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Time-dependent Hamiltonians with 100% evolution speed efficiency

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Abstract

The evolution speed in projective Hilbert space is considered for Hermitian Hamiltonians and for non-Hermitian (NH) ones. Based on the Hilbert–Schmidt norm and the spectral norm of a Hamiltonian, resource-related upper bounds on the evolution speed are constructed. These bounds are valid also for NH Hamiltonians and they are illustrated for an optical NH Hamiltonian and for a NH $\mathcal{PT}$-symmetric matrix Hamiltonian. Furthermore, the concept of quantum speed efficiency is introduced as measure of the system resources directly spent on the motion in the projective Hilbert space. A recipe for the construction of time-dependent Hamiltonians which ensure 100% speed efficiency is given. Generally these efficient Hamiltonians are NH but there is a Hermitian efficient Hamiltonian as well. Finally, the extremal case of a NH non-diagonalizable Hamiltonian with vanishing energy difference is shown to produce a 100% efficient evolution with minimal resources consumption.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Non-Hermitian (NH) models naturally emerge in many fields of physics as efficient tools for the description of complicated large systems in terms of smaller effective subsystems [1, 2]. Examples range from atomic/molecular physics [3, 4], light propagation in optically active crystals [5] and media with anisotropic pumping and absorption [6–16] over microwave cavities [17, 18], coupled electronic circuits [19, 20] up to mechanics [21, 22], hydrodynamics [23, 24] and magnetohydrodynamics [25–28]. Apart from the spectral properties of the Hamiltonians, the evolution processes generated by NH Hamiltonians can significantly differ from those generated by Hermitian Hamiltonians [5, 29–32]. In this regard it appears natural to ask what new possibilities NH evolution entails and what bounds can be broken when a Hamiltonian is no longer Hermitian.

For example, it was shown in [29] that a NH 2×2 matrix Hamiltonian with some predefined energy difference can generate a much faster evolution than a Hermitian Hamiltonian with the same energy difference. The evolution speed is the rate in which a state changes into other states (e.g. the angular speed of the state vector on a Bloch sphere). In the Hermitian case, the energy difference sets an upper bound on the evolution speed (the Fleming bound [33]). The clear violation of this bound in BH system demonstrated in [29] leads to the conclusion that these energy-difference based bounds should be replaced by some more adequate bounds for NH systems.

The first goal of this paper is to derive an upper bound on the evolution speed that works for any Hamiltonian, be it Hermitian or NH. The bounds derived here may not be tight for some Hamiltonians and/or for some initial conditions, but this statement is equally true for the Hermitian Fleming bound.

This leads directly to the second goal of this paper: to show how to construct Hamiltonians for which the evolution speed of the state of interest reaches the upper bound for any instant of the evolution. We call these Hamiltonians 'maximal efficiency' Hamiltonians (or maximally efficient Hamiltonians). For every state evolution there exists a family of Hamiltonians that are maximally efficient. This family contains both Hermitian and NH Hamiltonians.

Our third goal is to explore a very special Hamiltonian in this family which is of rank-1, non-diagonalizable and similar to a Jordan block with zero-eigenvalue. This Hamiltonian corresponds to a NH degeneracy called exceptional point (EP) which has only one (geometric) eigenvector [1, 21, 34]. We show that any state evolution can be generated solely by such NH degeneracies yielding an EP-driven evolution (EP-DE). This special evolution minimizes the Hilbert–Schmidt norm $\sum_{i,j}|\mathcal{H}_{i,j}|^2$ of the matrix Hamiltonian $\mathcal{H}$.

We note that the second goal strongly differs conceptually from the so-called quantum brachistochrone problem. In the quantum brachistochrone problem the goal is to find the (time independent) Hamiltonian which evolves some predefined initial state into some predefined final state in a minimal time. This problem was the subject of intensive studies during the last few years for Hermitian systems [35–37] as well as for NH ones [29, 30, 38–44]. As shown in [35, 36, 45] for Hermitian systems, the corresponding minimal-passage-time trajectories correspond to geodesics in projective Hilbert space (PHS). For the evolution problems we are investigating here, the trajectories in PHS are predefined and not necessarily geodesic. Instead, we are searching for Hamiltonians capable of producing evolution processes which exactly follow these predefined trajectories with minimal resources. That is, we look for efficient evolution and not for a fast evolution. In fact, our optimization problem is closer in spirit to the reverse engineering approach used to quicken adiabatic evolution [46–48]. Yet there are two main differences: the first difference is that in our case we seek only Hamiltonians which yield maximal efficiency. The second difference is that we take as input only the evolution of a single state (the state of interest), while in [46–48] the number of states needed to be specified is equal to the Hilbert space dimension (number of levels in the system).

The paper is organized as follows: section 2 contains some basic facts on the evolution speed in PHS. In section 3, the Hilbert–Schmidt norm and the spectral norm of a Hamiltonian are introduced as upper bounds on the evolution speed. In section 4, the concept of speed efficiency is introduced, and for the predefined evolution of a given state a family of Hamiltonians is constructed which ensure a speed efficiency of 100%. The generic properties of maximally efficient evolutions are explored. Section 5 is devoted to the special case of a maximally efficient evolution which is driven by a Hamiltonian at an EP (a NH degeneracy). In appendix A, for completeness we briefly discuss the relation between Bloch sphere and

PHS. In appendix B, the norm speed bounds are illustrated for a Hamiltonian that describes two optical systems recently studied. In appendix C, the norm speed bound is applied to the matrix Hamiltonian of a $\mathcal{PT}$-symmetric quantum brachistochrone.

2. Preliminaries—the evolution speed in projective Hilbert space $\mathbb{P}(\mathfrak{H})$

Let $|\psi\rangle \in \mathfrak{H} = \mathbb{C}^N$ be a solution of the time-dependent Schrödinger equation (TDSE):

$$\mathrm{i}\partial_t|\psi\rangle = H(t)|\psi\rangle, \tag{1}$$

where $H(t) \neq H^\dagger(t) \in \mathbb{C}^{N\times N}$ is the matrix of the corresponding time-dependent NH Hamiltonian. Defining the bra-vector $\langle\psi|$ in a standard way[4] as $\langle\psi| = |\psi\rangle^\dagger$, the adjoint TDSE has the form

$$-\mathrm{i}\partial_t\langle\psi| = \langle\psi|H^\dagger(t)\,. \tag{2}$$

Our main interest is to study the rate at which states evolve into different states. Phase evolution is irrelevant for this purpose. It makes sense, then, to study the evolution of states in a space where the phase is eliminated. The so-called PHS is exactly suited for this purpose. The well known Bloch sphere for two-level systems is closely related to PHS (see appendix A), but strictly speaking it is not a PHS. For the reader not familiar with PHS we provide a simplified and very limited presentation of the basic ideas needed to understand the present work. For a more complete and rigorous treatment see, e.g., [49].

