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$\mathcal{PT}$-symmetric quantum state discrimination

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The objective of this paper is to explain and elucidate the formalism of $\mathcal{PT}$ quantum mechanics by applying it to a well-known problem in conventional Hermitian quantum mechanics, namely the problem of state discrimination. Suppose that a system is known to be in one of two quantum states, $|\psi_1\rangle$ or $|\psi_2\rangle$. If these states are not orthogonal, then the requirement of unitarity forbids the possibility of discriminating between these two states with one measurement; that is, determining with one measurement what state the system is in. In conventional quantum mechanics, there is a strategy in which successful state discrimination can be achieved with a single measurement but only with a success probability p that is less than unity. In this paper, the state-discrimination problem is examined in the context of $\mathcal{PT}$ quantum mechanics and the approach is based on the fact that a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian determines the inner product that is appropriate for the Hilbert space of physical states. It is shown that it is always possible to choose this inner product so that the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal. Using $\mathcal{PT}$ quantum mechanics, one cannot achieve a better state discrimination than in ordinary quantum mechanics, but one can instead perform a *simulated* quantum state discrimination, in which with a single measurement a perfect state discrimination is realized with probability p.

1. Introduction

The problem of quantum state discrimination is important in many applications of quantum information technology. Typically, one wants to extract information that is encoded in the unknown state of a quantum system. Therefore, one measures an observable, the outcome of which provides some information about the state of the system. Solving the state-discrimination problem consists of finding (i) the optimal choice for the observable and (ii) the optimal strategy to infer the state of the system, given the outcome of the measurement.

Here, we consider an idealized state-discrimination problem: an experimentalist who wishes to determine the state of the system is given *a priori* information that the system is in one of two possible states, $|\psi_1\rangle$ or $|\psi_2\rangle$. Thus, the task reduces to making a quantum binary decision. Of course, if $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal, one can determine with certainty the state of the system with just one measurement. However, in general these states are not orthogonal, and thus it is not possible to ascertain the state of the system with a single measurement.

In the context of conventional quantum mechanics, there are various optimal strategies for solving the idealized binary state-discrimination problem (see [1–3], for example). The choice of strategy naturally depends on the criteria of optimality.

In this paper, we describe a strategy for quantum state discrimination that is based on $\mathcal{PT}$ *quantum mechanics*, the quantum mechanics of systems governed by non-Dirac–Hermitian, but $\mathcal{PT}$-symmetric, Hamiltonians. (By a *Dirac–Hermitian* Hamiltonian, we mean one that is symmetric under combined complex conjugation and matrix transposition.) We will see that, while $\mathcal{PT}$ quantum mechanics cannot outperform conventional quantum mechanics, using it to solve the *effective* (or simulated) state-discrimination problem provides insight into the physical content and mathematical structure of $\mathcal{PT}$ quantum mechanics. In §2, we review the basic idea of quantum state discrimination in the context of Hermitian quantum mechanics. Then, in §3, we show how to perform quantum state discrimination using $\mathcal{PT}$ quantum mechanics. Section 4 contains brief concluding remarks.

2. Hermitian quantum state discrimination

One optimal strategy for quantum state discrimination, which was proposed by Helstrom [4], minimizes the error probability associated with a single measurement. Suppose that the *a priori* probabilities of the input state being $|\psi_1\rangle$ or $|\psi_2\rangle$ are equal, where the angle between $|\psi_1\rangle$ and $|\psi_2\rangle$ is such that $|\langle\psi_1|\psi_2\rangle| = \cos\epsilon$. The experimentalist performs a binary measurement of the observable $A \equiv |E_1\rangle\langle E_1| - |E_2\rangle\langle E_2|$, where

$$\left.\begin{aligned} |E_1\rangle &= \frac{1}{2\sqrt{2}}\left[\left(\frac{1}{\sin(\epsilon/2)}+\frac{1}{\cos(\epsilon/2)}\right)|\psi_1\rangle - \left(\frac{1}{\sin(\epsilon/2)}-\frac{1}{\cos(\epsilon/2)}\right)|\psi_2\rangle\right] \\ \text{and}\qquad |E_2\rangle &= \frac{1}{2\sqrt{2}}\left[\left(\frac{1}{\sin(\epsilon/2)}+\frac{1}{\cos(\epsilon/2)}\right)|\psi_2\rangle - \left(\frac{1}{\sin(\epsilon/2)}-\frac{1}{\cos(\epsilon/2)}\right)|\psi_1\rangle\right]. \end{aligned}\right\} \tag{2.1}$$

