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FAST TRACK COMMUNICATION

$\mathcal{PT}$-symmetry, Cartan decompositions, Lie triple systems and Krein space-related Clifford algebras

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Abstract
Gauged $\mathcal{PT}$ quantum mechanics (PTQM) and corresponding Krein space setups are studied. For models with constant non-Abelian gauge potentials and extended parity inversions compact and noncompact Lie group components are analyzed via Cartan decompositions. A Lie-triple structure is found and an interpretation as $\mathcal{PT}$-symmetrically generalized Jaynes–Cummings model is possible with close relation to recently studied cavity QED setups with transmon states in multilevel artificial atoms. For models with Abelian gauge potentials a hidden Clifford algebra structure is found and used to obtain the fundamental symmetry of Krein space-related J-self-adjoint extensions for PTQM setups with ultra-localized potentials.

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Introduction

During the last 10 years many of the basic features of quantum mechanics with $\mathcal{PT}$-symmetric Hamiltonians (PTQM) [1, 2] have been worked out in detail and are now to a certain degree well understood. This concerns the mapping of the PTQM sector of exact $\mathcal{PT}$-symmetry to conventional (von-Neumann) quantum mechanics with Hermitian Hamiltonians [3], the relevance of the $\mathcal{C}$-operator as dynamically adapted mapping [4] between Krein space-related indefinite metric structures [5] and positive definite metrics of usual Hilbert spaces (required for a sensible probabilistic interpretation of the related wavefunctions) as well as the understanding of $\mathcal{PT}$-symmetric Hamiltonians as self-adjoint operators in Krein spaces [6–10].

Here, we will discuss some up to now unnoticed structural links of PTQM, and Krein space-related models in general, to Lie algebra and Lie group-related Cartan decompositions

[11], Lie triple systems [12–17] as well as to Clifford algebras [18]. Identifying these underlying structures will help in recognizing hidden $\mathcal{PT}$-like involutory structures in physical models which are up to now not related with $\mathcal{PT}$-symmetry and to deeper understand these models and the role of $\mathcal{PT}$-symmetry in general.

We start from the simplest $\mathcal{PT}$-symmetric Hamiltonian H, $[\mathcal{PT}, H] = 0$, of differential operator type:

$$H = p^2 + V(x), \quad p := -\mathrm{i}\partial_x, \quad V(-x) = V^*(x), \quad \mathcal{P}x\mathcal{P} = -x, \quad \mathcal{P}p\mathcal{P} = -p$$
$$\mathcal{T}\mathrm{i}I\mathcal{T} = -\mathrm{i}I, \quad \mathcal{T}x\mathcal{T} = x, \quad \mathcal{T}p\mathcal{T} = -p. \tag{1}$$

In general, this Hamiltonian is a $\mathcal{P}$-self-adjoint operator in a Krein space $(\mathcal{K}, [\cdot,\cdot]_\mathcal{P})$ (see, e.g., [19, 20]) with $[\cdot,\cdot]_\mathcal{P} := (\cdot, \mathcal{P}\cdot)$ being the $\mathcal{PT}$ inner product [2], $[H\phi, \psi]_\mathcal{P} = [\phi, H\psi]_\mathcal{P}$, i.e.

$$\mathcal{P}H = H^\dagger\mathcal{P}. \tag{2}$$

Because of $\mathcal{P}p = -p\mathcal{P} = -p^\dagger\mathcal{P}$, i.e. $[p\phi, \psi]_\mathcal{P} = -[\phi, p\psi]_\mathcal{P}$, this $\mathcal{P}$-self-adjointness is spoilt for the gauged Hamiltonian

$$H_g = (p - A)^2 + V(x), \qquad A(-x) = A^*(x), \qquad \mathcal{P}H_g \neq H_g^\dagger\mathcal{P}. \tag{3}$$

Instead the gauge transformation (Kummer–Liouville transformation)

$$U: H_g \mapsto H = UH_gU^{-1}, \qquad U = \mathrm{e}^{-\mathrm{i}\int_0^x A(s)\,\mathrm{d}s} \tag{4}$$

together with (2), $\mathcal{P} = \mathcal{P}^\dagger$ and $[U^\dagger]^{-1} = [U^{-1}]^\dagger$ leads to the pseudo-Hermiticity condition

$$\eta H_g = H_g^\dagger\eta, \qquad \eta := U^\dagger\mathcal{P}U, \qquad \eta = \eta^\dagger. \tag{5}$$

$\mathcal{PT}$-symmetry of the system remains preserved under the gauge transformation U:

$$[\mathcal{PT}, U] = 0, \qquad [\mathcal{PT}, H_g] = 0, \qquad [\mathcal{PT}, H] = 0. \tag{6}$$

These facts are well known and have been widely discussed for various PTQM models [21–25].