The angle Θ between two complex vectors $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^N$ can be obtained from the standard inner product of the two vectors:

$$\cos\Theta = \frac{|\langle\psi_1|\psi_2\rangle|}{\sqrt{\langle\psi_1|\psi_1\rangle}\sqrt{\langle\psi_2|\psi_2\rangle}}. \tag{3}$$

The angle Θ acts as a measure of distance between two states: $\Theta = 0$ means the two states are identical up to a complex factor and $\Theta = \pi/2$ indicates the states are mutually orthogonal[5].

Now imagine that a state is infinitesimally changed from $|\psi\rangle$ to $|\psi\rangle + |\mathrm{d}\psi\rangle$ where $\langle\psi|\psi\rangle \gg \langle\mathrm{d}\psi|\mathrm{d}\psi\rangle$. The angle between the original state and the modified state, $\mathrm{d}\Theta$, can be obtained from (3). Keeping leading orders in $|\mathrm{d}\psi\rangle$ and $\mathrm{d}\Theta$ we get:

$$\mathrm{d}\Theta^2 = \frac{\langle\mathrm{d}\psi|\mathrm{d}\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\mathrm{d}\psi|\psi\rangle\langle\psi|\mathrm{d}\psi\rangle}{\langle\psi|\psi\rangle^2} = \mathrm{d}s_{\mathrm{FS}}^2. \tag{4}$$

$\mathrm{d}s_{\mathrm{FS}}^2$ is known as the Fubini–Study metric [49] which describes the length of an infinitesimal arc traced on a unit hypersphere by changing a state by $|\mathrm{d}\psi\rangle$. To quantify the rate at which states change we will look at the evolution speed defined by: $\left|\frac{\mathrm{d}\Theta}{\mathrm{d}t}\right|$ (or equivalently $\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right|$), which can be interpreted as angular speed/frequency. This hypersphere is related to the PHS associated with the Fubini–Study metric. All states which differ by a complex number are mapped to the same point on the hypersphere (hence phase is immaterial in this space). The details of this mapping are not important for the present paper. What is important is that the distance between two states on the hypersphere which differ by $|\mathrm{d}\psi\rangle$ is given by (4). Formally, the PHS of an N-level system is denoted by $\mathbb{P}(\mathfrak{H}) = \mathbb{CP}^{N-1} = \mathbb{C}_*^N/\mathbb{C}_*$ where $\mathbb{C}_*^N = \mathbb{C}^N - \{(0,0,\ldots,0)\}$, $\mathbb{C}_* := \mathbb{C} - \{0\}$. As the state $|\psi\rangle$ evolves in time it traces a certain trajectory in $\mathbb{P}(\mathfrak{H})$ (i.e. on the hypersphere associated with the PHS). We denote the trajectory induced by $|\psi\rangle$ by $\pi(|\psi\rangle) \in \mathbb{P}(\mathfrak{H})$. Notice that for any complex function of time, $c(t)$ $(c(t) \neq 0)$:

$$\pi(|\psi\rangle) = \pi(c(t)|\psi\rangle) \tag{5}$$

[4] Unlike other choices often made for NH Hamiltonians in order to exploit the bi-orthogonality relations of the eigenstates [1].

[5] This is different from the Bloch sphere construction discussed in appendix A, where orthogonal states correspond to the angle of π between antipodal points on the sphere.

Next we wish to establish a relation between the evolution speed $\left|\frac{ds_{\mathrm{FS}}}{dt}\right|$ and the Hamiltonian. Making use of the TDSE (1) and its adjoint (2), and introducing the normalized state vectors $|\Psi\rangle := |\psi(t)\rangle/\sqrt{\langle\psi(t)|\psi(t)\rangle}$, we find the squared evolution speed in $\mathbb{P}(\mathfrak{H})$ is given by:

$$\left(\frac{ds_{\mathrm{FS}}}{dt}\right)^2 = \langle\Psi|H^\dagger(t)H(t)|\Psi\rangle - \langle\Psi|H^\dagger(t)|\Psi\rangle\langle\Psi|H(t)|\Psi\rangle =: K(t). \tag{6}$$

Henceforth, we refer to $K(t)$ as 'kinetic scalar' because it plays a structurally similar role to that of the kinetic energy in classical mechanical systems. The expression (6) is a straightforward generalization for NH Hamiltonians of the corresponding evolution speed discussed in [45, 49] for Hermitian systems. We note that for those systems $K(t)$ just reduces to the instantaneous energy variance $K(t) = \langle\Psi|H^2(t)|\Psi\rangle - \langle\Psi|H(t)|\Psi\rangle^2$. Further insight into this expression can be obtained by introducing an instantaneous orthonormal basis set $\{|k(t)\rangle\}_{k=1}^N \in \mathfrak{H} = \mathbb{C}^N$, with $|\Psi\rangle$ identified with one of its elements $|\Psi(t)\rangle = |j(t)\rangle$, the kinetic scalar (6) in this basis set takes the form:

$$K(|j(t)\rangle) = \sum_{k=1}^{N}\langle j|H^\dagger|k\rangle\langle k|H|j\rangle - \langle j|H^\dagger|j\rangle\langle j|H|j\rangle = \sum_{k\neq j}|\langle k|H|j\rangle|^2 \geqslant 0. \tag{7}$$

Obviously, K is characterizing the total rate for transitions from the given state $|\Psi\rangle = |j\rangle$ to other states of the system. Splitting off the trace of the Hamiltonian

$$H = \mathcal{H} + \mu I, \qquad \mu := \mathrm{Tr}(H)/N \tag{8}$$

one immediately sees that K is invariant with regard to trace shifts[6], $K[H] = K[\mathcal{H}]$. Hence, we can subsequently restrict our attention to traceless Hamiltonians $\mathcal{H}$. In appendix A, we discuss the dynamics on the PHS-related Bloch sphere and obtain its relation to the kinetic scalar.

