Asymptotic methods for spherically symmetric MHD α^2-dynamos

U. Günther[1,*] and **O.N. Kirillov**[1,2**]

[1] Research Center Dresden-Rossendorf, POB 51 01 19, D-01314 Dresden, Germany.
[2] Institute of Mechanics, Moscow State Lomonosov University, Michurinskii pr. 1, 119192 Moscow, Russia.

We consider two models of spherically-symmetric MHD α^2-dynamos; one with idealized boundary conditions (BCs); and one with physically realistic BCs. As it has been shown in our previous work, the eigenvalues λ of a model with idealized BCs and constant $\alpha-$profile α_0 are linear functions of α_0 and form a mesh in the $(\alpha_0, \lambda)-$plane. The nodes of the spectral mesh correspond to double-degenerate eigenvalues of algebraic and geometric multiplicity 2 (diabolical points). It was found that perturbations of the constant $\alpha-$profile lead to a resonant unfolding of the diabolical points with selection rules of the resonant unfolding defined by the Fourier coefficients of the perturbations. In the present contribution we present new exact results on the spectrum of the model with physically realistic BCs and constant α. For non-degenerate (simple) eigenvalues perturbation gradients are found at any particular α_0. We briefly discuss the spectral behavior of the α^2-dynamo operator over a family of homotopic deformations of the BCs between idealized ones and physically realistic ones. Furthermore, we demonstrate that although the spectral singularities are lifted, a memory about their locations remains deeply imprinted in the homotopic family of spectral deformations due to a hidden underlying invariance.

1 Introduction

The boundary eigenvalue problem for the mean field α^2-dynamo of magnetohydrodynamics (MHD) in its kinematic regime and with spherically symmetric $\alpha-$profile $\alpha(r)$ is usually equipped with either idealized or physically realistic boundary conditions [1–3]. Although the latter are more relevant for the use in applications, the former allow for an almost completely analytical treatment of the problem and in this way provide immediate deep insights into underlying structures and effects.

In order to understand the relationship between models with idealized BCs and physically realistic BCs and to use the advantages of the idealized BCs in studies of physically realistic models, we consider a homotopic interpolation family of boundary eigenvalue problems depending on an auxiliary parameter $\beta \in [0, 1]$

$$\mathfrak{A}_\alpha \mathfrak{u} = \lambda \mathfrak{u}, \ \mathfrak{u}(r \searrow 0) = \mathfrak{B}\mathfrak{u}(1) = 0, \quad \mathfrak{A}_\alpha := \begin{pmatrix} -A_l & \alpha(r) \\ A_{l,\alpha} & -A_l \end{pmatrix}, \quad \mathfrak{B} := \begin{pmatrix} \beta[\partial_r + l] + 1 - \beta & 0 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where

$$A_l := -\partial_r^2 + \frac{l(l+1)}{r^2}, \quad A_{l,\alpha} := -\partial_r \alpha(r) \partial_r + \alpha(r) \frac{l(l+1)}{r^2} = \alpha(r) A_l - \alpha'(r) \partial_r \, . \tag{2}$$

For $\beta = 0$ the BCs in (1) are idealized ones. They may be related to the high conductivity limit of the dynamo maintaining fluid/plasma [1, 3]. The other end point of the homotopic deformation family ($\beta = 1$) corresponds to physically realistic BCs [1, 2].

2 Spectral and gradient function patterns

In [3] it was shown that for $\alpha(r) = \alpha_0 = \text{const}$ and idealized BCs, i.e. $\beta = 0$, the eigenvalues and eigenvectors of the boundary eigenvalue problem are given as

$$\lambda_n^\pm = \lambda_n^\pm(\alpha_0) = -\rho_n \pm \alpha_0 \sqrt{\rho_n} \in \mathbb{R}, \quad n \in \mathbb{Z}^+, \quad \mathfrak{u}_n^\pm = \begin{pmatrix} 1 \\ \pm\sqrt{\rho_n} \end{pmatrix} u_n \in \mathbb{R}^2 \otimes L_2(0, 1) \, , \tag{3}$$

where $\rho_n > 0$ are squares of Bessel function zeros

$$J_{l+\frac{1}{2}}(\sqrt{\rho_n}) = 0, \qquad 0 < \sqrt{\rho_1} < \sqrt{\rho_2} < \cdots \, . \tag{4}$$

The branches $\lambda_n^\pm$ of the spectrum are real-valued linear functions of the parameter α_0 with slopes $\pm\sqrt{\rho_n}$ and form a mesh-like structure in the $(\alpha_0, \lambda)-$plane (Fig. 1 (a)). The nodes of the mesh correspond to two-fold degenerate semi-simple eigenvalues (diabolical points).

* Corresponding author E-mail: u.guenther@fzd.de, Phone: +49 351 260 2415, Fax: +49 351 260 2007
** E-mail: o.kirillov@fzd.de; kirillov@imec.msu.ru