Next we assume, for simplicity, a purely real coordinate dependence $x \in \Omega \subseteq \mathbb{R}$ with Ω any $\mathcal{P}$-symmetric interval. Then splitting $A(x) = A_+(x) + \mathrm{i}A_-(x)$ into even and odd components, $\mathcal{P}A_\pm(x) = A_\pm(-x) = \pm A_\pm(x)$, leads to a factorization of U into unitary and Hermitian $\mathcal{P}$-self-adjoint factors

$$U = U_uU_h, \qquad U_u = \mathrm{e}^{-\mathrm{i}\int_0^x A_+(s)\,\mathrm{d}s}, \qquad U_h = \mathrm{e}^{\int_0^x A_-(s)\,\mathrm{d}s} \tag{7}$$

$$U_u^\dagger = U_u^{-1}, \qquad U_h^\dagger = U_h, \qquad \mathcal{P}U = U^\dagger\mathcal{P}, \qquad \mathcal{P}U_u = U_u^\dagger\mathcal{P}, \qquad \mathcal{P}U_h = U_h\mathcal{P}. \tag{8}$$

This is just the simplest (Abelian) version of a polar decomposition which here is naturally associated with the corresponding decomposition of the metric $\eta = J|\eta|$ into the modulus $|\eta| := \sqrt{\eta^2} = U_h^2$ and involution $J := \eta|\eta|^{-1} = U_u^{-1}\mathcal{P}U_u = J^\dagger = J^{-1}$. It shows that H_g is J-self-adjoint in the weighted ($|\eta|$-deformed) Hilbert space $L_2(|\eta|\mathrm{d}x)$ with the inner product $(\phi, \psi)_{|\eta|} := \int_\mathbb{R} \psi(x)\phi^*(x)\,\mathrm{e}^{2\int_0^x A_-(s)\,\mathrm{d}s}\,\mathrm{d}x$

$$(H_g\phi, J\psi)_{|\eta|} = (\phi, JH_g\psi)_{|\eta|}. \tag{9}$$

Obviously, the unitary component U_u of the gauge transformation $U(x)$ rotates the original involution (Krein space metric) $\mathcal{P}$ into the new involution $J = U_u^{-1}\mathcal{P}U_u$ whereas the Hermitian component U_h induces the new integration weight $|\eta|$, i.e. we have a Krein space mapping $U: (\mathcal{K}_\mathcal{P}, [\cdot,\cdot]_\mathcal{P}) \mapsto (\tilde{\mathcal{K}}_J, [\cdot,\cdot]_{|\eta|J})$.

A further mapping ρ will be needed to pass from $L_2(|\eta|\mathrm{d}x)$ in (9) to a Hilbert space $\mathcal{H}$ where a Hamiltonian H_g with a real spectrum (exact $\mathcal{PT}$-symmetry) will be not only J-self-adjoint but self-adjoint [3, 26]. This ρ will strongly depend on the concrete form of the $\mathcal{PT}$-symmetric potentials $A(x)$, $V(x)$ and, in general, it will be highly nonlocal [2, 27].

Subsequently, we mainly concentrate on the symmetry structures inherent in the model and we will not focus on the nonlocalities as the latter are typical, e.g., for the construction of $\mathcal{C}$ operators for Hamiltonians built over differential operators [28].

The above decomposition (7) indicates on two ways of possible model generalizations based (i) on a generalization of the Abelian gauge potential to a non-Abelian one or, via slightly different structures, (ii) on the direct use of a hidden Clifford algebra.

Non-Abelian gauge potentials, Cartan decompositions and Lie triple systems

First we note that the decomposition (7) of the gauge transformation U into unitary and Hermitian components can be regarded as the trivial Abelian version of a Cartan decomposition of a Lie group into a compact subgroup and a noncompact homogeneous coset space. Subsequently we demonstrate the interrelation of $\mathcal{PT}$-symmetry and Cartan decompositions of Lie groups (and Lie algebras) on the simplest example of a matrix Hamiltonian with non-Abelian but constant[3] gauge potential A. The parity inversion $\mathbf{P}$ is assumed to be of tensor product type, i.e. we set

$$H_g = (p - A)^2 + V(x), \qquad A \in \mathbb{C}^{m\times m}, \qquad V(x) \in \mathbb{C}^{m\times m} \otimes L_1(\mathbb{R}) \tag{10}$$

$$[\mathbf{P}\mathcal{T}, H_g] = 0, \qquad \mathbf{P} = \Theta \otimes \mathcal{P}, \qquad \Theta \in \mathbb{R}^{m\times m}, \qquad \Theta^2 = I_m, \qquad \mathbf{P}^2 = I_m \otimes I. \tag{11}$$

