Asymptotical AdS space from nonlinear gravitational models with stabilized extra dimensions

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We consider nonlinear gravitational models with a multidimensional warped product geometry. Particular attention is payed to models with quadratic scalar curvature terms. It is shown that for certain parameter ranges, the extra dimensions are stabilized if the internal spaces have a negative constant curvature. In this case, the four-dimensional effective cosmological constant as well as the bulk cosmological constant become negative. As a consequence, the homogeneous and isotropic external space is asymptotically AdS_4. The connection between the D-dimensional and the four-dimensional fundamental mass scales sets a restriction on the parameters of the considered nonlinear models.

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I. INTRODUCTION

The multidimensionality of our Universe is one of the most intriguing assumptions in modern physics. It follows naturally from theories unifying different fundamental interactions with gravity, e.g., M- or string theory [1]. The idea has received a great deal of renewed attention over the last few years within the "brane-world" description of the Universe. In this approach the $SU(3)\times SU(2)\times U(1)$ standard model (SM) fields are localized on a three-dimensional spacelike hypersurface (brane) whereas the gravitational field propagates in the whole (bulk) space-time. The framework also implies that usual four-dimensional physics is located on the brane (i.e., our Universe). Moreover, brane-world physics provides a possible solution of the hierarchy problem due to the well known connection between the Planck scale $M_{Pl(4)}$ and the fundamental scale $M_{*(4+D')}$ of the four-dimensional and the $(4+D')$-dimensional space-time, respectively:

$$M^2_{Pl(4)} \sim V_{D'} M^{2+D'}_{*(4+D')}. \tag{1}$$

Here $V_{D'}$ denotes the volume of the compactified D' extra dimensions. It was realized in [2–4] that the localization of the SM fields on the brane allows one to lower $M_{*(4+D')}$ down to the electroweak scale $M_{EW}\sim 1$ TeV without contradicting present observations. Therefore the compactification scale of the internal space can be of the order of

$$r \sim V_{D'}^{1/D'} \sim 10^{(32/D')-17} \ \text{cm}. \tag{2}$$

In this Arkani-Hamed–Dimopoulos–Dvali (ADD) model [2], physically acceptable values correspond to $D' \geqslant 3$ (see, e.g., [5]), and for $D'=3$ one arrives at a submillimeter compactification scale $r\sim 10^{-6}$ cm of the internal space. Additionally, the geometry is assumed to be factorizable as in the standard Kaluza-Klein (KK) model. That is, the topology is the direct product of a nonwarped external space-time manifold and internal space manifolds with warp factors which depend on the external coordinates. In addition to this, the M-theory inspired Randall-Sundrum (RS) scenario [6] represents an interesting approach with a nonfactorizable geometry and $D'=1$. Here, the four-dimensional space-time is warped with a factor $\tilde{\Omega}$ which depends on the extra dimension and Eq. (1) is modified as follows: $M_{Pl(4)} \sim \tilde{\Omega}^{-1} M_{EW}$. In our paper we shall concentrate on the factorizable geometry of the ADD model.

According to observations the internal space should be static or nearly static at least from the time of primordial nucleosynthesis (otherwise the fundamental physical constants would vary). This means that at the present evolutionary stage of the Universe the compactification scale of the internal space should either be stabilized and trapped at the minimum of some effective potential, or it should be slowly varying (similar to the slowly varying cosmological constant in the quintessence scenario [7]). In both cases, small fluctuations over stabilized or slowly varying compactification scales (conformal scales/geometrical moduli) are possible.

Stabilization of extra dimensions (moduli stabilization) in models with large extra dimensions (ADD models) has been

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considered in a number of papers (see, e.g., Refs. [4,8–14]).[1] In the corresponding approaches, a product topology of the $(4+D')$-dimensional bulk space-time was constructed from Einstein spaces with scale (warp) factors depending only on the coordinates of the external four-dimensional component. As a consequence, the conformal excitations have the form of massive scalar fields living in the external space-time. Within the framework of multidimensional cosmological models (MCM) such excitations were investigated in [15–17] where they were called gravitational excitons. Later, since the ADD compactification approach, these geometrical moduli excitations are known as radions [4,9]. It should be noted that over the last years the term radion has been used to describe quite different forms of metric perturbations within brane-world models. In MCM with warped product topology of the internal spaces they are understood as conformal excitations of the additional dimensions (gravitational excitons), whereas in RS-I-type models they describe the relative motion of branes [18].[2] The differences between these two frameworks have been pointed out in [22,23].

All of the above mentioned papers are devoted to the stabilization of large extra dimension in theories with linear multidimensional gravitational action. String theory suggests that the usual linear Einstein-Hilbert action should be extended with higher order nonlinear curvature terms. In the present paper we use a simplified approach with a multidimensional Lagrangian of the form $L=f(R)$, where $f(R)$ is an arbitrary smooth function of the scalar curvature. Without connection to stabilization of the extra-dimensions, such models (four-dimensional as well as multidimensional ones) were considered, e.g., in Ref. [24]. There, it was shown that the nonlinear models are equivalent to models with linear gravitational action plus a minimally coupled scalar field with self-interaction potential.