To conclude this section we would like to point out that equation (6) implies that if H is Hermitian and time-independent, then the evolution speed is constant: $\frac{d}{dt}\left|\frac{ds_{FS}}{dt}\right|=0$. This, however, is not true for time-independent NH Hamiltonians.

3. Upper norm bounds on the evolution speed

The absolute values of the matrix elements of Hermitian or NH Hamiltonians are directly defined by the intensities of interactions and the strength of the corresponding fields. Naturally, a model remains valid only within the region of applicability of the corresponding underlying theory and/or the applicability of the approximations made in the derivation of the model. Hence, it is natural to ask for the maximal evolution speed achievable by a given quantum system when the resources are limited.

3.1. The Hilbert–Schmidt-norm upper bound on the evolution speed

To obtain the first simple upper bound on the evolution speed we notice that:

$$K \leqslant \langle\Psi|\mathcal{H}^\dagger\mathcal{H}|\Psi\rangle \leqslant \mathrm{tr}(\mathcal{H}^\dagger\mathcal{H}) \equiv ||\mathcal{H}||_{\mathrm{HS}}^2, \tag{9}$$

where $||\mathcal{H}||_{\mathrm{HS}}^2$ is the Hilbert–Schmidt norm of the Hamiltonian [50]. It is also known as Euclidean norm, l_2-norm, Schatten 2-norm, Frobenius norm and Schur norm [50]. It was

[6] A time-dependent trace $\mathrm{tr}(H) = N\mu(t) \in \mathbb{C}$ can be removed from the Hamiltonian H by the transformation $|\psi\rangle \to e^{i\int_0^t \mu(t')dt'}|\psi\rangle$. Since this transformation involves only a multiplication by a complex function the motion in the PHS is not affected by this transformation (5). Setups with non-vanishing complex traces have been considered, e.g., in [38].

possible to use the trace in the last inequality since $\mathcal{H}^\dagger\mathcal{H}$ is a positive operator. Equations (6) and (9) set a bound on the evolution speed:

$$\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right| = \sqrt{K} \leqslant \|\mathcal{H}\|_{\mathrm{HS}}. \tag{10}$$

Upon writing the HS norm explicitly in terms of matrix elements

$$\|\mathcal{H}\|_{\mathrm{HS}}^2 = \sum_{i,j} |\mathcal{H}_{ij}|^2, \tag{11}$$

it becomes clear that the evolution speed is limited by the size of the Hamiltonian elements and not just by the eigenvalues difference. This becomes very important in the vicinity of NH degeneracies as shown in appendix C.

3.2. The spectral norm upper bound—a tighter bound on the evolution speed

To get a tighter bound on the evolution speed we use the following inequality:

$$K \leqslant \langle\Psi|\mathcal{H}^\dagger\mathcal{H}|\Psi\rangle = \sum_{k=1}^{N} \lambda_k |\langle\Psi|k\rangle|^2 \leqslant \max(\lambda_k) \equiv ||\mathcal{H}||_{\mathrm{SP}}^2, \tag{12}$$

where $\lambda_k \geqslant 0$ and $|k\rangle$ are the eigenvalues and eigenstates of the matrix $\mathcal{H}^\dagger\mathcal{H}$, $\mathcal{H}^\dagger\mathcal{H}|k\rangle = \lambda_k|k\rangle$. $||\mathcal{H}||_{\mathrm{SP}} = \sqrt{\max(\lambda_k)}$ is known as the spectral norm of $\mathcal{H}$ (also known as Ky Fan 1-norm [50]). To understand the second inequality in (12), notice that the states $\{|k\rangle\}$ constitute a complete orthonormal basis set and $\langle\Psi|\Psi\rangle = 1$, so that the projection sum satisfies $\sum_k |\langle\Psi|k\rangle|^2 = 1$. Thus, $\sum_{k=1}^N \lambda_k|\langle\Psi|k\rangle|^2$ is just a weighted average of positive numbers λ_k. Such a weighted average is always smaller or equal to the largest element.

Obviously, the Hilbert–Schmidt norm $||\mathcal{H}||_{\mathrm{HS}}^2$ and the spectral norm $||\mathcal{H}||_{\mathrm{SP}}^2$ can be represented in terms of the eigenvalues λ_k . This implies the following useful relation:

$$||\mathcal{H}||_{\mathrm{SP}}^2 \leqslant ||\mathcal{H}||_{\mathrm{HS}}^2 \leqslant \mathrm{rank}(H)||\mathcal{H}||_{\mathrm{SP}}^2 . \tag{13}$$

Therefore, the spectral norm bound on the evolution speed

$$\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right| = \sqrt{K} \leqslant \|\mathcal{H}\|_{\mathrm{SP}} \tag{14}$$

is always tighter than the Hilbert–Schmidt norm bound. Simple and useful lower and upper bounds on $\|\mathcal{H}\|_{\mathrm{SP}}$ for $\mathcal{H}(t) \in \mathbb{C}^{N\times N}$ (but not on the evolution speed!) are given by:

$$\max(|\mathcal{H}_{i,j}|) \leqslant \|\mathcal{H}\|_{\mathrm{SP}} \leqslant N \max(|\mathcal{H}_{i,j}|). \tag{15}$$

The values $\sqrt{\lambda_k}$ are known as the singular values and they play an essential role in singular value decompositions [51]. The spectral norm, then, is the largest singular value of $\mathcal{H}$. Finally, we briefly comment on two extremal cases of traceless 2×2 matrix Hamiltonians.