Involution property $\Theta^2 = I_m$ and reality $\Theta \in \mathbb{R}^{m\times m}$ imply diagonalizability and symmetry of the matrix $\Theta = \Theta^T$. This means that without loss of generality, i.e. modulo a global $SO(m,\mathbb{R})$ rotation, we may fix henceforth $\Theta = I_{p,q} = \mathrm{diag}(I_p, -I_q)$, $p + q = m$. Furthermore, we assume for simplicity that $\mathcal{T}$ acts as the same complex conjugation as for the scalar Hamiltonian (3), i.e. $\mathcal{T} \cong I_m \otimes \mathcal{T}$ so that involution commutativity concerning the extended parity inversion $\mathbf{P}$ is fulfilled trivially[4] $[\mathbf{P}, \mathcal{T}] = 0$. In this case $\mathbf{P}\mathcal{T}$-symmetry, $[\mathbf{P}\mathcal{T}, H_g] = 0$, implies

$$\Theta A^* \Theta = A, \qquad \Theta V^*(-x)\Theta = V(x) \tag{12}$$

whereas $\mathbf{P}$-self-adjointness $\mathbf{P}H^\dagger\mathbf{P} = H$ of the globally re-gauged Hamiltonian

$$H = UH_gU^{-1} = p^2 + \mathrm{e}^{-\mathrm{i}Ax}V(x)\,\mathrm{e}^{\mathrm{i}Ax}, \qquad U = \mathrm{e}^{-\mathrm{i}Ax} \tag{13}$$

leads to the additional conditions

$$\Theta A^\dagger \Theta = -A, \qquad \Theta V^\dagger(-x)\Theta = V(x). \tag{14}$$

Together (12) and (14) give $A = -A^T$, $V = V^T$, and they fix via (13) the Lie group structure of the gauge transformation U. Denote the set of corresponding Lie group elements by $G_\Theta \ni U$ and the vector space of its Lie algebra elements by g_Θ. Then for the elements $a \in g_\Theta$, because of $a := -\mathrm{i}A$, it holds

$$a = -a^T, \qquad \Theta a^\dagger \Theta = a. \tag{15}$$

[3] In case of non-Abelian local (coordinate-dependent) gauge potentials in theories over a spacetime manifold $\mathcal{M}$ (e.g. over usual Minkowski space) finite gauge transformation operators U will have the form of path-ordered exponentials. For simplicity we restrict our consideration here to constant gauge transformations only.

[4] In general, the time involution $\mathcal{T}$ may be extended nontrivially to any anti-linear involution $\mathbf{T} = \mu \otimes \mathcal{T}$ with $\mu^2 = I_m$, $\mu \in \mathbb{C}^{m\times m}$. In the simplest case of $\mu \in \mathbb{R}^{m\times m}$, involution commutativity $[\mathbf{P}, \mathbf{T}] = 0$ together with fixed $\Theta = I_{p,q}$ implies a block-diagonal $\mu = \mathrm{diag}(\mu_p, \mu_q) = SI_{r,s}S^{-1}$, $S \in SO(m,\mathbb{R})$ with a possibly different signature $(r, s) \neq (p, q)$. Moreover, even involution commutativity may be violated, $[\mathbf{P}, \mathbf{T}] \neq 0$ as, e.g., for the pinor-representations [29] of the Dirac equation. We leave corresponding considerations to future research and restrict our attention here to the simplest ansatz $\mathbf{T} = I_m \otimes \mathcal{T}$ only.

Hence, g_Θ is constituted by the Θ-Hermitian elements of $so(m,\mathbb{C})$. In order to understand the role of this Θ-Hermiticity condition we first note that the compact subgroup of the special complex orthogonal group $SO(m,\mathbb{C})$ is the real orthogonal group $SO(m,\mathbb{R})$, whereas the (homogeneous) coset space $SO(m,\mathbb{C})/SO(m,\mathbb{R})$ parameterizes the noncompact ('boost'-type) transformations. This is well known (see, e.g. [11], chapter 9, section II) and follows trivially from the Cartan decomposition of general $GL(m,\mathbb{C})$ matrices into unitary compact components and Hermitian noncompact components (i.e. from their polar decomposition). In fact, the corresponding Cartan involution τ for the Lie algebra $gl(m,\mathbb{C}) \ni a$ is $\tau(a) = -a^\dagger$ and $gl(m,\mathbb{C})$ can be decomposed as $gl(m,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ with $\tau\mathfrak{k} = \mathfrak{k}$, $\tau\mathfrak{p} = -\mathfrak{p}$ for compact subalgebra $\mathfrak{k}$ and the set of noncompact coset elements $\mathfrak{p}$, respectively. Imposing the additional antisymmetry restriction $a = -a^T$ for $so(m,\mathbb{C})$ elements the Cartan involution reduces to complex conjugation $\tau(a) = -a^\dagger = a^* = \mathcal{T}a$. Accordingly, $\mathcal{T}$ splits $so(m,\mathbb{C})$ just into real and purely imaginary components

$$so(m,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}, \qquad \mathfrak{k} = so(m,\mathbb{R}), \qquad \mathfrak{p} = \{b \in so(m,\mathbb{C}) | b = \mathrm{i}f,\ f \in so(m,\mathbb{R})\} \tag{16}$$

$$\mathcal{T}\mathfrak{k} = \mathfrak{k}, \qquad \mathcal{T}\mathfrak{p} = -\mathfrak{p}. \tag{17}$$