(It is easy to verify that the vectors $|E_1\rangle$ and $|E_2\rangle$ are orthogonal and are normalized to unity.) A measurement of A gives the value either $+1$ or -1. If this measurement returns the value $+1$ (-1), the experimentalist makes the educated guess that the unknown state is $|\psi_1\rangle$ ($|\psi_2\rangle$). Of course, if the two states are nearly identical, so that $\epsilon \ll 1$, then the result of a single measurement is hardly conclusive. Nevertheless, this strategy minimizes the probability of making the wrong inference, and thus is optimal on average [4–6].

For this strategy, the probability of error can be reduced if there are many identically prepared samples, all of which are either in the state $|\psi_1\rangle$ or else in the state $|\psi_2\rangle$. Now, each time a measurement is performed, the experimentalist gains some information, which can then be used to optimize the next measurement. By following the optimal strategy described in Brody & Meister [7] for sequential measurements, the experimentalist can discriminate the states with high confidence; that is, the error probability approaches zero at an optimal rate as the number of measurements increases. Of course, this strategy involves repeated quantum measurements and

repeated measurements on a single system are not in general permissible because a measurement can change the state of the system. Thus, to identify the state of the system with high confidence, a large number of identically prepared samples is needed. Naturally, preparing many identical samples can be costly, and determining the state will be time consuming.

This approach to the quantum state-discrimination problem has a simple classical analogue. Suppose that one is given a coin and is told that it is either (i) a fair coin with the probability of heads being 50 per cent or else (ii) an unfair coin with the probability of heads being 50.1 per cent. To distinguish between these two possibilities with confidence would require a long string of coin tosses. Moreover, with only a finite number of coin tosses, one could never be certain which possibility is the truth.

A second strategy for quantum state discrimination, known as *unambiguous state discrimination*, was introduced by Ivanovic [8] and investigated by others [9–11]. The unambiguous state-discrimination procedure is suboptimal in the sense that it does not minimize the error probability, but it has the advantage that with a non-zero probability the experimentalist can ascertain the state with just a *finite* number of sample states. Thus, this procedure has no direct classical coin-tossing analogue.

There are two ways to implement the unambiguous state-discrimination procedure. In the first scheme, we take the two-dimensional Hilbert subspace spanned by the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$, and embed this space in a three-dimensional Hilbert subspace spanned by a set of orthogonal basis vectors $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$. These (non-unique) basis vectors are chosen so that in the three-dimensional Hilbert space $|\psi_1\rangle$ is the linear combination

$$|\psi_1\rangle = \alpha_1|e_1\rangle + \beta_1|e_3\rangle \tag{2.2}$$

and $|\psi_2\rangle$ is the linear combination

$$|\psi_2\rangle = \alpha_2|e_2\rangle + \beta_2|e_3\rangle. \tag{2.3}$$

The experimentalist then performs a measurement of the observable

$$F = f_1|e_1\rangle\langle e_1| + f_2|e_2\rangle\langle e_2| + f_3|e_3\rangle\langle e_3|, \tag{2.4}$$

where f_1, f_2 and f_3 are the three distinct eigenvalues of F. If the measurement yields the value f_3, then the outcome is a *null* result that provides no information concerning the input state. However, if the measurement yields the value f_k ($k = 1, 2$), then the experimentalist concludes with *certainty* that the input state is $|\psi_k\rangle$. There is an optimal choice of basis vectors that minimizes the likelihood of a null result. However, we emphasize that even with the optimal choice of basis, with a single measurement of a suitable observable in the higher-dimensional Hilbert space, the experimentalist can only *sometimes* determine the input state with certainty; thus, unambiguous state discrimination can be viewed as an *effective* (or *simulated*) approach to quantum state discrimination.