- For a NH two-level Hamiltonian $\mathcal{H}$ which is similar to a Jordan block with zero-eigenvalue $\mathcal{H} \sim J_2(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ it holds $\mathrm{rank}(\mathcal{H}) = 1$ so that
$$\mathcal{H} \sim J_2(0) \qquad \Longrightarrow \qquad \|\mathcal{H}\|_{\mathrm{SP}} = \|\mathcal{H}\|_{\mathrm{HS}}. \tag{16}$$
This fact will be important later on in section 5.
- For a Hermitian traceless two-level Hamiltonian with energy separation ΔE, the spectral norm is $\|\mathcal{H}\|_{\mathrm{SP}} = |\Delta E|\,/2$ and we obtain:
$$\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right| \leqslant |\Delta E|/2. \tag{17}$$

As discussed in appendix A the Bloch unit vector, $\hat{n}$, is related to the kinetic scalar via: $\left|\frac{\mathrm{d}\hat{n}}{\mathrm{d}t}\right| = 2\sqrt{K}$. Therefore the corresponding upper bound on the evolution speed over the Bloch sphere in the *Hermitian* case is:

$$\left|\frac{\mathrm{d}\hat{n}}{\mathrm{d}t}\right| = 2\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right| \leqslant |\Delta E| \tag{18}$$

which is known as Fleming bound [33]. That is, for a two-level Hermitian operator the spectral bound coincides with the known Hermitian bound.

For explicit examples of the speed bound we refer the reader to appendices B and C. In appendix B, a Hamiltonian that describes certain optical systems is analyzed. Appendix C studies a $\mathcal{PT}$-symmetric Hamiltonian that was introduced in [29], in the context of the $\mathcal{PT}$-symmetric brachistochrone problem. In the next section we introduce the notion of speed efficiency which quantifies how close the actual motion in $\mathbb{P}(\mathfrak{H})$ is to the speed bound just derived. Later we show how to construct a Hamiltonian which reaches the spectral bound for a given motion in $\mathbb{P}(\mathfrak{H})$ at all times.

4. Speed efficiency of quantum evolution

In this section we introduce the notion of a maximally efficient evolution. We wish to compare the actual speed of motion in the PHS $\mathbb{P}(\mathfrak{H})$ to the speed bound given by the spectral norm $\|\mathcal{H}\|_{\mathrm{SP}}$ characterizing the available resources of the system. We use $\|\mathcal{H}\|_{\mathrm{SP}}$ since it is tighter than the Hilbert–Schmidt norm $\|\mathcal{H}\|_{\mathrm{HS}}$. Let $|\psi\rangle \in \mathbb{C}^N$ be a time-dependent state in an N-level system that induces some predefined evolution $\pi(|\psi\rangle)$ in the corresponding PHS $\mathbb{P}(\mathfrak{H}) = \mathbb{CP}^{N-1} \ni \pi(|\psi\rangle)$. We define the efficiency to be:

$$\eta(\mathcal{H}, |\psi\rangle) = \frac{\sqrt{K(\psi)}}{\|\mathcal{H}\|_{\mathrm{SP}}} \leqslant 1\,. \tag{19}$$

It is important to realize that this efficiency is an instantaneous (or local) property of $\mathcal{H}$ and its solution $|\psi\rangle$. The shape of the curve in $\mathbb{P}(\mathfrak{H})$ alone has nothing to do with efficiency. For example, a geodesic in $\mathbb{P}(\mathfrak{H})$ can have efficiency smaller than 1, and on the other hand, non-geodesic curves can have 100% efficiency.

Loosely speaking, the value of η quantifies to what extent the Hamiltonian really uses all its resources to generate motion in $\mathbb{P}(\mathfrak{H})$. That is why we call an ($\eta = 1$)-evolution, a 'maximally efficient evolution'. As an example of inefficient evolution, consider a spin in a magnetic field which is not exactly perpendicular to the spin direction. The part of the magnetic field which is parallel to the spin is wasted as it does not contribute to the precession motion. As we shall demonstrate in the next section, this inefficiency can be fixed by making the Hamiltonian time-dependent (rotating the magnetic field in time). In the NH case, reaching 100% efficiency becomes even more difficult. As explained at the end of section 2, for NH systems the condition $\frac{\mathrm{d}}{\mathrm{d}t}H = 0$ does not guarantee a constant evolution speed, i.e. $\left|\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right| \neq \mathrm{const}$. On the other hand, the spectral norm is fixed if $\frac{\mathrm{d}}{\mathrm{d}t}H = 0$. Equation (19), then, implies that η varies with time and, therefore, the evolution cannot be maximally efficient at all times.

In the next subsection we show how to construct Hamiltonians that are designed to generate maximally efficient evolution for a given predefined motion in PHS at all times. We will demonstrate that such an ($\eta = 1$)-evolution can be either Hermitian or NH.

4.1. Maximally efficient evolution

Our goal in this section is to find a Hamiltonian $\mathcal{H}_0$ that generates the same motion $\pi(|\psi\rangle)$ in $\mathbb{P}(\mathfrak{H})$ as $|\psi\rangle$ but with 100% efficiency. The solution $|m\rangle$ corresponding to the maximally efficient Hamiltonian $\mathcal{H}_0$ may differ from $|\psi\rangle$ only by a time dependent complex factor. In short, we look for $|m\rangle$ and $\mathcal{H}_0$ that satisfy:

$$\pi(|m\rangle) = \pi(|\psi\rangle) \tag{20}$$

$$\mathrm{i}\partial_\mathrm{t}|\mathrm{m}\rangle = \mathcal{H}_0|\mathrm{m}\rangle \tag{21}$$

$$\eta(t) = 1. \tag{22}$$

The first requirement is that the states $|m\rangle$ and $|\psi\rangle$ have the same motion in $\mathbb{P}(\mathfrak{H})$. The second requirement states that $\{\mathcal{H}_0, |m\rangle\}$ satisfy the TDSE, whereas the third requirement simply means that we are searching for maximal efficiency. To satisfy the first requirement we set:

$$|m\rangle = c(t)|\psi\rangle \tag{23}$$

where $c(t)$ is a complex differentiable function of time (see equation (5)). For reasons that will become clear shortly, we fix $c(t)$ by choosing $|m\rangle$ to be normalized to unity and to be parallel transported[7]:

$$\langle m|m\rangle = 1, \tag{24}$$

$$\langle m|\partial_t|m\rangle = (\partial_t\langle m|)|m\rangle = 0. \tag{25}$$