The Θ-Hermiticity condition in (15) refines this decomposition by an additional Θ-related block structure. Explicitly $\Theta a^\dagger \Theta = a$ implies

$$a =: \begin{pmatrix} \mathrm{i}u & v \\ -v^T & \mathrm{i}w \end{pmatrix}, \qquad u \in \mathbb{R}^{p\times p}, \qquad v \in \mathbb{R}^{p\times q}, \qquad w \in \mathbb{R}^{q\times q} \tag{18}$$

$$\mathfrak{k}_\Theta = \left\{ b \in so(m,\mathbb{R}) | b = \begin{pmatrix} 0 & v \\ -v^T & 0 \end{pmatrix} \right\}, \tag{19}$$

$$\mathfrak{p}_\Theta = \left\{ c \in so(m,\mathbb{C}) | c = \mathrm{i}f = \begin{pmatrix} \mathrm{i}u & 0 \\ 0 & \mathrm{i}w \end{pmatrix},\ f \in so(p,\mathbb{R}) \oplus so(q,\mathbb{R}) \right\} \tag{20}$$

$$b^\dagger = -b, \quad b \in \mathfrak{k}_\Theta, \qquad c^\dagger = c, \qquad c \in \mathfrak{p}_\Theta. \tag{21}$$

Denoting the Cartan decomposition of $su(p,q)$ by[5]

$$su(p,q) = \mathfrak{l} \oplus \mathfrak{q}, \qquad \mathfrak{l} = s(u(p) \oplus u(q)), \qquad \mathfrak{q} = su(p,q) \ominus \mathfrak{l} \tag{22}$$

we see from $a = -\mathrm{i}A$ with $A = -A^T$ and $\Theta A^\dagger \Theta = -A$, i.e. $A \in so(m,\mathbb{C}) \cap su(p,q)$, that

$$\begin{aligned} g_\Theta = \{a \in so(m,\mathbb{C}) | a = \mathrm{i}f, \qquad f \in so(m,\mathbb{C}) \cap su(p,q)\} = \mathfrak{k}_\Theta \oplus \mathfrak{p}_\Theta \\ \mathfrak{k}_\Theta = so(m,\mathbb{C}) \cap \mathrm{i}\mathfrak{q}, \qquad \mathfrak{p}_\Theta = so(m,\mathbb{C}) \cap \mathrm{i}\mathfrak{l}. \end{aligned} \tag{23}$$

This means that g_Θ can be considered as a 'Wick rotated' $so(m,\mathbb{C}) \cap su(p,q)$, an $so(m,\mathbb{C}) \cap su(p,q)$ with Weyl unitary trick applied not only to the noncompact component $\mathfrak{q}$ but to the algebra as a whole. Correspondingly the roles of compact and noncompact components in $su(p,q) \cap so(m,\mathbb{C})$ and g_Θ are interchanged $\mathfrak{l}, \mathfrak{q} \rightleftarrows \mathfrak{p}_\Theta, \mathfrak{k}_\Theta$. The latter fact explains the block-diagonal decomposition of the noncompact $\mathfrak{p}_\Theta$ in (19) and the off-diagonal block form of $\mathfrak{k}_\Theta$.

Next we note that the intersection set g_Θ is not a Lie algebra itself. Rather this Lie algebra subspace g_Θ forms a Lie triple system (LTS) (see, e.g., [14], section 1.1; [17], section 10). To see this we follow standard techniques [12–16] and denote by κ the Lie algebra involution

$$\kappa(a) := -\Theta a^\dagger \Theta. \tag{24}$$

[5] Recall that the compact subgroup of $SU(p,q)$ is $S(U(p) \times U(q))$ (see, e.g., [11]).

Then the Θ-Hermiticity condition in (15) defines g_Θ as κ-odd subspace in $so(m,\mathbb{C})$

$$g_\Theta = \{a \in so(m,\mathbb{C})|\kappa(a) = -a\}, \tag{25}$$

whereas the commutator $[g_\Theta, g_\Theta]$ is κ-even $\kappa([g_\Theta, g_\Theta]) = [g_\Theta, g_\Theta]$, i.e. g_Θ does not close under the Lie bracket $[g_\Theta, g_\Theta] \nsubseteq g_\Theta$. It only closes under the ternary composition[6]

$$a, b, c \in g_\Theta : \qquad [a, [b, c]] \in g_\Theta \tag{26}$$

so that g_Θ is indeed a Lie triple system (LTS) $[[g_\Theta, g_\Theta], g_\Theta] \subseteq g_\Theta$.

