{c1}^2\delta\bar{R} > 0 \tag{B.5}$$

whereas the quartic equation (59) and the definition (66) of z yield

$$\delta\Lambda_D = 3\gamma\frac{D-8}{D}\bar{R}_{c1}^2(\delta\bar{R})^2, \qquad \delta z = 12\gamma w(D)(8/3)^3\Lambda_{D,c1}^2\delta\Lambda_D \tag{B.6}$$

and hence

$$\delta z = \frac{2^{11}}{3}\frac{D^2(D-8)^2}{(D-2)^4}(\gamma\bar{R}_{c1}\Lambda_{D,c1})^2(\delta\bar{R})^2 \geqslant 0. \tag{B.7}$$

We observe that, although inequality (B.5) implies

$$D > 8: \quad \delta\bar{R} > 0, \qquad D < 8: \quad \delta\bar{R} < 0, \tag{B.8}$$

independently of the signs of γ and $\bar{R}_{c1}$, the variety $\mathcal{V}$ is for $\gamma \neq 0$, $D \neq 8$ and (because of $\delta z > 0$) located over the region

$$z(\Lambda_D, \gamma) > z_{c1}(\Lambda_D, \gamma) = -1 \tag{B.9}$$

of the (Λ_D, γ) plane. This means that by any perturbation (motion) on the variety $\mathcal{V}$ we cannot pass across the critical value $z_{c1}(\Lambda_D, \gamma) = -1$. Hence, $z_{c1}(\Lambda_D, \gamma) = -1$ must be a boundary segment of the projection $\pi\mathcal{V}$ of $\mathcal{V}$ onto the (Λ_D, γ) plane: $z_{c1} \subset \partial(\pi\mathcal{V})$. The latter fact is confirmed by the observation that the critical surface $\Xi_{c1} \subset \mathcal{M}$ coincides with the singular surface

$$\partial_{\bar{R}}h[\Lambda_D, \gamma, \bar{R}] = (D-2) + 4(D-8)\gamma\bar{R}^3 = 0$$

of the projection π of $\mathcal{V}$ onto the (Λ_D, γ) plane[22].

Appendix C. Mapping $f' > 0$ into parameter space

The inequality $f' = 1 + 4\gamma\bar{R}^3 > 0$ can be analysed with the same technique as the minimum-ensuring inequality (60) (see relations (B.1)–(B.9)): we obtain the intersection of the critical surface

$$\Xi_{c2} = \{(\Lambda_D, \gamma, \bar{R}) \in \mathcal{M} | \; \xi_{c2}[\Lambda_D, \gamma, \bar{R}] = 1 + 4\gamma\bar{R}^3 = 0\} \tag{C.1}$$

with the algebraic variety $\mathcal{V}$, i.e., $\Xi_{c2} \cap \mathcal{V}$, and study the behaviour of small parameter perturbations off Ξ_{c2} and along the variety $\mathcal{V}$.

Explicitly, this means that we resolve (C.1) for $\bar{R}$ to obtain

$$\bar{R}_{c2} = -(4\gamma)^{-1/3} \tag{C.2}$$

and plug this $\bar{R}_{c2}$ into the quartic equation (59). From the intermediate result

$$4\gamma(8\Lambda_D/3)^3 = -1 \tag{C.3}$$

we find by multiplication with $w(D)$ that the intersection $\Xi_{c2} \cap \mathcal{V}$ corresponds to the critical value

$$z_{c2}(\Lambda_D, \gamma) = -w(D). \tag{C.4}$$

Substituting, furthermore, the perturbation ansatz

$$\begin{aligned} \Lambda_D &= \Lambda_{D,c2} + \delta\Lambda_D, & \bar{R} &= \bar{R}_{c2} + \delta\bar{R}, \\ z &= z_{c2} + \delta z, & f' &= \delta f', \end{aligned} \tag{C.5}$$

[22] Singularities of smooth projections are extensively discussed, e.g., in [53].