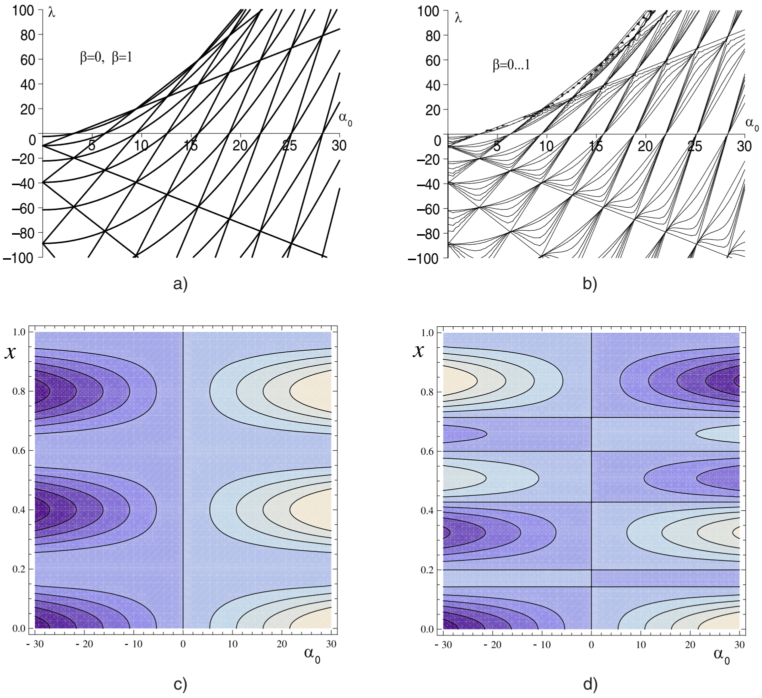

Fig. 1 $l = 0$: Spectral mesh for $\beta = 0$ and separated parabolic branches for $\beta = 1$ (a); homotopic deformation of the spectral mesh for $\beta \in [0, 1]$ (b); $\beta = 1$: landscape of the perturbation gradient $g_{mn}(r)$ over the $(\alpha_0, r)-$plane calculated for parabolic eigenvalue branches with index-pairing $m = n = 5$ (c) and $m = 5,\ n = 7$ (d).

In the hyper-idealized case of zero spherical harmonics, $l = 0$, these diabolical points (at the pairwise intersections of eigenvalue branches (3) with indices n and m) are additionally located on parabolic curves [3]

$$\lambda_0^\nu = \frac{1}{4}\left(\alpha_0^{\nu\,2} - \pi^2 m^2\right), \tag{5}$$

where $\alpha_0^\nu = \pi(2n + m)$, see Fig. 1 (a).

In case of physically realistic BCs, $\beta = 1$, and constant $\alpha(r) = \alpha_0$ the spectrum of the problem (1) consists of simple real eigenvalues which form non-intersecting curves (branches) in the $(\alpha_0, \Re\lambda)-$plane. For $l = 0$ these curves are parabolas

$$\lambda = \frac{1}{4}\left(\alpha_0^2 - \pi^2 m^2\right) \tag{6}$$

labeled by the integer index m (see Fig. 1 (a,b)). Although the spectrum (6) is regular it, nevertheless, preserves an imprinted memory about the locations of the diabolical points of the setup with idealized BCs. The effect is clearly visible in the homotopic deformation of the spectral mesh (Fig. 1 (b)).

For $\alpha = \text{const}$ and $l = 0$ the homotopic deformation (Fig. 1 (b)) between the spectral mesh at $\beta = 0$ and the parabolas at $\beta = 1$ is governed by the characteristic equation

$$4\beta\lambda\sin(\sqrt{\alpha_0^2 - 4\lambda}) + (1-\beta)2\sqrt{\alpha_0^2 - 4\lambda}\left[\cos(\sqrt{\alpha_0^2 - 4\lambda}) - \cos(\alpha_0)\right] = 0. \tag{7}$$

It is remarkable that the diabolical points (5) of the mesh are fixed points of this homotopy — a fact which indicates on their 'deep imprint' in the differential expression of the matrix differential operator (1) independently of the concrete BCs.

The magnetic field of a dynamo is maintained by its non-decaying modes ($\Re\lambda > 0$). When additionally one of the dominant modes is oscillating ($\Im\lambda \neq 0$) then via nonlinear back-reactions ($\alpha-$quenching) the dynamo becomes prone to polarity reversals [2]. In order to gain a deeper insight into the structural features of these reversals numerics-based dynamical studies can be supplemented by spectral methods based on perturbation gradients [3,4]. The relevant eigenvalue branches with $\Re\lambda > 0, \Im\lambda \neq 0$ can be induced by deforming (perturbing) the original $\alpha-$profile $\alpha_0 = \text{const}$ into an inhomogeneous one, $\alpha(r) \neq \text{const}$. The corresponding deformation process is best controlled (and optimized) with the help of Fréchet gradient techniques. Fréchet gradients $g_m(r, \alpha_0)$ for simple eigenvalues on particular parabolas of index m [see Eq. (6)] were found via perturbation theory in [3, 4] and allow for deformations of whole eigenvalue branches. The gradient landscapes over the $(\alpha_0, r)-$plane show typical periodical patterns similar to those for the parabolas with $m = 3$ and $m = 4$ depicted in Fig. 1(c,d). The periodic pattern implicitly reflects the structure of the spectral mesh for the dynamo with the idealized BCs. The highest sensitivity with regard to $\alpha-$perturbations can be gained for the regions in the $(\alpha_0, r)-$plane with strongest Fréchet gradients. For the considered model with physically realistic BCs ($\beta = 1$) these regions are located close to the diabolical points (nodes of the spectral mesh) of the model with idealized BCs ($\beta = 0$). This allows for the conclusion that models with idealized BCs can be expected to provide further insight even into the realistic polarity reversal regimes of [2].

Acknowledgements The work has been partially supported by the German Research Foundation DFG, grant GE 682/12-3, (U.G.), by the Alexander von Humboldt Foundation (O.K.) and by the Saxonian Ministry of Science, grant 4-7531.50-04-844-07/8 (O.K.).

References

[1] U. Günther and F. Stefani, J. Math. Phys. **44**, 3097 (2003).
[2] F. Stefani and G. Gerbeth, Phys. Rev. Lett. **94**, 184506 (2005).
[3] U. Günther and O.N. Kirillov, J. Phys. A: Math. Gen. **39**, 10057 (2006).
[4] O.N. Kirillov and A.P. Seyranian A.P. SIAM J. Appl. Math. **64**(4), 1383 (2004).