The first condition determines $|c(t)|$, and the second one yields the phase of $c(t)$ (up to a time-independent constant). To find the Hamiltonian that drives $|m\rangle$ with maximal efficiency we choose the following ansatz:

$$\mathcal{H}_0 = \mathrm{i}|\partial_t m\rangle\langle m| - \mathrm{i}g|m\rangle\langle\partial_t m|, \tag{26}$$

where, in general, g can be time-dependent. For $g = 1$ the Hamiltonian is Hermitian. Notice that $|\partial_t m\rangle \equiv \partial_t\,|m\rangle$ is not normalized and not parallel transported. Moreover, in contrast to $|m\rangle$, $|\partial_t m\rangle$ is not a solution of the TDSE (with $\mathcal{H}_0$ as Hamiltonian). However, $|m\rangle$ and $|\partial_t m\rangle$ are mutually orthogonal by virtue of the parallel transport we imposed. By applying (26) to the state $|m\rangle$ we see that the requirement (21) is immediately satisfied. To fulfil the remaining third requirement we note that in the basis $\{|m\rangle\,, |\partial_t m\rangle\}$ the Hamiltonian $\mathcal{H}_0$ has only off-diagonal elements so that $\mathcal{H}_0$ is traceless by construction. To calculate the efficiency we first calculate the spectral bound and then the kinetic scalar. Evaluating $\mathcal{H}_0^\dagger\mathcal{H}_0$ we get

$$\mathcal{H}_0^\dagger\mathcal{H}_0 = \langle\partial_t m|\partial_t m\rangle|m\rangle\langle m| + |g|^2|\partial_t m\rangle\langle\partial_t m|. \tag{27}$$

The eigenstates of $\mathcal{H}_0^\dagger\mathcal{H}_0$ are $|m\rangle$ and $|\partial_t m\rangle$ and the corresponding eigenvalues are $\langle\partial_t m|\partial_t m\rangle$ and $|g|^2\,\langle\partial_t m|\partial_t m\rangle$, respectively. The spectral norm is given by:

$$\|\mathcal{H}_0\|_{\mathrm{SP}} = \sqrt{\langle\partial_t m|\partial_t m\rangle}\,\max(1, |g|). \tag{28}$$

Using the fact that $\langle n\,|\mathcal{H}_0|\,n\rangle = 0$ and equations (6) and (27) we get that:

$$K(\mathcal{H}_0, |m\rangle) = \langle\partial_t m|\partial_t m\rangle. \tag{29}$$

The efficiency, then, is given by:

$$\eta = \frac{\sqrt{K(\mathcal{H}_0, |m\rangle)}}{\|\mathcal{H}_0\|_{\mathrm{SP}}} = \frac{1}{\max(1, |g|)}. \tag{30}$$

Clearly, maximal efficiency $\eta = 1$ (third requirement (22)) is achieved provided that:

$$|g| \leqslant 1. \tag{31}$$

[7] If $|\chi\rangle$ is a normalized state, $\langle\chi|\chi\rangle = 1$, then its parallel transported form is given by $|\tilde{\chi}\rangle = \mathrm{e}^{-\int_0^t\langle\chi|\partial_{t'}\chi\rangle\mathrm{d}t'}\,|\chi\rangle$. $|\tilde{\chi}\rangle$ satisfies $\langle\tilde{\chi}|\partial_{t'}\tilde{\chi}\rangle = \langle\partial_{t'}\tilde{\chi}|\tilde{\chi}\rangle = 0$. This fixes the phase of the state up to a constant determined by the choice of the lower limit of the time integral.

In summary, given any arbitrary state $|\psi\rangle$, equations (23)–(26) together with (31) show how to construct maximal efficiency Hamiltonians that generate the same motion in PHS $\mathbb{P}(\mathfrak{H})$ as $|\psi\rangle$.

It is instructive to look on the instantaneous eigenvalues of $\mathcal{H}_0^2$. From

$$\mathcal{H}_0^2 = g|\partial_t m\rangle\langle\partial_t m| + g\langle\partial_t m|\partial_t m\rangle|m\rangle\langle m| \tag{32}$$

and $\mathrm{tr}(\mathcal{H}_0) = 0$ it follows that the instantaneous eigenvalues $E_\pm(t)$ of $\mathcal{H}_0$ are:

$$E_\pm(t) = \pm\sqrt{g}\sqrt{\langle\partial_t m|\partial_t m\rangle}. \tag{33}$$

In case of $g \in \mathbb{R}$, these eigenvalues are real for $g > 0$ and purely imaginary for $g < 0$, i.e. $E_\pm(t) \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$. This indicates a hidden instantaneous pseudo-Hermiticity of $\mathcal{H}_0$, which is related and analogous to the considerations[8] in appendix C. One of the key points of this work is that the evolution can be maximally efficient regardless of whether the Hamiltonian is Hermitian or not.

Finally, we note that the Hamiltonian $\mathcal{H}_0$ in (26) shows some structural analogy to the brachistochrone Hamiltonians for Hermitian systems (constructed in [36]). In fact, $\mathcal{H}_0$ extends the geodesic-trajectory paradigm of [35, 36, 45] to maximally efficient evolution regimes over arbitrarily predefined time-dependent trajectories in $\mathbb{P}(\mathfrak{H})$. Moreover, the constraint $\Delta E = \mathrm{const}$ is replaced by the constraint $\eta = 1$.

4.2. Inherent properties of the maximally efficient evolution

Here we wish to highlight three points which are generic for maximally efficient evolutions. The first point concerns the fact that $|m\rangle$ is normalized and parallel transported. In the construction of $\mathcal{H}_0$ we demanded $\langle m(t)|m(t)\rangle = 1$ and $\langle m|\partial_t m\rangle = 0$. Now we wish to show that if these constraints are relaxed the spectral norm will increase even though the motion in $\mathbb{P}(\mathfrak{H})$ remains unaltered. According to (19) the efficiency will drop below 100% by this modification. Assume we wish to change the amplitude and phase of $|m\rangle$ by some complex factor $\mathrm{e}^{-\mathrm{i}\varphi(t)}$ where $\varphi(t)$ is some complex number. This is accomplished by adding $\mathcal{H}_0$ a diagonal term so that:

$$\mathcal{H}_{\mathrm{new}} = \mathcal{H}_0 + |m\rangle\langle m|\partial_t\varphi(t). \tag{34}$$