($f'_{c2} = 0$ holds) into the defining relation (66) for z, the quartic equation (59), and the equality $f' = 1 + 4\gamma \bar{R}^3$, we get for fixed γ

$$\delta z = 12\gamma w(D)(8/3)^3 \Lambda_{D,c2}^2 \delta\Lambda_D, \qquad \delta\Lambda_D = \frac{3}{D}\delta\bar{R},$$
$$\delta f' = 12\gamma \bar{R}_{c2}^2 \delta\bar{R}, \tag{C.6}$$

respectively, and by the combination of these results also

$$\delta\Lambda_D = \frac{1}{4\bar{R}^2 D\gamma}\delta f', \qquad \delta z = \frac{3}{D}\left(\frac{8}{3}\right)^3 \left(\frac{\Lambda_{D,c2}}{\bar{R}_{c2}}\right)^2 w(D)\delta f'. \tag{C.7}$$

From the definition (66) of $w(D)$ and its implication

$$w(D < 8) < 0, \qquad 0 < w(D > 8) < 1 \tag{C.8}$$

we find[23] for $\delta f' > 0$, $f' > 0$

$$D < 8: \quad z_{c2}(\Lambda_D, \gamma) = |w(D)| > 0, \qquad z < -w(D) = |w(D)|, \tag{C.9}$$

$$D > 8: \quad z_{c2}(\Lambda_D, \gamma) = -w(D) < 0, \qquad -w(D) < z. \tag{C.10}$$

Appendix D. Parameter limits

In section 5.3 it has been found that for the ($f' > 0$) sector the boundary segments $z(\Lambda_D, \gamma) = -w(D) \subset \partial\Theta_{(\Lambda_D,\gamma)}$ of the projection $\Theta_{(\Lambda_D,\gamma)} := \pi\Upsilon$ of the stability region $\Upsilon \subset \mathcal{V} \subset \mathcal{M}$ onto the (Λ_D, γ) plane correspond to the limit $\phi \to -\infty$. Here, we clarify the behaviour of the system in the vicinity of the other boundary segments $\partial\Theta_{(\Lambda_D,\gamma)} \supset \{\Lambda_D = 0 \cup \gamma = 0\}$ for $D > 8$ (see figure 4) and $\partial\Theta_{(\Lambda_D,\gamma)} \supset \{\Lambda_D = 0 \cup \gamma = 0 \cup z = -1\}$ for $D < 8$ (see figure 3).

$\Lambda_D \to 0, \gamma \neq 0$. In this limit, we obtain from equations (63), (66) and (68)

$$q \to 0, \quad z \to 0, \quad Q \to r^2 \neq 0, \quad v_1 \to 2^{1/3} \tag{D.1}$$

and hence from equations (76), (77) and (92)

$$\gamma > 0: \quad \bar{R}(\Lambda_D \to -0) \to -0 \quad \Longleftarrow \quad \begin{cases} \bar{R}_{-,+} \to -0 & \text{for} \quad D > 8, \\ \bar{R} \to -0 & \text{for} \quad D = 8, \\ \bar{R}_{+,-} \to -0 & \text{for} \quad D < 8, \end{cases} \tag{D.2}$$

[23] For completeness, we note that a formal crossing of the critical surface Ξ_{c2} into the sector $f' < 0$ would lead to the restrictions

$$D < 8 : z > |w(D)|, \qquad D > 8 : z < -w(D).$$

The sector $f' < 0$ itself would correspond to a negative effective gravitational constant in the BD frame (an antigravity sector) [54, 55]. Antigravity effects are described for various types of higher dimensional set-ups. In certain SUGRA configurations, spatially bounded antigravity regions are known as repulsons (white holes) [56]. Several brane-world models show radion induced scalar antigravity at ultra-large distances [57]. In our model, the antigravity sector would fill a complete external spacetime which necessarily would be disconnected from our own observable universe with $f' > 0$ by a conformal singularity at $f' = 0$ (see section 5.3 and also [55]). There remains an interesting open question of whether the formal correspondence between the strong-curvature sector of the $\bar{R}$ nonlinear model and the emerging antigravity sector of the R linear model in the BD frame simply signals an inconsistency of the theory and whether it can find a physically meaningful interpretation in an enlarged (extended) set-up. The repulson in SUGRA models was reinterpreted as an unphysical region and its resolution was found by the enhançon mechanisms (excision of the antigravity region and placing a heavy shell of wrapped D-branes on its boundary so that the former antigravity region in the interior is shielded by the D-brane source and replaced by a segment of flat space) [58]. In analogy to the enhançon mechanism in the repulson case, one can expect a resolution of the conformal singularity at $f' = 0$ by some kind of quantum gravity mechanism.

$$\gamma < 0: \quad \bar{R}_{-,-}(\Lambda_D \to 0) \to -(2r)^{1/6} \qquad \text{for} \quad D < 8. \tag{D.3}$$