For completeness, we display the Cartan decomposition of the group elements of the set $G_\Theta = K_\Theta \Pi_\Theta$. Separately considered the compact and the noncompact subset, $K_\Theta \subset SO(m,\mathbb{R})$ and $\Pi_\Theta \subset SO(m,\mathbb{C})/SO(m,\mathbb{R})$, have parameterizations induced by the corresponding Lie algebra elements in (19), (20) (see e.g. [11], chapter 9, section IV)

$$K_\Theta = \left\{ U_\mathfrak{k} \in SO(m,\mathbb{R})|\ U_\mathfrak{k} = \mathrm{e}^{bx} = \begin{pmatrix} \cos(\sqrt{vv^T}x) & v\frac{\sin(\sqrt{v^Tv}x)}{\sqrt{v^Tv}} \\ -\frac{\sin(\sqrt{v^Tv}x)}{\sqrt{v^Tv}}v^T & \cos(\sqrt{v^Tv}x) \end{pmatrix},\ b \in \mathfrak{k}_\Theta \right\},$$
$$\Pi_\Theta = \{U_\mathfrak{p} \in SO(m,\mathbb{C})/SO(m,\mathbb{R})|\ U_\mathfrak{p} = \mathrm{e}^{cx} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}ux}, \mathrm{e}^{\mathrm{i}wx}),\ c \in \mathfrak{p}_\Theta\}. \tag{27}$$

Furthermore, it follows from (21) that

$$U_\mathfrak{k}^\dagger = U_\mathfrak{k}^{-1}, \qquad U_\mathfrak{p}^\dagger = U_\mathfrak{p} \tag{28}$$

as the generalization of decomposition (7) for the Abelian gauge transformation.

In the trivial case of $\Theta = I_m$ there is no compact subgroup present at all and the global gauge transformations U are pure boosts

$$U = \mathrm{e}^{\mathrm{i}ux} = \mathrm{e}^{-\mathrm{i}Ax} \in \Pi_I, \qquad A = -A^T \in \mathbb{R}^{m\times m}, \qquad U = U^\dagger. \tag{29}$$

This fact is due to the obvious anti-Hermiticity of the gauge potential $A = -A^\dagger$ which is in clear contrast to the Hermitian gauge potentials present in the Hermitian Hamiltonians of conventional (von Neumann) quantum mechanics. For $m = 2$, e.g., it holds $\mathrm{i}u = \alpha\sigma_2$, $\alpha \in \mathbb{R}$ with $A = \mathrm{i}\alpha\sigma_2$ so that $U = \mathrm{e}^{\alpha\sigma_2 x} = \cosh(\alpha x)I_2 + \sinh(\alpha x)\sigma_2$ similar to earlier findings e.g. in [33, 34].

In contrast, for $\Theta \neq I_m$, $m \geqslant 2$ and vanishing noncompact component, we find the gauge potentials A as antisymmetric Hermitian matrices $A \in \mathrm{i}\mathfrak{k}_\Theta = \{A \in so(m,\mathbb{C})|A = \mathrm{i}b,\ b \in so(m,\mathbb{R})\}$. In the simplest case, $m = 2$, this reduces to $\Theta = \sigma_3$, $A = \alpha\sigma_2$, $\alpha \in \mathbb{R}$ and $U_\mathfrak{k} = \mathrm{e}^{-\mathrm{i}\alpha\sigma_2 x} \in SO(2,\mathbb{R}) \subset U(2)$.

For general Θ the gauge potential A will be composed simultaneously of anti-Hermitian as well as Hermitian components corresponding to non-compact and compact components of the Lie algebra element a, respectively.

The global gauge transformations $U \in G_\Theta$ are $\mathbf{P}\mathcal{T}$-symmetry preserving

$$[\mathbf{P}\mathcal{T}, U] = 0, \qquad [\mathbf{P}\mathcal{T}, H_g] = 0, \qquad [\mathbf{P}\mathcal{T}, H] = 0, \tag{30}$$

in analogy to (6) for Abelian systems. In contrast, the $\mathbf{P}$-symmetry properties of the $U \in G_\Theta$ components are reversed compared to that for the Abelian U in (8):

$$U \in G_\Theta : \qquad \mathbf{P}U_\mathfrak{k} = U_\mathfrak{k}\mathbf{P}, \qquad \mathbf{P}U_\mathfrak{p} = U_\mathfrak{p}^{-1}\mathbf{P}. \tag{31}$$

This reversed behavior can be traced back to the special interplay of complex conjugation and the antisymmetry of the gauge potential as an $so(m,\mathbb{C})$ element. On its turn it implies (via $\mathbf{P}$-Hermiticity of the re-gauged Hamiltonian H in (13), the relation to the original Hamiltonian

[6] From the large number of recent studies on ternary and n-ary Lie algebras as well as metric Lie 3- and n-algebras we note as few examples [17, 30–32].

H_g, as well as (28), (31) and the decomposition $U = U_{\mathfrak{k}}U_{\mathfrak{p}}$) that H_g itself is **P**-Hermitian as well:

$$\mathbf{P}H = H^\dagger\mathbf{P} \quad\Longrightarrow\quad \eta H_g = H_g^\dagger\eta, \qquad \eta = U^\dagger\mathbf{P}U = U_{\mathfrak{p}}U_{\mathfrak{k}}^{-1}\mathbf{P}U_{\mathfrak{k}}U_{\mathfrak{p}} = \mathbf{P}. \tag{32}$$