It is worth remarking, in the above procedure, that if the original vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are elements of a two-dimensional Hilbert space (e.g. spin states of a single electron) then, in order to realize the measurement of F, it is necessary to introduce the so-called ancilla particle. That is, we couple the original system characterized by the state $|\psi_1\rangle$ or $|\psi_2\rangle$ to another system so that the Hilbert space of states for the combined system, obtained as a tensor product, is larger than two. One can then perform orthogonal measurement in the larger-dimensional Hilbert space, which can be described in the two-dimensional subspace in terms of an over-complete basis that forms a positive operator-valued measure. (See [1,2] for further details.)

A second way to implement unambiguous state discrimination is to take $|e_1\rangle$ and $|e_2\rangle$ to be a pair of orthogonal vectors in the two-dimensional Hilbert subspace spanned by $|\psi_1\rangle$ and $|\psi_2\rangle$, so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are given as linear combinations of $|e_1\rangle$ and $|e_2\rangle$. We then perform a unitary rotation U in the *three*-dimensional Hilbert subspace spanned by the orthogonal vectors $|e_1\rangle$, $|e_2\rangle$ and $|e_3\rangle$ in such a way that $|\psi_1\rangle$ transforms into a vector of the form

$$U|\psi_1\rangle = \alpha_1|e_1\rangle + \beta_1|e_3\rangle \tag{2.5}$$

and $|\psi_2\rangle$ transforms into a vector of the form

$$U|\psi_2\rangle = \alpha_2|e_2\rangle + \beta_2|e_3\rangle \tag{2.6}$$

with some coefficients $\alpha_{1,2}$ and $\beta_{1,2}$. We then measure the observable F in (2.4) and make the same inferences as in the first procedure for unambiguous state discrimination. Needless to say, if the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are originally elements of a two-dimensional Hilbert space, then, to implement this second procedure, we are required to introduce an ancillary particle, as in the case of the first procedure.

Let us compare these two ways of implementing unambiguous state discrimination. The first is passive in character in the sense that the objective is simply to find a set of three basis vectors that minimizes the chance of a null measurement. The second is active and dynamical in nature; that is, one chooses an arbitrary fixed basis and then transforms unitarily the original pair of states $|\psi_1\rangle$ and $|\psi_2\rangle$ into a new pair of states that minimizes the likelihood of a null measurement. Both approaches are equivalent because they have the same chance of a successful outcome.

The possibility of a null result in unambiguous state discrimination can be understood mathematically in terms of a non-unitary similarity transformation. Suppose that the unitary operation U in (2.5) and (2.6) belongs to a one-parameter family U_t of unitary evolution operators such that, at time $t = 0$, U_0 is the identity, and that, at time $t = T$, $U_T = U$. The projection of the vectors $U_t|\psi_k\rangle$ $(k = 1, 2)$ onto the Hilbert subspace spanned by $|e_1\rangle$ and $|e_2\rangle$ is

$$|\varphi_k(t)\rangle = (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)U_t|\psi_k\rangle. \tag{2.7}$$

Note that the norm of the vector $|\varphi_k(t)\rangle$ decreases in time because its probability is *leaking* into the third dimension. Thus, in effect, the dynamics $|\varphi_k(t)\rangle = \mathrm{e}^{-Qt/2}|\varphi_k(0)\rangle$ is governed by a non-unitary similarity transformation generated by some Hermitian operator Q.