Obviously, the system behaves differently in the upper and lower (Λ_D, γ) plane. In the case of $\gamma < 0$, the system behaves regularly for $\Lambda_D \to 0$ and the half-line $(\Lambda_D = 0, \gamma < 0)$ is not distinguished from its vicinity. In contrast to this, the limit $(\Lambda_D \to -0, \gamma > 0)$ corresponds to a flat-space limit $\bar{R} \to -0$ which via (55) is associated with a freezing of the nonlinearity field ϕ: $f' \to 1$ at $\phi_0 \to 0$. From equation (41) it follows that

$$\partial^2_\phi U\big|_{\phi_0} \approx \frac{D-2}{24(D-1)}\frac{1}{\gamma \bar{R}^2} \to +\infty \tag{D.4}$$

so that for the mass of the nonlinearity field $m^2_\phi \to +\infty$ holds. We note that the nonlinearity field in an R^2 model has a finite mass m_ϕ in the limit $\Lambda_D \to -0$ (see, e.g., [33, 34]). The different behaviour of the models is caused by the different powers of the term $(e^{A\phi} - 1)$ in $U(\phi)$: for an R^2 model this power equals 2, whereas for an R^4 model it equals $4/3$. Hence, in the latter case the second derivative $d^2U(\phi)/d\phi^2|_{\phi_{(0)}}$ diverges in the limit $\phi_{(0)} \to 0$.

We arrived at the interesting fact that in the considered toy model the extremum condition in the form of the quartic equation (59) relates the scalar curvature $\bar{R}$ at the minimum and the bare cosmological constant Λ_D in the case of $\gamma > 0$ so strongly that for stabilized internal spaces the limit $\Lambda_D \to 0$ corresponds to the flat-space limit $\bar{R} \to -0$. As it should, the flat-space limit of the total scalar curvature $\bar{R} \to -0$ implies via (43), i.e., $U(\phi_0) \to -0$, and (32), $\hat{R}_i = 2d_i U(\phi_0)/(D-2)$, also a decompactification of the internal space components $\hat{R}_i = e^{-2\beta^i_0} R_i \to -0$, $\beta^i_0 \to +\infty$ (the R_i are held fixed).

$\gamma \to 0$, $\Lambda_D \neq 0$. The definitions (63) and (66) show that for fixed $\Lambda_D \neq 0$ the limit $\gamma \to 0$ implies

$$r, |q|, Q \to +\infty, \qquad z \to 0. \tag{D.5}$$

With the help of an expansion in terms of small $z \approx 0$ the rescaled curvatures (77) are easily obtained from (69) as

$$\begin{aligned} T_{+,-}(z \to 0) &\approx -3 \times 2^{-5/2} z^{1/3} \\ T_{-,+}(z \to 0) &\approx 3 \times 2^{-5/2} z^{1/3} \\ T_{-,-}(z \to 0) &\approx -2^{1/6} - 2^{-5/2} z^{1/3} \end{aligned} \tag{D.6}$$

so that the curvatures $\bar{R}_{\epsilon,\pm}$ themselves can be estimated via $\bar{R}_{\epsilon,\pm} = r^{1/6} T_{\epsilon,\pm}$ as[24]

$$\left.\begin{aligned} \bar{R}_{-,+}(\gamma \to +0; D > 8) \\ \bar{R}_{+,-}(\gamma \to +0; D < 8) \end{aligned}\right\} \approx \frac{2D\Lambda_D}{D-2}, \tag{D.7}$$

$$\bar{R}_{-,-}(\gamma \to -0; D < 8) \approx -\left|\frac{D-2}{\gamma(D-8)}\right|^{1/3} \to -\infty. \tag{D.8}$$

In the exceptional $(D = 8)$ case the scalar curvature in the minimum does not depend on γ and is given by equation (92)

$$\bar{R} = \frac{8}{3}\Lambda_D = \frac{2D\Lambda_D}{D-2}. \tag{D.9}$$

[24] The general limiting behaviour $\bar{R}(\gamma \to 0)$ without the identification of the concrete solution branch $\bar{R}_{\epsilon,\pm}$ can be easily obtained from the quartic equation (59). Assuming $\bar{R}(\gamma \to 0) < \infty$ and taking the limit $\gamma \to 0$ in equation (59) gives $\bar{R} = 2D\Lambda_D/(D-2)$, whereas division of (59) by $\bar{R}$ for behaviour $|\bar{R}(\gamma \to 0)| \to \infty$ yields $\bar{R} = -(\frac{D-2}{\gamma(D-8)})^{1/3}$.