A simple explicit comparison of the $\mathcal{P}$- and **P**-pseudo-Hermiticity conditions for the gauged Hamiltonians in (3) and (10) shows that the violation of the $\mathcal{P}$-Hermiticity for a scalar H_g with an Abelian gauge potential is due to the non-vanishing derivative term $\mathrm{i}\partial_x A(x)$ in H_g. The vanishing of this term $\mathrm{i}\partial_x A = 0$ for the constant (global) gauge potential A removes this obstruction and leads to preserved **P**-self-adjointness of H_g in (10), $[H_g\phi, \psi]_{\mathbf{P}} = [\phi, H_g\psi]_{\mathbf{P}}$. Effectively, this results from the sign invariance of the Ap-term under the simultaneous action of $\mathcal{P}p = -p\mathcal{P}$ and $\Theta A = -A^\dagger\Theta$ used for the construction of the Krein space adjoint with regard to $[\cdot,\cdot]_{\mathbf{P}}$.

Before we turn to the discussion of Clifford algebra-related structures in the $\mathcal{PT}$-symmetric scalar Schrödinger equation, we note that the $\mathbf{P}\mathcal{T}$-symmetric matrix Hamiltonian H_g in (10) with the constant gauge potential A and appropriately chosen $V(x)$ can be related to a Jaynes–Cummings type Hamiltonian[7] with additional non-Hermitian $\mathbf{P}\mathcal{T}$-symmetric degrees of freedom. To see this we introduce creation and annihilation operators $d^\dagger := (-\mathrm{i}p + x)/\sqrt{2}$, $d := (\mathrm{i}p + x)/\sqrt{2}$ and split the Lie algebra element a (see equation (18)) in strictly upper and lower triangular (nilpotent) components

$$a = c - c^T, \qquad c := \begin{pmatrix} \mathrm{i}\tilde{u} & v \\ 0 & \mathrm{i}\tilde{w} \end{pmatrix}, \qquad c^m = 0 \tag{33}$$

with $\tilde{u}$, $\tilde{w}$ the strictly upper triangular components of u, w. For

$$V(x) = (x^2 - 1)I_m + 2(c + c^T)x + a^2 + 2\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \mathrm{diag}[\omega_1, \ldots, \omega_m], \qquad \omega_j \in \mathbb{R} \tag{34}$$

and particle number operator $N = d^\dagger d$ this yields, e.g.,

$$\tfrac{1}{2}H_g = N + \sqrt{2}(cd + c^T d^\dagger) + \boldsymbol{\omega} \tag{35}$$

describing a special type of $\mathbf{P}\mathcal{T}$-symmetry preserving (gain-loss-balanced[8]) d-particle-induced excitation process in a multi-level quantum system. Models of this type can be considered, e.g., as $\mathbf{P}\mathcal{T}$-symmetric generalization of the recently studied circuit and cavity QED setups [43, 44] allowing for the interaction of a single (d-)mode of the cavity electromagnetic field with a set of transmon states of a multilevel artificial atom with level energies ω_j.

Krein space-related hidden Clifford algebra

The analysis of the scalar $\mathcal{PT}$-symmetric Hamiltonian (3) with the local Abelian gauge potential $A(x)$ can be pursued in another direction by concentrating on the symmetry properties of the unitary factor $U_u = \mathrm{e}^{-\mathrm{i}\mathcal{Q}}$, $\mathcal{Q} := \int_0^x A_+(s)\,\mathrm{d}s$ in (7) which was responsible for the rotation of the involution as $U_u : \mathcal{P} \mapsto J = U_u^{-1}\mathcal{P}U_u$. Representing $\mathcal{Q}$ as

$$\mathcal{Q} = \mathcal{R}q, \qquad \mathcal{R} := \mathrm{sign}(\mathcal{Q}), \qquad q := |\mathcal{Q}| \tag{36}$$

we see that the essential structure underlying the $\mathcal{P}$-Hermiticity condition $\mathcal{P}U = U^\dagger\mathcal{P}$ together with $\mathcal{PQ} = -\mathcal{QP}$ and $\mathcal{P}q = q\mathcal{P}$ is the anticommutation of space reflection operator $\mathcal{P}$ and sign operator $\mathcal{R}$:

$$\mathcal{PR} = -\mathcal{RP}. \tag{37}$$

[7] For recent discussions of Jaynes–Cummings models see, e.g., [35, 36].
[8] For other $\mathcal{PT}$-symmetric gain-loss-balanced systems see, e.g., [37–42].