To keep the trace zero, another diagonal term must be added as well, in principle, but it is of no importance to the present discussion. This transformation does not change the value of K. The spectral norm squared is the largest expectation of $\mathcal{H}_{\mathrm{new}}^\dagger\mathcal{H}_{\mathrm{new}}$, so:

$$\|\mathcal{H}_{\mathrm{new}}\|_{\mathrm{SP}}^2 \geqslant \langle m|\mathcal{H}_{\mathrm{new}}^\dagger\mathcal{H}_{\mathrm{new}}|m\rangle = |\partial_t\varphi(t)|^2 + \langle\partial_t m|\partial_t m\rangle > \langle\partial_t m|\partial_t m\rangle. \tag{35}$$

Since K remained the same and the spectral norm increased, we see that the efficiency is now:

$$\eta_{\mathrm{new}} = \frac{\sqrt{\langle\partial_t m|\partial_t m\rangle}}{\|\mathcal{H}_{\mathrm{new}}\|_{\mathrm{SP}}} \leqslant \frac{\sqrt{\langle\partial_t m|\partial_t m\rangle}}{\sqrt{|\partial_t\varphi(t)|^2 + \langle\partial_t m|\partial_t m\rangle}} < 1. \tag{36}$$

This decrease in efficiency with respect to $\mathcal{H}_0$ expresses the simple fact that changes in phase and/or amplitude also require spectral norm resources from the Hamiltonian. In order to direct all resources to motion in $\mathbb{P}(\mathfrak{H})$, any phase and amplitude changes should be avoided.

The second point concerns the role of g. While the $\mathcal{H}_0$ found earlier conserves the norm of $|m\rangle$, it does not do so for other initial states (with the exception of the Hermitian case $g = 1$). Moreover, while $|m\rangle$ evolves exactly in the same way for all values of g, the evolution of other states strongly depends on the value of g. This is demonstrated in the example shown in figure 1. The state of interest in this example was chosen to be: $|m\rangle = \cos(t)\,|\uparrow\rangle + \sin(t)\,|\downarrow\rangle$. We considered two different maximally efficient Hamiltonians. The first one is Hermitian

[8] We leave a corresponding detailed investigation to future research.

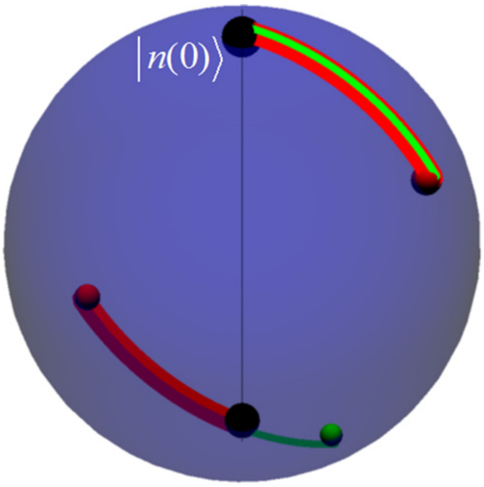

Figure 1. The value of g has no effect on the evolution when applied to the initial state $\mathcal{H}_0$ was designed to propagate efficiently (north pole of the Bloch sphere in this example). Yet when applying $\mathcal{H}_0$ to a different initial state (south pole) the value of g may completely change the evolution. The large black dots mark the starting points of the evolution. See text for details.

($g = 1$) and the other is not ($g = -0.8$). The Hamiltonian is constructed using the recipe in section 4.1. If the initial state is $|\psi(t = 0)\rangle = |m(t = 0\rangle = |\uparrow\rangle$ we observe that, as expected, both Hamiltonians generate the same evolution. Yet, if $|\psi(t = 0)\rangle = |\downarrow\rangle$, the different Hamiltonians generate different evolutions and the effect of g becomes apparent.

The third point that we note is that for periodic motion in $\mathbb{P}(\mathfrak{H})$ the state $|m\rangle$ accumulates only an Anandan–Aharonov phase [52], since the dynamical phase $\langle m|H|m\rangle$ is zero for maximally efficient evolution. Once again, this is the result of wasting no resources on phase accumulation.

5. EP driven evolution

A special case of great interest is $g = 0$. Equation (32) shows that in this case $\mathcal{H}_0^2 = 0$. This can only happen if $\mathcal{H}_0$ is similar to a rank-1 Jordan block with zeros on the diagonal, i.e. when $\mathcal{H}_0 \sim J_2(0)$. That is, $\mathcal{H}_0(g = 0)$ describes an EP operator—a NH degeneracy. The dynamics can be fast even though at each instant the instantaneous eigenvalue difference is zero. *This degeneracy is time-dependent*. Moreover, the orientation of the single geometric eigenvector of a Hamiltonian $\mathcal{H}_0 \sim J_2(0)$ associates a preferred directionality to this degeneracy. The directionality of the EP at each instant of time is chosen such that it induces the desired dynamics. Thus it appears natural to name this evolution an exceptional point driven evolution (EP-DE). Another unique feature of this evolution can be seen by evaluating the Hilbert–Schmidt norm. For a general value of g the HS norm is:

$$\|\mathcal{H}_0\|_{\mathrm{HS}}^2 = (1 + |g|^2)\langle\partial_t m|\partial_t m\rangle. \tag{37}$$

Obviously, the HS norm takes its minimal value for EP-DE ($g = 0$), i.e. from all the possible maximally efficient evolutions the EP-DE has the minimal HS norm. At this point the HS norm $\|\mathcal{H}_0\|_{\mathrm{HS}}$ and the spectral norm $\|\mathcal{H}_0\|_{\mathrm{SP}}$ coincide, a fact mentioned in (16). Equation (11) shows that the EP-DE provides the minimal value of $\sum_{i,j} |\mathcal{H}_{ij}|^2$ for a given trajectory in $\mathbb{P}(\mathfrak{H})$.

Conclusion

The concept of speed efficiency was defined using spectral speed bounds derived for NH or Hermitian systems. A recipe for the construction of 100% efficiency Hamiltonians for any evolution in the projective Hilbert space was given. These Hamiltonians contain a free parameter, 'g'. 100% efficiency is obtained for $|g| \leqslant 1$. The Hermitian case corresponds to $g = 1$. We conclude that it is possible to have a Hamiltonian which is both 100% efficient and Hermitian. The $g = 0$ case corresponds to a Hamiltonian which is not diagonalizable, i.e. the evolution is driven solely by a time-dependent NH degeneracy (exceptional point). This particular evolution minimizes the quantity $\sum_{i,j} |\mathcal{H}_{ij}|^2$ with respect to all other 100% efficiency evolutions considered in this work.