Again, the system behaves qualitatively differently in the upper and the lower (Λ_D, γ) plane. Because of the finite asymptotics (D.7) and equation (D.9), in the upper half-plane it holds (for $D < 8$ and $D \geqslant 8$)

$$f'(\gamma \to +0) \to 1, \qquad \phi_0 \to 0 \tag{D.10}$$

$$\partial^2_\phi U(\gamma \to +0)|_{\phi_0} \approx \frac{(D-2)^3}{96(D-1)D^2}\frac{1}{\gamma\Lambda_D^2} \to +\infty \tag{D.11}$$

$$U(\phi_0; \gamma \to +0) \to \Lambda_D \tag{D.12}$$

and the nonlinearity field ϕ undergoes a freezing stabilization at $\phi_0 = 0$ with diverging mass $m_\phi \to +\infty$ but finite scalar curvature (D.7) and finite minimum position $U(\phi_0)$. Hence, under the freezing stabilization of the nonlinearity field for $\gamma \to +0$ the system turns smoothly into a system with linear scalar curvature term $\bar{R}$, i.e. into a system with an Einstein–Hilbert action in $\bar{R}$. This is a generic feature of models with nonlinear scalar curvature terms and was earlier described for R^2 models in [33, 34]. Figure 5 gives a rough illustration of the corresponding deformation of the potential $U(\phi)$ under variation of γ and for a fixed value of Λ_D.

The behaviour of the system is completely different in the lower-half-plane limit $\gamma \to -0$. Here we have to distinguish the dimensions $D = 8$ and $D < 8$. For $D = 8$ equations (D.11) and (D.12) extend to the lower half-plane $\gamma \to -0$, i.e., the system is completely unstable in this limit $\partial^2_\phi U\big|_{\phi_0}(\gamma \to -0) \to -\infty$. Obviously, $\partial^2_\phi U\big|_{\phi_0} \sim 1/\gamma$ in (D.11) encounters a pole singularity with respect to γ. This is different for $\bar{R}_{-,-}$, $D < 8$. Here, one finds from (55) and (D.8)

$$\mathrm{e}^{A\phi_0} = f'(\gamma \to -0) \to \frac{3D}{|D-8|} \tag{D.13}$$

and from (41) and (43)

$$\partial^2_\phi U(\gamma \to -0)|_{\phi_0} \approx -\frac{1}{\gamma^{1/3}}\frac{(D-2)^{1/3}(D-8)^{2/3}}{8(D-1)}\left|\frac{3D}{D-8}\right|^{-\frac{2}{D-2}} \to +\infty \tag{D.14}$$

$$U(\phi_0; \gamma \to -0) \approx \frac{1}{\gamma^{1/3}}\frac{(D-2)^{4/3}}{2D|D-8|)}\left|\frac{3D}{D-8}\right|^{-\frac{2}{D-2}} \to -\infty. \tag{D.15}$$

From these equations and their rough illustration in figure 6 we read off that in the limit $\gamma \to -0$ the minimum of the potential $U(\phi)$ lowers infinitely, $U(\phi_0; \gamma \to -0) \to -\infty$, and becomes fixed at the finite value $\phi_0 = \frac{1}{A}\ln\left|\frac{3D}{D-8}\right|$. The corresponding scalar curvature diverges as $\bar{R}_{-,-}(\gamma \to -0) \to -\infty$ and is separated from the conformal singularity $f'(\phi \to -\infty) \to 0$ by a barrier whose top is defined by the associated maximum branch $\bar{R}_{-,+}$ and tends to the value

$$\bar{R}_{-,+}(\gamma \to -0) \to \frac{2D\Lambda_D}{D-2}, \qquad U|_{\max}(\gamma \to -0) \to \Lambda_D \tag{D.16}$$

(see also figure 6). This means that the metastable sector in the lower (Λ_D, γ) plane is separated from absolutely stable systems in the upper half-plane and their limiting linear $(\gamma \to +0)$ models by an infinite gap. Considering the behaviour of the system over the (Λ_D, γ) plane we have to conclude that it possesses an essential singularity in the limit $\gamma \to -0$, i.e., the scalar curvature $\bar{R}$ encounters an infinite jump between $\gamma \to +0$ $(\bar{R} \to 2D\Lambda_D/(D-2))$ and $\gamma \to -0$ $(\bar{R} \to -\infty)$ whereas $\partial^2_\phi U\big|_{\phi_0}$ has for $\gamma > 0$ a pole-like singularity $\sim\gamma^{-1}$ and for $\gamma < 0$ it behaves like a branching point singularity $\sim\gamma^{-1/3}$. This is a strong indication that the description of a physical system in terms of the considered oversimplified toy model breaks down in the limit $\gamma \to -0$. The search for possibly existing physically realistic metastable systems with $\gamma < 0$ is out of the scope of the present work and we leave the corresponding investigation to future research.