3. State discrimination using $\mathcal{PT}$ quantum mechanics

In this paper, we present an *effective* (or *simulated*) approach to quantum state discrimination in the context of $\mathcal{PT}$ quantum mechanics. Our approach is similar to the technique of unambiguous state discrimination in Hermitian quantum mechanics. The idea is to introduce a carefully chosen non-Dirac–Hermitian $\mathcal{PT}$-symmetric (space–time–reflection-symmetric) Hamiltonian whose $\mathcal{PT}$ symmetry is *unbroken* (that is, its eigenvalues are real [12–14]). Such a Hamiltonian H determines the inner product in the Hilbert space on which it is defined, and relative to this inner product its eigenvectors are orthonormal [15–17]. If H is chosen correctly, the inner product of the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ can be made arbitrarily small. In this Hilbert space, there exists an observable that perceives these two states as being orthogonal. Such an observable is unitarily equivalent to H itself. Thus, in this Hilbert space, a single quantum measurement can be performed whose outcome discriminates the two states. This corresponds to a successful (non-null) outcome in the case of unambiguous state discrimination.

We do *not* claim that this non-Hermitian procedure is superior to conventional Hermitian quantum-mechanical state discrimination. This is because the procedure requires a *change of Hilbert space* from the original two-dimensional Hilbert space in which the vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ are specified to a new two-dimensional Hilbert space spanned by the eigenstates of the $\mathcal{PT}$-symmetric Hamiltonian H. This change of Hilbert space is realized by means of a similarity transformation, and this similarity transformation is the same as that discussed in §2 in the context of unambiguous state discrimination. The similarity transformation introduces the likelihood that a measurement will give a null result. Thus, while $\mathcal{PT}$-symmetric quantum state discrimination in principle requires only one measurement, some of the time it produces a null result. It is in this sense that $\mathcal{PT}$-symmetric quantum state discrimination can be regarded as a *simulated* quantum state discrimination.

The techniques used in this paper are similar to those used by Bender *et al.* [18] to achieve arbitrarily fast quantum state evolution. The situation considered by Bender *et al.* [18] is as follows: an experimentalist is faced with the task of transforming one specified quantum state

into another specified quantum state in the shortest possible time. With an unbounded energy resource, one can achieve arbitrarily fast time evolution. However, one is given the constraint that the difference between the largest eigenvalue E_+ and smallest eigenvalue E_- of the Hamiltonian is a fixed energy ω: $E_+ - E_- = \omega$. Thus, the *quantum brachistochrone* problem is to find the Hamiltonian H that achieves the fastest time evolution while at the same time satisfying this energy constraint. Among all Hermitian Hamiltonians, the Hamiltonian that achieves the fastest time evolution requires a non-vanishing amount of time [19,20]. However, it has been shown previously [18,21–23] that a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian can perform this time evolution in an arbitrarily short time.

Arbitrarily fast quantum evolution can only be achieved if the matrix elements of the Hamiltonian are arbitrarily large complex numbers. If this is the case, then for a Dirac–Hermitian Hamiltonian, the ω constraint cannot be satisfied. The mechanism exploited by Bender *et al.* [18] is that, when H is $\mathcal{PT}$ symmetric, the constraint that ω be fixed *can* be satisfied even though the matrix elements of H are arbitrarily large and complex. Faster-than-Hermitian quantum evolution using a $\mathcal{PT}$-symmetric Hamiltonian has been seen experimentally [24]. However, because a change of Hilbert space is required, a successful fast time evolution can only be achieved with probability less than 1, and thus in this experiment the fast time evolution is referred to as a *simulated* quantum brachistochrone.

We propose two different but related solutions to the non-Hermitian simulated unambiguous state-discrimination problem, and these correspond to the two different ways described above to implement Hermitian unambiguous quantum state discrimination. The first is simply to apply a binary measurement. Depending on the outcome, the state of the system can then, in principle, be determined with certainty. The second way is to implement a dynamical evolution governed by a complex Hamiltonian having real eigenvalues. This evolution can be achieved in a system in which there is a delicate and precise balance of loss and gain. (Systems of this kind have been constructed in many different laboratory experiments [25–32].) A system of this type is not Dirac–Hermitian; however, such a system is described by a $\mathcal{PT}$-symmetric Hamiltonian that *is* Hermitian with respect to an alternative Hilbert-space inner product. Thus, the time evolution is unitary in the appropriate Hilbert space. This suggests a second and alternative solution whereby a unitary evolution using a complex Hamiltonian in a suitably defined Hilbert space is applied so that the two input states $|\psi_1\rangle$ and $|\psi_2\rangle$ evolve into a pair of states that are perceived as being orthogonal in the conventional Hermitian inner product Hilbert space. A real binary measurement can then be applied to distinguish the states with certainty.