From the fact that $\mathcal{R}$ and $\mathcal{P}$ are involutions, $\mathcal{R}^2 = \mathcal{P}^2 = I$, we find that they can be interpreted as basis (generating) elements of the real Clifford algebra

$$R_{2,0} = \mathrm{span}_{\mathbb{R}}\{I, \mathcal{P}, \mathcal{R}, \mathcal{P}\mathcal{R}\} \tag{38}$$

or its complex extension

$$Cl_2 = \mathrm{span}_{\mathbb{C}}\{I, \mathcal{P}, \mathcal{R}, \mathcal{P}\mathcal{R}\}. \tag{39}$$

We recall that a real Clifford algebra $R_{m,n}$ with generating elements $\{e_k\}_{k=1}^{m+n}$

$$\begin{aligned} &\{e_i, e_k\} := e_i e_k + e_k e_i = 0 \quad \forall \mathrm{i} \neq k \\ &e_i^2 = I \quad \forall \mathrm{i} = 1, \ldots, m, \qquad e_i^2 = -I \quad \forall \mathrm{i} = m+1, \ldots, m+n \end{aligned} \tag{40}$$

is naturally related to an indefinite form $B(x, y) = \sum_{k=1}^{m} x_k y_k - \sum_{k=m+1}^{m+n} x_k y_k$ over $\mathbb{R}^{m+n} \ni x, y$ (see, e.g. [18], section I.1.1). By embedding $R_{m,n}$ into a complex Clifford algebra, Cl_{m+n}, (complexifying it) the indefinite metric structure becomes irrelevant and it holds $R_{m,n} \hookrightarrow R_{m,n} \times \mathbb{C} \simeq Cl_{m+n}$ for any metric signature (m, n) with fixed value $m + n$. For Cl_{m+n} it suffices to work with basis elements of positive type $e_k^2 = I$, $\forall k = 1, \ldots, m+n$ so that the concrete interpretation as (38) or (39) depends only on whether one works with an $\mathbb{R}$- or a $\mathbb{C}$-span.

For a gauged scalar Hamiltonian H_g the Clifford algebra structures become especially clearly pronounced, e.g. when the potentials $A(x)$ and $V(x)$ in (3) under appropriate regularization are shrunken to an ultra-local support of delta-function type (see e.g. [45, 46]). Below we demonstrate this fact on a Hamiltonian with general regularized zero-range potential at the point $x = 0$ as studied, e.g., in [45, 46]:

$$H_{\mathrm{reg}} = p^2 + t_{11}\langle\delta, \cdot\rangle\delta + t_{12}\langle\delta', \cdot\rangle\delta + t_{21}\langle\delta, \cdot\rangle\delta' + t_{22}\langle\delta', \cdot\rangle\delta'. \tag{41}$$

The concrete operator realization H_T $(T = \|t_{ij}\|)$ in $L_2(\mathbb{R})$ can be defined by setting

$$H_T = H_{\mathrm{reg}} \restriction \mathcal{D}(H_T), \qquad \mathcal{D}(H_T) = \left\{ f \in W_2^2(\mathbb{R}\backslash\{0\}) : H_{\mathrm{reg}} f \in L_2(\mathbb{R}) \right\}, \tag{42}$$

where the derivative $p^2 = -\partial_x^2$ acts on $W_2^2(\mathbb{R}\backslash\{0\})$ in the distributional sense and the regularized delta-function δ and its derivative δ' (with support at 0) are defined on the piecewise continuous functions $f \in W_2^2(\mathbb{R}\backslash\{0\})$ as (for more details see, e.g., [46])

$$\langle\delta, f\rangle = [f(+0) + f(-0)]/2, \qquad \langle\delta', f\rangle = -[f'(+0) + f'(-0)]/2.$$

Denoting the set of $\mathcal{PT}$-symmetric operators H_T, $[\mathcal{PT}, H_T] = 0$, by $\mathcal{N}_{\mathcal{PT}}$ one immediately verifies that $H_T \in \mathcal{N}_{\mathcal{PT}} \iff t_{11}, t_{22} \in \mathbb{R}, \quad t_{12}, t_{21} \in \mathrm{i}\mathbb{R}$. $\mathcal{N}_{\mathcal{PT}}$ contains the subset of $\mathcal{P}$-self-adjoint Hamiltonians which are determined by the condition $t_{12} = t_{21}$. For their $\mathcal{PT}$-symmetric potentials $V = t_{11}\langle\delta, \cdot\rangle\delta + t_{12}\langle\delta', \cdot\rangle\delta + t_{21}\langle\delta, \cdot\rangle\delta' + t_{22}\langle\delta', \cdot\rangle\delta'$ it additionally holds

$$\mathcal{P}V^\dagger = V\mathcal{P}, \qquad \langle Vu, v\rangle = \langle u, V^\dagger v\rangle, \quad u, v \in W_2^2(\mathbb{R}\backslash\{0\}). \tag{43}$$

In analogy to the gauged Hamiltonians (3), this $\mathcal{P}$-self-adjointness can be modified toward a $\mathcal{P}_\phi$-self-adjointness with Clifford-rotated involution

$$\mathcal{P}_\phi = \mathcal{P}\,\mathrm{e}^{\mathrm{i}\phi\mathcal{R}} = \mathrm{e}^{-\mathrm{i}\phi\mathcal{R}/2}\mathcal{P}\,\mathrm{e}^{\mathrm{i}\phi\mathcal{R}/2}, \qquad \mathcal{R}f(x) := \mathrm{sign}(x)f(x) \tag{44}$$

so that an appropriate Krein space involution can be constructed for any parameter combination $t_{12} \neq t_{21}$ as well. The Clifford rotation angle ϕ is fixed by the parameters of the matrix T and can be defined from the relation

$$\mathrm{i}\sin(\phi)\,[\det(T) + 4] = 2\cos(\phi)(t_{12} - t_{21}). \tag{45}$$

The derivation of this relation is based on the interpretation of the $\mathcal{PT}$-symmetric operators H_T as extensions of the symmetric operator

$$H_{\rm sym} = -\partial_x^2, \qquad \mathcal{D}(H_{\rm sym}) = \left\{u(x) \in W_2^2(\mathbb{R}\backslash\{0\}) \mid u(0) = u'(0) = 0\right\}. \qquad (46)$$