$\boldsymbol{z \to -1}$. Relations (88)–(91) show that this limit can only be reached by metastable configurations $\gamma < 0, D < 8, \bar{R}_{-,-}$. According to (66) it corresponds to a vanishing discriminant $Q = 0$ of the cubic equation (62) so that this equation has two coinciding solutions $u_{1,2} = 2r^{1/3}$ ($v_{1,2}(z = -1) = 2$ holds). From equations (76) and (77) we observe that this leads to a coalescence of the minimum branch $\bar{R}_{-,-}$ and the associated maximum branch $\bar{R}_{-,+}$ in an inflection point at $z = -1$

$$\bar{R}_{-,\pm}(z = -1) = -\frac{1}{2}\sqrt{u_1(z = -1)} = -\frac{r^{1/6}}{\sqrt{2}} = -2^{-2/3}\left(\frac{D-2}{\gamma(D-8)}\right)^{1/3}. \tag{D.17}$$

The situation is also obvious from figure 2. Configurations with $z \leqslant -1$ have no extremum at all and are necessarily unstable.

References

[1] Perlmutter S, Turner M S and White M J 1999 *Phys. Rev. Lett.* **83** 670–3 (*Preprint* astro-ph/9901052)
[2] Jaffe A H *et al* (Boomerang Collaboration) 2001 *Phys. Rev. Lett.* **86** 3475–9 (*Preprint* astro-ph/0007333)
[3] Gasperini M and Veneziano G 1993 *Astropart. Phys.* **1** 317–39 (*Preprint* hep-th/9211021)
Gasperini M and Veneziano G 2003 *Phys. Rep.* **373** 1–212 (*Preprint* hep-th/0207130)
[4] Dvali G R and Tye S H H 1999 *Phys. Lett.* B **450** 72–82 (*Preprint* hep-ph/9812483)
Alexander S H S 2002 *Phys. Rev.* D **65** 023507 (*Preprint* hep-th/0105032)
Burgess C P *et al* 2001 *J. High Energy Phys.* JHEP07(2001)047 (*Preprint* hep-th/0105204)
Gomez-Reino M and Zavala I 2002 *J. High Energy Phys.* JHEP09(2002)020 (*Preprint* hep-th/0207278)
[5] Khoury J, Ovrut B A, Steinhardt P J and Turok N 2001 *Phys. Rev.* D **64** 123522 (*Preprint* hep-th/0103239)
Steinhardt P J and Turok N 2002 *Phys. Rev.* D **65** 126003 (*Preprint* hep-th/0111098)
Khoury J, Steinhardt P J and Turok N 2004 *Phys. Rev. Lett.* **92** 031302 (*Preprint* hep-th/0307132)
[6] Kanno S, Sasaki M and Soda J 2003 *Prog. Theor. Phys.* **109** 357–69 (*Preprint* hep-th/0210250)
[7] Townsend P K and Wohlfarth M N R 2003 *Phys. Rev. Lett.* **91** 061302 (*Preprint* hep-th/0303097)
Ohta N 2003 *Phys. Rev. Lett.* **91** 061303 (*Preprint* hep-th/0303238)
Ohta N 2003 *Prog. Theor. Phys.* **110** 269–83 (*Preprint* hep-th/0304172)
Roy S 2003 *Phys. Lett.* B **567** 322–9 (*Preprint* hep-th/0304084)
Gutperle M, Kallosh R and Linde A 2003 *J. Cosmol. Astropart. Phys.* JCAP07(2003)001 (*Preprint* hep-th/0304225)
Chen C-M, Ho P-M, Neupane I P, Ohta N and Wang J E 2003 *J. High Energy Phys.* JHEP10(2003)058 (*Preprint* hep-th/0306291)
[8] Kallosh R, Linde A, Prokushkin S and Shmakova M 2002 *Phys. Rev.* D **66** 123503 (*Preprint* hep-th/0208156)
[9] Kachru S, Kallosh R, Linde A and Trivedi S P 2003 *Phys. Rev.* D **68** 046005 (*Preprint* hep-th/0301240)
[10] Kachru S *et al* 2003 *J. Cosmol. Astropart. Phys.* JCAP10(2003)013 (*Preprint* hep-th/0308055)
Hsu J P, Kallosh R and Prokushkin S 2003 *J. Cosmol. Astropart. Phys.* JCAP12(2003)009 (*Preprint* hep-th/0311077)
Hsu J P and Kallosh R 2004 *J. High Energy Phys.* JHEP04(2004)042 (*Preprint* hep-th/0402047)
[11] Kallosh R and Prokushkin S 2004 SuperCosmology *Preprint* hep-th/0403060
Burgess C P, Cline J M, Stoica H and Quevedo F 2004 *J. High Energy Phys.* JHEP09(2004)033 (*Preprint* hep-th/0403119)
[12] Navarro I and Santiago J 2004 *J. Cosmol. Astropart. Phys.* JCAP09(2004)005 (*Preprint* hep-th/0405173)
Biswas T and Jaikumar P 2004 *J. High Energy Phys.* JHEP08(2004)053 (*Preprint* hep-th/0407063)
[13] Bojowald M 2002 *Phys. Rev. Lett.* **89** 261301 (*Preprint* gr-qc/0206054)
Bojowald M and Vandersloot K 2003 *Phys. Rev.* D **67** 124023 (*Preprint* gr-qc/0303072)
Tsujikawa S, Singh P and Maartens R 2004 *Class. Quantum Grav.* **21** 5767–75 (*Preprint* astro-ph/0311015)
Bojowald M, Lidsey J E, Mulryne D J, Singh P and Tavakol R 2004 *Phys. Rev.* D **70** 043530 (*Preprint* gr-qc/0403106)
Bojowald M, Maartens R and Singh P 2004 *Phys. Rev.* D **70** 083517 (*Preprint* hep-th/0407115)