Needless to say, in a closed system discriminating two non-orthogonal states with one measurement is forbidden by unitarity. Hence, using a non-Hermitian Hamiltonian for quantum-state manipulation necessarily involves an open system. Most of the experiments reported previously [25–32] involve classical systems whose dynamics is still described by the Schrödinger equation. At present, we are unaware of whether a precise balance of loss and gain can be realized experimentally in an open quantum system. However, if this is possible, then the schemes described below for perfect simulated quantum state discrimination can be realized, at least in principle.

(a) Solution 1: finding a $\mathcal{PT}$-symmetric Hamiltonian whose inner product interprets $|\psi_1\rangle$ and $|\psi_2\rangle$ as being orthogonal

Consider the two-dimensional subspace spanned by the two vectors $|\psi_1\rangle$ and $|\psi_2\rangle$. Let the angular distance between the two states in the Bloch sphere be 2ϵ. Without loss of generality, we can re-parametrize the Bloch sphere so that both states lie on the same meridian; that is, $|\psi_1\rangle$ lies at the angles (θ, ϕ), and $|\psi_2\rangle$ lies at $(\theta + 2\epsilon, \phi)$:

$$|\psi_1\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ \mathrm{e}^{i\phi}\sin\dfrac{\theta}{2} \end{pmatrix} \quad \text{and} \quad |\psi_2\rangle = \begin{pmatrix} \cos\dfrac{\theta+2\epsilon}{2} \\ \mathrm{e}^{i\phi}\sin\dfrac{\theta+2\epsilon}{2} \end{pmatrix}. \tag{3.1}$$

We still have the freedom to choose specific values for θ and ϕ, and for simplicity we choose $\phi = -\pi/2$ and $\theta = \pi/2 - \epsilon$.

Let us consider the general 2×2 $\mathcal{PT}$-symmetric Hamiltonian [16]

$$H = \begin{pmatrix} r\,\mathrm{e}^{i\beta} & s \\ s & r\,\mathrm{e}^{-i\beta} \end{pmatrix} = r\cos\beta\,\mathbf{1} + \boldsymbol{\sigma}\cdot(s,\,0,\,ir\sin\beta)\,, \tag{3.2}$$

where the parameters r, s and β are real, and $\boldsymbol{\sigma}$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This Hamiltonian commutes with $\mathcal{PT}$, where the parity reflection operator is given by

$$\mathcal{P} = \sigma_1 \tag{3.3}$$

and the time-reversal operator $\mathcal{T}$ is complex conjugation.

For H in (3.2), the parametric region of unbroken $\mathcal{PT}$ symmetry in which the eigenvalues are real is $s^2 > r^2\sin^2\beta$. In this region, we can calculate the $\mathcal{C}$ operator,

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}, \tag{3.4}$$

where $\sin\alpha = (r/s)\sin\beta$. Then, using the $\mathcal{CPT}$ operator, we can calculate the bra vectors corresponding to ket vectors. Specifically, we find that for $|\psi_1\rangle$ in (3.1) the corresponding $\langle\psi_1|$ is the row vector

$$\langle\psi_1| = \frac{1}{\cos\alpha}\left(\cos\frac{\pi-2\epsilon}{4} - \sin\alpha\,\sin\frac{\pi-2\epsilon}{4},\, -i\sin\alpha\,\cos\frac{\pi-2\epsilon}{4} + i\sin\frac{\pi-2\epsilon}{4}\right). \tag{3.5}$$