It will be presented in full detail in [47]. For the specific angle ϕ the $\mathcal{PT}$-symmetric Hamiltonian H_T in (42) is $\mathcal{P}_\phi$-self-adjoint, $\mathcal{P}_\phi H_T^\dagger = H_T \mathcal{P}_\phi$. Accordingly, for the $\mathcal{PT}$-symmetric potential V it holds (conf. (43))

$$\mathcal{P}_\phi V^\dagger = V\mathcal{P}_\phi, \qquad \langle Vu, v\rangle = \langle u, V^\dagger v\rangle, \qquad u, v \in W_2^2(\mathbb{R}\backslash\{0\}) \qquad (47)$$

with the rotated involution $\mathcal{P}_\phi = \mathrm{e}^{-\mathrm{i}\phi\mathcal{R}/2}\mathcal{P}\mathrm{e}^{\mathrm{i}\phi\mathcal{R}/2}$ built from the Clifford algebra elements (involutions) $\mathcal{P}$ and $\mathcal{R}$. In the special case of $\phi = 0$ equation (45) implies $t_{12} = t_{21}$ so that (47) indeed coincides with (43), and $\mathcal{P}_{\phi=0} = \mathcal{P}$.

Concluding remarks

- The Cartan decomposition used here for the structure analysis of the gauge potentials A can also be applied to the similarity transformation[9] ρ which maps a spectrally diagonalizable $\mathcal{PT}$-symmetric Hamiltonian H with real spectrum into its equivalent Hermitian operator $h = \rho H\rho^{-1}$. Although, in general, ρ is a highly nonlocal operator, as similarity transformation it can nevertheless be understood as the Lie group element. Within the framework of generalized Cartan decompositions the Hermiticity $\rho = \rho^\dagger$ and positivity $\rho > 0$ clearly indicate that ρ should be an element of some noncompact coset space. For the simple finite-dimensional matrix setups of [33, 34, 37] this non-compactness of ρ was clearly visible in the corresponding $SO(m, \mathbb{C})$ 'boost'-type.
- The possible use of the generalized Jaynes–Cummings setup of [43, 44] as reliable experimental candidate for the implementation of qubit states, together with the structural links indicated here, seems to open a new and interesting playground for experimental implementations of $\mathcal{PT}$-symmetric and Lie-triple setups as well.
- The symmetric operator $H_{\rm sym}$ in (46) commutes with both generating involutions $\mathcal{P}$ and $\mathcal{R}$ from the Clifford algebra Cl_2 in (39). It will be shown in [48] that for any involution J constructed in an arbitrary way from Cl_2-involution elements there necessarily exists a very special subclass of J-self-adjoint extensions of $H_{\rm sym}$ which will have a spectrum filling the whole complex plane $\mathbb{C}$.
- It is known (see, e.g., section I.3.5 in [18]) that a Clifford algebra Cl_m with m basis elements $\{e_1, \ldots, e_m\}$ has a faithful representation as matrix algebra $Cl_{2k} \sim \mathbb{C}^{2^k\times 2^k}$, $Cl_{2k+1} \sim \mathbb{C}^{2^k\times 2^k} \oplus \mathbb{C}^{2^k\times 2^k}$. Furthermore, it is known that the J-self-adjoint extensions of a symmetric operator with deficiency indices $\langle n, n\rangle$ are parameterized by unitary matrices $U \in U(n) \subset \mathbb{C}^{n\times n}$. Once, the extension-related Clifford elements act via a representation in this $\mathbb{C}^{n\times n}$ matrix space the maximal number m of Clifford basis elements in Cl_m is bounded by the dimensionality of this matrix space and, hence, by $2^k \leqslant n$ for $m = 2k$ and $2^{k+1} \leqslant n$ for $m = 2k + 1$. The Hamiltonian H_T in (42) is related to the symmetric operator $H_{\rm sym}$ in (46) with deficiency indices $\langle 2, 2\rangle$ and parameter matrix $U \in U(2)$ [5]. This means that not more than the two Clifford basis elements $\mathcal{P}$ and $\mathcal{R}$ can be naturally associated with this operator extension.

[9] We use the notations from [2, 4, 5, 27] with $\rho^2 = \mathrm{e}^{-Q} = \mathcal{PC}$ and the $\mathcal{C}$-operator, as usual, as dynamical symmetry $[\mathcal{C}, H] = 0$ and involution $\mathcal{C}^2 = I$.

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