[14] Capozziello S, Carloni S and Troisi A 2003 Quintessence without scalar field *Preprint* astro-ph/0303041
Carrol S M, Duvvuri V, Trodden M and Turner M S 2004 *Phys. Rev.* D **70** 043528 (*Preprint* astro-ph/0306438)
Vollick D N 2003 *Phys. Rev.* D **68** 063510 (*Preprint* astro-ph/0306630)
Dick R 2004 *Gen. Rel. Grav.* **36** 217 (*Preprint* gr-qc/0307052)
Dolgov A D and Kawasaki M 2003 *Phys. Lett.* B **573** 1 (*Preprint* astro-ph/0307285)
Nojiri S and Odintsov S D 2003 *Phys. Rev.* D **68** 123512 (*Preprint* hep-th/0307288)
Nojiri S and Odintsov S D 2004 *Gen. Rel. Grav.* **36** 1765–80 (*Preprint* hep-th/0308176)
Chiba T 2003 *Phys. Lett.* B **575** 1 (*Preprint* astro-ph/0307338)
Meng X and Wang P 2003 *Class. Quantum Grav.* **20** 4949–62 (*Preprint* astro-ph/0307354)
[15] Nojiri S and Odintsov S D 2003 *Phys. Lett.* B **576** 5–11 (*Preprint* hep-th/0307071)
[16] Kerner R 1982 *Gen. Rel. Grav.* **14** 453–69
Barrow J D and Ottewill A C 1983 *J. Phys. A: Math. Gen.* **16** 2757–76
Duruisseau J P, Kerner R and Eysseric P 1983 *Gen. Rel. Grav.* **15** 797–807
Whitt B 1984 *Phys. Lett.* B **145** 176–8
Maeda K, Stein-Schabes J A and Futamase T 1989 *Phys. Rev.* D **39** 2848–53
Magnano G and Sokolowski L M 1994 *Phys. Rev.* D **50** 5039–59 (*Preprint* gr-qc/9312008)
Wands D 1994 *Class. Quantum Grav.* **11** 269–79 (*Preprint* gr-qc/9307034)
[17] Barrow J D and Cotsakis S 1988 *Phys. Lett.* B **214** 515–8
[18] Maeda K 1989 *Phys. Rev.* D **39** 3159–62
[19] Ellis J, Kaloper N, Olive K A and Yokoyama J 1999 *Phys. Rev.* D **59** 103503 (*Preprint* hep-ph/9807482)
[20] Günther U and Zhuk A 2001 *Class. Quantum Grav.* **18** 1441–60 (*Preprint* hep-ph/0006283)
[21] Cline J M and Vinet J 2003 *Phys. Rev.* D **68** 025015 (*Preprint* hep-ph/0211284)
[22] Forgacs P and Horvath Z 1979 *Gen. Rel. Grav.* **10** 931–40
Forgacs P and Horvath Z 1979 *Gen. Rel. Grav.* **11** 205
[23] Günther U, Starobinsky A and Zhuk A 2004 *Phys. Rev.* D **69** 044003 (*Preprint* hep-ph/0306191)
[24] Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 *Phys. Lett.* B **429** 263–72 (*Preprint* hep-ph/9803315)
Antoniadis I, Arkani-Hamed N, Dimopoulos S and Dvali G R 1998 *Phys. Lett.* B **436** 257–63 (*Preprint* hep-ph/9804398)
[25] Arkani-Hamed N, Dimopoulos S and March-Russell J 2001 *Phys. Rev.* D **63** 064020 (*Preprint* hep-th/9809124)
Banks T, Dine M and Nelson A E 1999 *J. High Energy Phys.* JHEP06(1999)014 (*Preprint* hep-th/9903019)