Thus, we can calculate the inner product $\langle\psi_1|\psi_2\rangle$, and if we require this inner product to vanish, we obtain the condition

$$\sin\alpha = \cos\epsilon. \tag{3.6}$$

Finally, to distinguish between the two states $|\psi_1\rangle$ and $|\psi_2\rangle$, we need only construct projection operators that leave one state invariant and annihilate the other state. To do so we must normalize these states. A straightforward calculation gives

$$\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = \sin\epsilon. \tag{3.7}$$

Hence, the *normalized* state $|\psi_1\rangle$ and its complex conjugate are given by

$$\left.\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{\sin\epsilon}}\begin{pmatrix} \cos\dfrac{\pi-2\epsilon}{4} \\ -i\sin\dfrac{\pi-2\epsilon}{4} \end{pmatrix} \\ \text{and} \quad \langle\psi_1| &= \frac{1}{\sin^{3/2}\epsilon}\left(\cos\frac{\pi-2\epsilon}{4} - \cos\epsilon\sin\frac{\pi-2\epsilon}{4}, -i\cos\epsilon\cos\frac{\pi-2\epsilon}{4} + i\sin\frac{\pi-2\epsilon}{4}\right). \end{aligned}\right\} \tag{3.8}$$

The results for $|\psi_2\rangle$ and $\langle\psi_2|$ are obtained by replacing $\pi - 2\epsilon$ with $\pi + 2\epsilon$.

We then construct the projection operators

$$\left.\begin{aligned} |\psi_1\rangle\langle\psi_1| &= \frac{1}{2\sin\epsilon}\begin{pmatrix} 1+\sin\epsilon & -i\cos\epsilon \\ -i\cos\epsilon & -1+\sin\epsilon \end{pmatrix} \\ \text{and} \quad |\psi_2\rangle\langle\psi_2| &= \frac{1}{2\sin\epsilon}\begin{pmatrix} -1+\sin\epsilon & i\cos\epsilon \\ i\cos\epsilon & 1+\sin\epsilon \end{pmatrix}. \end{aligned}\right\} \tag{3.9}$$

It is straightforward to verify that these operators are $\mathcal{PT}$ *observables* because they are $\mathcal{CPT}$ self-adjoint [15,16]; that is, they commute with the $\mathcal{CPT}$ operator. Furthermore, these projection operators constitute a resolution of the identity

$$|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = \mathbf{1}. \tag{3.10}$$

The projection operators in (3.9) can be expressed as a linear combination of Pauli sigma matrices,

$$|\psi_1\rangle\langle\psi_1| = \frac{1}{2}\mathbf{1} + \boldsymbol{\sigma}\cdot\left(-\frac{i}{2}\cot\epsilon, 0, \frac{1}{\sin\epsilon}\right), \tag{3.11}$$

and so can the Hamiltonian

$$H = \sqrt{r^2 - s^2\cos^2\epsilon}\,\mathbf{1} + \boldsymbol{\sigma}\cdot(s, 0, is\cos\epsilon). \tag{3.12}$$

Thus, we see that these operators are equivalent to applying a magnetic field in a complex direction. A single application of one of the projection measurements in (3.9) distinguishes the states $|\psi_1\rangle$ and $|\psi_2\rangle$ with certainty.

(b) Solution 2: finding a $\mathcal{PT}$-symmetric Hamiltonian under which $|\psi_1\rangle$ and $|\psi_2\rangle$ evolve into orthogonal states

Recent experimental results [25–32] indicate that it may be easier to implement a non-Hermitian Hamiltonian than to implement a non-Hermitian observable. In such cases there is an alternative strategy to accomplish state discrimination: we construct a Hamiltonian under which the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ evolve into states that are orthogonal under the conventional Hermitian inner product. We then proceed to make a measurement using a conventionally Hermitian observable.