Appendix A. The Bloch sphere and the Fubini–Study metric

The dynamics of NH 2×2 matrix systems in $\mathfrak{H} = \mathbb{C}^2 \ni |\psi\rangle$ is conveniently analyzed as dynamics on the Bloch sphere. The latter is spanned by the unit vectors

$$\hat{n}(t) = \frac{\langle\psi|\vec{\sigma}|\psi\rangle}{\langle\psi|\psi\rangle} = \langle\Psi|\vec{\sigma}|\Psi\rangle \in S^2 \subset \mathbb{R}^3. \tag{A.1}$$

Its close relationship to the PHS $\mathbb{CP}^1 = \mathbb{P}(\mathfrak{H}) = \mathbb{C}^2_*/\mathbb{C}_* \sim S^2$ can be seen by the explicit comparison of the Bloch sphere metric with the Fubini–Study metric of $\mathbb{CP}^1$. For a qubit $|\Psi\rangle \in \mathbb{C}^2$ parametrized as $|\Psi\rangle = \left(\cos(\theta/2), \mathrm{e}^{\mathrm{i}\phi}\sin(\theta/2)\right)^T$, $\theta \in [0,\pi]$, $\phi \in [0, 2\pi]$ it holds $\hat{n} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))^T$ and the Bloch sphere metric reads

$$\mathrm{d}\hat{n}^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2. \tag{A.2}$$

For the *same state* $|\Psi\rangle$ the Fubini–Study metric (6) reduces to

$$\mathrm{d}s^2_{\mathrm{FS}} = \frac{1}{4}(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2) \tag{A.3}$$

and therefore:

$$\left(\frac{\mathrm{d}\hat{n}}{\mathrm{d}t}\right)^2 = 4\left(\frac{\mathrm{d}s_{\mathrm{FS}}}{\mathrm{d}t}\right)^2 = 4K(t). \tag{A.4}$$

This is closely related to the fact that orthogonal states are antipodal on the Bloch sphere having a geodesic distance π, whereas the corresponding Fubini–Study distance as discussed in section 2 is $s_{\mathrm{FS}} = \Theta = \pi/2$.

The main results of this paper can be expressed using the Bloch sphere and the NH Bloch equations (see for example [5]) formalism, but we found that the results are more neatly described by the 'ket-bra' operator formalism and the Schrödinger equation. Moreover, unlike the NH Bloch equation formalism the 'ket-bra' formalism is applicable to a multilevel system without any alterations.

Appendix B. Speed bounds in optical systems

Let us examine the NH evolution in optical systems where the Hamiltonians are explicitly known and a two-level description is either a good approximation or even exact. Consider the Hamiltonian $\mathcal{H}_0$ introduced and studied in [31] in the context of 'EP cycling' [3, 5, 16, 32]:

$$\mathcal{H}(z) = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i}q(z) & 0 \end{pmatrix}, \tag{B.1}$$

where z, the propagation coordinate, plays the role of time. This Hamiltonian can describe different physical systems. In [16], it was used to describe the evolution of the transverse electric field and its spatial derivative in a waveguide. In this system $q(z)$ is proportional to the change in the index of refraction with respect to vacuum. In [5] $\mathcal{H}(z)$ describes the evolution of the two optical polarizations in crystals. $q(z)$ in this case is related to the change in the transverse part of the reciprocal dielectric tensor. The eigenvalue difference of $\mathcal{H}(z)$ is $\Delta E = 2\sqrt{q(z)}$, and at $q = 0$ a NH degeneracy (an EP) forms which is experimentally well accessible in these systems.

From the structure of $\mathcal{H}_0(z)$ in (B.1), it is obvious that for $q = 0$ and $\Delta E(q = 0) = 0$ the evolution speed does not vanish for all states. Using (A.4) and the kinetic scalar definition (6), one finds that the non-vanishing angular velocity is $\left|\frac{\mathrm{d}\theta}{\mathrm{d}z}\right|=|\dot{\theta}| = 2\,|q(z)|$ for the spin-up state $|\uparrow\rangle = (1, 0)^T$, and $|\dot{\theta}| = 2$ for the spin-down state $|\downarrow\rangle = (0, 1)^T$. In particular, at the degeneracy $q = 0$ the spin-up state becomes an eigenstate ($\dot{\theta} = 0$), whereas the spin-down state still has a speed $\dot{\theta} = 2$ regardless of $\Delta E = 0$. For $q \neq 0$ the evolution speed of a general state (not necessarily spin-up or spin-down) can be shown to be limited by

$$|\dot{\theta}| \leqslant 2\,\max(1, |q(z)|) \leqslant 2\sqrt{1 + |q|^2}\,, \tag{B.2}$$

where the last inequality simply follows from $\max(|a|\,, |b|) \leqslant \sqrt{|a|^2 + |b|^2}$. From the norms of the Hamiltonian (B.1),$\|\mathcal{H}\|_{\mathrm{SP}} = \max(|\mathrm{i}|\,, |-\mathrm{iq(z)}|)$, $\|\mathcal{H}\|_{\mathrm{HS}} = \sqrt{1 + |q|^2}$, we see that

$$|\dot{\theta}| \leqslant 2\,\|\mathcal{H}\|_{\mathrm{SP}} \leqslant 2\,\|\mathcal{H}\|_{\mathrm{HS}}. \tag{B.3}$$

Hence, the maximal speed exactly fits within the spectral norm bound of $\mathcal{H}(z)$. Other optical systems whose evolution speeds near EPs can easily be studied are discussed, e.g., in [15].