Arkani-Hamed N, Dimopoulos S, Kaloper N and March-Russell J 2000 *Nucl. Phys.* B **567** 189–228 (*Preprint* hep-ph/9903224)
Carroll S M, Geddes J, Hoffman M B and Wald R M 2002 *Phys. Rev.* D **66** 024036 (*Preprint* hep-th/0110149)
Geddes J 2002 *Phys. Rev.* D **65** 104015 (*Preprint* gr-qc/0112026)
Demir D A and Shifman M 2002 *Phys. Rev.* D **65** 104002 (*Preprint* hep-ph/0112090)
Nasri S, Silva P J, Starkman G D and Trodden M 2002 *Phys. Rev.* D **66** 045029 (*Preprint* hep-th/0201063)
Perivolaropoulos L and Sourdis C 2002 *Phys. Rev.* D **66** 084018 (*Preprint* hep-ph/0204155)
Perivolaropoulos L 2003 *Phys. Rev.* D **67** 123516 (*Preprint* hep-ph/0301237)
[26] Antoniadis I and Bachas C 1999 *Phys. Lett.* B **450** 83–91 (*Preprint* hep-th/9812093)
Antoniadis I, Benakli K, Laugier A and Maillard T 2003 *Nucl. Phys.* B **662** 40–62 (*Preprint* hep-ph/0211409)
[27] Giddings S B, Kachru S and Polchinski J 2002 *Phys. Rev.* D **66** 106006 (*Preprint* hep-th/0105097)
[28] Kachru S, Schulz M B and Trivedi S 2003 *J. High Energy Phys.* JHEP10(2003)007 (*Preprint* hep-th/0201028)
[29] Kaya A 2004 *J. Cosmol. Astropart. Phys.* JCAP08(2004)014 (*Preprint* hep-th/0405099)
[30] Günther U and Zhuk A 1997 *Phys. Rev.* D **56** 6391–402 (*Preprint* gr-qc/9706050)
[31] Günther U and Zhuk A 1998 *Class. Quantum Grav.* **15** 2025–35 (*Preprint* gr-qc/9804018)
[32] Günther U and Zhuk A 2000 *Phys. Rev.* D **61** 124001 (*Preprint* hep-ph/0002009)
[33] Günther U, Moniz P and Zhuk A 2002 *Phys. Rev.* D **66** 044014 (*Preprint* hep-th/0205148)
[34] Günther U, Moniz P and Zhuk A 2003 *Phys. Rev.* D **68** 044010 (*Preprint* hep-th/0303023)
[35] Arkani-Hamed N, Dimopoulos S and Dvali G 1999 *Phys. Rev.* D **59** 086004 (*Preprint* hep-ph/9807344)
[36] Günther U and Zhuk A 2004 Remarks on dimensional reduction of multidimensional cosmological models *Preprint* gr-qc/0401003 (*Proc. 10th Marcel Grossmann Meeting on General Relativity* at press)
[37] Günther U and Zhuk A *S*-brane scenarios: late-time acceleration and variations of the fine structure constant In preparation
[38] Giddings S B and Myers R C 2004 *Phys. Rev.* D **70** 046005 (*Preprint* hep-th/0404220)
[39] Hannestad S 1999 *Phys. Rev.* D **60** 023515 (*Preprint* astro-ph/9810102)
Kaplinghat M, Scherrer R J and Turner M S 1999 *Phys. Rev.* D **60** 023516 (*Preprint* astro-ph/9810133)
Webb J K *et al* 2001 *Phys. Rev. Lett.* **87** 091301 (*Preprint* astro-ph/0012539)
Srianand R, Chand H, Petitjean P and Aracil B 2004 *Phys. Rev. Lett.* **92** 121302 (*Preprint* astro-ph/0402177)