In conventional Hilbert space the standard inner product is based on the Hermitian adjoint (transpose and complex conjugate). Thus, at time t the inner product is simply $\langle\psi_1|e^{iH^\dagger t}\,e^{-iHt}|\psi_2\rangle$, where H is given in (3.2), $H^\dagger$ denotes the Hermitian adjoint of H, and we have taken $\hbar = 1$. To evaluate this inner product, we use the standard matrix identity to simplify the exponential of H in (3.2),

$$\exp(i\phi\,\boldsymbol{\sigma}\cdot\mathbf{n}) = \cos\phi\,\mathbf{1} + i\sin\phi\,\boldsymbol{\sigma}\cdot\mathbf{n}. \tag{3.13}$$

From this identity we obtain the result

$$\cos^2\alpha\,e^{iH^\dagger t}\,e^{-iHt} = \begin{pmatrix} \cos^2(\omega t - \alpha) + \sin^2(\omega t) & -2i\sin^2(\omega t)\sin\alpha \\ 2i\sin^2(\omega t)\sin\alpha & \cos^2(\omega t + \alpha) + \sin^2(\omega t) \end{pmatrix}, \tag{3.14}$$

in which $\omega = \sqrt{s^2 - r^2\sin^2\beta}$. (Note that in the Hermitian limit $\alpha \to 0$; this becomes the identity matrix $\mathbf{1}$.)

We thus calculate the inner product at time t,

$$\begin{aligned}\langle\psi_1, t|\psi_2, t\rangle &= \langle\psi_1|e^{iH^\dagger t}\,e^{-iHt}|\psi_2\rangle \\ &= \cos\epsilon[\cos^2\alpha + 2\sin^2(\omega t)\sin^2\alpha] - 2\sin^2(\omega t)\sin\alpha.\end{aligned} \tag{3.15}$$

This inner product vanishes when

$$\sin^2(\omega t) = \frac{\cos^2\alpha\,\cos\epsilon}{2\sin\alpha - 2\sin^2\alpha\cos\epsilon}, \tag{3.16}$$

which has a solution for t if $\epsilon \neq 0$.

Note that the time needed for this evolution becomes arbitrarily small and approaches 0 as $\cos\alpha \to 0$ (or $\alpha \to \pm\pi/2$). (This result is similar to what was found in the case of the non-Hermitian quantum brachistochrone [18,21–23].) From (3.2), we can see that the limit $\alpha \to \pi/2$ corresponds to applying a magnetic field in a complex direction and that the imaginary component of this magnetic field $r\sin\beta$ takes its highest possible value s. There may be practical constraints that

make it difficult to realize such a limit, in which case an experimentalist must wait some time until (3.16) is satisfied. At this point, a single Hermitian projection measurement using a classical apparatus can be applied to distinguish between the two possible input states with certainty.

4. Concluding remarks

In summary, we have presented two alternative ways to distinguish between a pair of non-orthogonal pure quantum states with a single measurement. To do so, we have exploited the complex degrees of freedom made available by $\mathcal{PT}$ symmetry. The projection operators in (3.9) that achieve exact state discrimination between arbitrarily close states require arbitrarily large complex matrix elements. However, the mechanism that was used to solve the quantum brachistochrone problem applies once again: because the projection operators are $\mathcal{PT}$ symmetric, the eigenvalues are still bounded (they are 0 and 1).

Of course, as we remarked earlier, we do not claim that we can achieve perfect quantum state discrimination using $\mathcal{PT}$-symmetric Hamiltonians. Rather, we can only achieve *simulated* quantum state discrimination. We do not know whether it is actually possible to build an experimental device that can achieve the mathematical results demonstrated in this paper. The experiments described previously [25–32] and especially that described by Zheng *et al.* [24] provide some hope that it may eventually be possible to realize at a quantum level complex Hamiltonians having large complex matrix elements. If one of the strategies formulated here can be implemented, then there may be considerable benefits in the area of quantum information theory. For example, in quantum computation it is known that an unstructured database search can be mapped to the problem of distinguishing exponentially close quantum states [33]. The reformulation of the database search can also be achieved using the method described here to search a database exponentially fast. This is because the method presented here can be applied to distinguish fast and accurately any pair of distinct states.

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