Appendix C. Spectral speed bounds for pseudo-Hermitian and $\mathcal{PT}$-symmetric two-level Hamiltonians

We start from a general type NH traceless Hamiltonian, $\mathcal{H}$, written in terms of the Pauli matrices $\vec{\sigma}$:

$$\mathcal{H}(t) = [\vec{\alpha}(t) + \mathrm{i}\tilde{\beta}(\mathrm{t})] \cdot \tilde{\sigma}\,, \tag{C.1}$$

where $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^3$ are some time-dependent real vectors. The eigenvalues are:

$$E_{\pm} = \pm\sqrt{(\vec{\alpha} + \mathrm{i}\tilde{\beta})^2} = \pm\sqrt{\vec{\alpha}^2 - \vec{\beta}^2 + \mathrm{i}2\tilde{\alpha}\cdot\tilde{\beta}}, \tag{C.2}$$

and become real or pairwise complex conjugate for

$$\vec{\alpha}\cdot\vec{\beta} = 0 \qquad \Longrightarrow \qquad E_{\pm} \in \mathbb{R} \cup \mathrm{i}\mathbb{R}. \tag{C.3}$$

Operators and matrices with this specific spectral behavior are known to be symmetric under an anti-unitary transformation [53], to be pseudo-Hermitian [54] and self-adjoint in a Pontryagin space [55] (a finite-dimension type version of a Krein space [56]).

For Hamiltonians $\mathcal{H}$ with $\vec{\alpha}\cdot\vec{\beta} = 0$ the SP norm and the eigenvalue difference reduce to

$$\|\mathcal{H}\|_{\mathrm{SP}} = |\vec{\alpha}| + |\vec{\beta}|\,, \qquad \Delta E = 2\sqrt{\vec{\alpha}^2 - \vec{\beta}^2}. \tag{C.4}$$

Obviously it is possible to have an arbitrary large $\|\mathcal{H}\|_{\mathrm{SP}}$ and a vanishing energy difference by choosing $|\vec{\alpha}| \to |\vec{\beta}|$. This choice corresponds to an EP limit for which $|\Delta E|\,/\,\|\mathcal{H}\|_{\mathrm{SP}} \to 0$, since the eigenvalues are very small near the degeneracy while $\|\mathcal{H}\|_{\mathrm{SP}}$ remains roughly constant and finite.

A simple example for a NH Hamiltonian with a similar type of behavior is the Hamiltonian $\mathcal{H}$ used in studies of the $\mathcal{PT}$-symmetric quantum brachistochrone problem [29]

$$\mathcal{H} = \begin{pmatrix} \mathrm{i}r\sin\chi & s \\ s & -\mathrm{i}r\sin\chi \end{pmatrix} = s\sigma_x + \mathrm{i}r\sin\chi\,\sigma_z, \qquad r, s, \chi \in \mathbb{R}. \tag{C.5}$$

This complex symmetric Hamiltonian is $\mathcal{PT}$-symmetric, $[\mathcal{PT}, \mathcal{H}] = 0$ and $\mathcal{P}$-pseudo-Hermitian, $\mathcal{PH} = \mathcal{H}^\dagger\mathcal{P}$, with the parity operation given as $\mathcal{P} = \sigma_x$ and the time reversal, $\mathcal{T}$, as complex conjugation. In [29], it was shown that for certain parameter combinations r, s, χ such Hamiltonians with fixed and purely real eigenvalues difference $\Delta E \in \mathbb{R}$ can evolve a given initial state $|\Psi_I\rangle \in \mathbb{C}^2$ into an orthogonal final state $|\Psi_F\rangle \in \mathbb{C}^2$, $\langle\Psi_I|\Psi_F\rangle = 0$, in an arbitrarily short time interval. Due to the finite geodesic distance π between these antipodal states $|\Psi_I\rangle$ and $|\Psi_F\rangle$ on the Bloch sphere, the corresponding evolution speed should diverge in this limit. The concrete relations can be easily obtained in terms of the simplifying reparametrization $r \sin\chi = s \sin\alpha$, which yields

$$\mathcal{H} = s\begin{pmatrix} \mathrm{i}\sin\alpha & 1 \\ 1 & -\mathrm{i}\sin\alpha \end{pmatrix}, \qquad \Delta E = 2s\cos\alpha, \qquad \|\mathcal{H}\|_{SP} = |s|(1 + |\sin\alpha|). \tag{C.6}$$

As demonstrated in [39], the ultra-fast evolution regime predicted in [29] corresponds to an EP-limit $\alpha \to \pm\pi/2$ so that for fixed $\Delta E = \text{const}$ it holds $s = \Delta E/(2\cos\alpha) \to \infty$ and

$$\mathcal{H} \to s\begin{pmatrix} \mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{pmatrix}, \qquad \|\mathcal{H}\|_{\mathrm{SP}} \approx 2|s| \to \infty, \qquad |\Delta E| \,/\, \|\mathcal{H}\|_{\mathrm{SP}} \approx \cos\alpha \to 0. \tag{C.7}$$

According to (A.4), this would indeed allow for diverging evolution speeds on the Bloch sphere

$$\left|\frac{\mathrm{d}\hat{n}}{\mathrm{d}t}\right| = 2\sqrt{K} \leqslant 2\|\mathcal{H}\|_{\mathrm{SP}} \to \infty. \tag{C.8}$$

From this diverging spectral norm one might be led to the conclusion that actually such ultra-high evolution speeds and corresponding ultra-short evolution times might be forbidden by the limited resources of the system and the validity region of the model used. Both would set some natural upper bounds (ultra-violet cut-offs) on the evolution speed. This would be true if one were keeping within the present NH setups. Nevertheless, the same ultra-high-speed evolution regimes can be induced in subsystems of entangled Hermitian systems in larger Hilbert spaces [43]. Due to geometric contraction effects the corresponding evolution speed of the associated (Naimark-dilated) Hermitian system in the larger Hilbert space will remain finite, well-behaved and much below any ultra-violet cutoffs.

For completeness we note that the present evolution speed considerations are closely related to questions for possible lower bounds on evolution times (quantum brachistochrone problems) and possible violations of such bounds. Corresponding intensive theoretical studies for Hermitian setups [35–37] in the early 2000s have been followed by investigations of various aspects of NH systems ($\mathcal{PT}$-symmetric [29, 30, 39–43] and quasi-Hermitian ones [44], as well as other of more general NH types [38]).

$\mathcal{PT}$-symmetric setups [57–59] have been experimentally studied via special arrangements of gain–loss components (active $\mathcal{PT}$-symmetry) and components of different loss (passive $\mathcal{PT}$-symmetry) in optical waveguide systems [9, 11], microwave billiards [18], electronic LRC-circuits [19, 20] and in mechanical systems of coupled pendulums [22].

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