[40] Ivashchuk V D, Melnikov V N and Zhuk A I 1989 *Nuovo Cimento* B **104** 575–82
[41] Rainer M and Zhuk A 1996 *Phys. Rev.* D **54** 6186–92 (*Preprint* gr-qc/9608020)
[42] Ivashchuk V D, Melnikov V N and Selivanov A B 2003 *J. High Energy Phys.* JHEP09(2003)059 (*Preprint* hep-th/0308113)
[43] Kofman L, Linde A, Liu X, Maloney A, McAllister L and Silverstein E 2004 *J. High Energy Phys.* JHEP05(2004)030 (*Preprint* hep-th/0403001)
[44] Maeda K and Ohta N 2004 *Phys. Lett.* B **597** 400–7 (*Preprint* hep-th/0405205)
[45] Tseytlin A A 2000 *Nucl. Phys.* B **584** 233–50 (*Preprint* hep-th/0005072)
Becker K and Becker M 2001 *J. High Energy Phys.* JHEP07(2001)038 (*Preprint* hep-th/0107044)
Frolov S and Tseytlin A A 2002 *Nucl. Phys.* B **632** 69–100 (*Preprint* hep-th/0111128)
[46] Howe P S 2004 R^4 terms in supergravity and M-theory *Preprint* hep-th/0408177
[47] Abramowitz M and Stegun A 1972 *Handbook of Mathematical Functions* (New York: Dover)
[48] Dvali G R, Gabadadze G, Kolanovic M and Nitti F 2002 *Phys. Rev.* D **65** 024031 (*Preprint* hep-th/0106058)
Burgess C P 2004 *Liv. Rev. Rel.* **7** 5 (*Preprint* gr-qc/0311082)
[49] Paul B C, Mukherjee S and Tavakol R K 2002 *Phys. Rev.* D **65** 064020 (*Preprint* hep-th/0201009)
[50] Linde A 1994 *Phys. Lett.* B **327** 208
Vilenkin A 1994 *Phys. Rev. Lett.* **72** 3137
[51] Blanco-Pillado J J *et al* 2004 *J. High Energy Phys.* JHEP11(2004)063 (*Preprint* hep-th/0406230)
[52] Nakamura T T and Stewart E D 1996 *Phys. Lett.* B **381** 413 (*Preprint* astro-ph/9604103)
Gong J-O and Stewart E D 2002 *Phys. Lett.* B **538** 213 (*Preprint* astro-ph/0202098)
Groot Nibbelink S and van Tent B J W 2000 Density perturbations arising from multiple field slow-roll inflation *Preprint* hep-ph/0011325
[53] Arnold V I (ed) 1994 *Dynamical Systems V: Bifurcation Theory and Catastrophe Theory* (Berlin: Springer)
[54] Linde A 1979 *Pis. Zh. Eksp. Teor. Fiz.* **30** 479–82
Linde A 1979 *JETP Lett.* **30** 447–9 (Engl. Transl.)
Linde A 1980 *Phys. Lett.* B **93** 394–6
[55] Starobinsky A 1981 *Sov. Astron. Lett.* **7** 36–8
[56] Kallosh R and Linde A 1995 *Phys. Rev.* D **52** 7137–45 (*Preprint* hep-th/9507022)
Behrndt K and Kallosh R 1996 *Phys. Rev.* D **53** R589–R592 (*Preprint* hep-th/9509102)
Cvetic M and Youm D 1995 *Phys. Lett.* B **359** 87–92 (*Preprint* hep-th/9507160)
Gaida I 1998 *Class. Quantum Grav.* **15** 2261–9 (*Preprint* hep-th/9803215)
Gaida I, Hollmann H R and Stewart J M 1999 *Class. Quantum Grav.* **16** 2231–46 (*Preprint* hep-th/9811032)
[57] Gregory R, Rubakov V and Sibiryakov S 2000 *Phys. Lett.* B **489** 203–6 (*Preprint* hep-th/0003045)
Csaki C, Erlich J, Hollowood T J and Terning J 2001 *Phys. Rev.* D **63** 065019 (*Preprint* hep-th/0003076)
[58] Johnson C V, Peet A W and Polchinski J 2000 *Phys. Rev.* D **61** 086001 (*Preprint* hep-th/9911161)
Järv L and Johnson C V 2000 *Phys. Rev.* D **62** 126010 (*Preprint* hep-th/0002244)
Johnson C V, Myers R C, Peet A W and Ross S F 2001 *Phys. Rev.* D **64** 106001 (*Preprint* hep-th/0105077)
Dyson L M, Järv L and Johnson C V 2002 *J. High Energy Phys.* JHEP05(2005)019 (*Preprint* hep-th/0112132)
Astefanesei D and Myers R 2002 *J. High Energy Phys.* JHEP02(2002)043 (*Preprint* hep-th/0112133)