In the present paper we advance this equivalence towards investigating the problem of extra dimensions stabilization. We find that the stabilization of extra dimensions takes place only if additional internal spaces have a compact hyperbolic geometry and the effective four-dimensional cosmological constant is negative. If the external space M_0 is homogeneous and isotropic this implies that M_0 becomes asymptotically an anti–de Sitter space (AdS_{D_0}). Additionally, we show that requiring the extra dimensions to be dynamically stabilized is a sufficient condition for the bulk space-time to acquire a constant negative curvature.

The paper is structured as follows. After explaining the general setup of our model in Sec. II, we concretize the geometry to a warped product of n internal spaces. We perform a dimensional reduction of the action functional to a four-dimensional effective theory with $(n+1)$ self-interacting minimally coupled scalar fields (Sec. III). The stabilization of the extra dimensions is then reduced to the condition that the obtained effective potential for these fields should have a minimum. In Sec. IV a detailed analysis of this problem is given for a model with one internal space. The main results are summarized and discussed in the concluding Sec. V.

II. GENERAL THEORY

We consider a $D=(4+D')$-dimensional nonlinear pure gravitational theory with action

$$S=\frac{1}{2\kappa_D^2}\int_M d^D x\sqrt{|\bar{g}|}f(\bar{R}), \tag{3}$$

where $f(\bar{R})$ is an arbitrary smooth function with mass dimension $\mathcal{O}(m^2)$ (m has the unit of mass) of a scalar curvature $\bar{R}=R[\bar{g}]$ constructed from the D-dimensional metric $\bar{g}_{ab}$ $(a,b=1,\ldots,D)$.

$$\kappa_D^2=8\pi/M_{*(4+D')}^{2+D'} \tag{4}$$

is the D-dimensional gravitational constant (subsequently, we assume that $M_{*(4+D')}\sim M_{EW}$). The equation of motion for this theory reads [24]

$$f'\bar{R}_{ab}-\frac{1}{2}f\,\bar{g}_{ab}-\bar{\nabla}_a\bar{\nabla}_b f'+\bar{g}_{ab}\bar{\Box}f'=0, \tag{5}$$

where $f'=df/d\bar{R}$, $\bar{R}_{ab}=R_{ab}[\bar{g}]$. $\bar{\nabla}_a$ is the covariant derivative with respect to the metric $\bar{g}_{ab}$; and the corresponding Laplacian is denoted by

$$\bar{\Box}=\Box[\bar{g}]=\bar{g}^{ab}\bar{\nabla}_a\bar{\nabla}_b=\frac{1}{\sqrt{|\bar{g}|}}\partial_a(\sqrt{|\bar{g}|}\ \bar{g}^{ab}\partial_b). \tag{6}$$

Equation (5) can be rewritten in the form

$$f'\bar{G}_{ab}+\frac{1}{2}\bar{g}_{ab}(\bar{R}f'-f)-\bar{\nabla}_a\bar{\nabla}_b f'+\bar{g}_{ab}\bar{\Box}f'=0, \tag{7}$$

where $\bar{G}_{ab}=\bar{R}_{ab}-\frac{1}{2}\overline{R g_{ab}}$. The trace of Eq. (5) is

$$(D-1)\bar{\Box}f'=\frac{D}{2}f-f'\bar{R} \tag{8}$$

and can be considered as a connection between $\bar{R}$ and f.

It is well known that for $f'(\bar{R})>0$ the conformal transformation

$$g_{ab}=\Omega^2\bar{g}_{ab}, \tag{9}$$

with

$$\Omega=[f'(\bar{R})]^{1/(D-2)}, \tag{10}$$

[1] In most of these papers, moduli stabilization was considered without regard to the energy-momentum localized on the brane. A brane matter contribution was taken into account, e.g., in [14].

[2] A detailed discussion of radion stabilization and dynamics in RS models is given, e.g., in [19,20]. An extended list of references on this topic can be found in [21].

reduces the nonlinear theory (3) to a linear one with an additional scalar field. The equivalence of the theories can be easily proven with the help of the following auxiliary formulas:

$$\square = \Omega^{-2}[\bar{\square} + (D-2)\bar{g}^{ab}\Omega^{-1}\Omega_{,a}\partial_b],$$

$$\bar{\square} = \Omega^2\square - (D-2)g^{ab}\Omega\Omega_{,a}\partial_b\,, \tag{11}$$

$$R_{ab} = \bar{R}_{ab} + \frac{D-1}{D-2}(f')^{-2}\bar{\nabla}_a f'\bar{\nabla}_b f' - (f')^{-1}\bar{\nabla}_a\bar{\nabla}_b f' - \frac{1}{D-2}\bar{g}_{ab}(f')^{-1}\bar{\square}f', \tag{12}$$

and

$$R = (f')^{2/(2-D)}\left\{\bar{R} + \frac{D-1}{D-2}(f')^{-2}\bar{g}^{ab}\partial_a f'\partial_b f' - 2\frac{D-1}{D-2}(f')^{-1}\bar{\square}f'\right\}. \tag{13}$$

Thus Eqs. (7) and (8) can be rewritten as

$$G_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}g^{mn}\phi_{,m}\phi_{,n} - \frac{1}{2}g_{ab}e^{(-D/\sqrt{(D-2)(D-1)})\phi}(\bar{R}f' - f) \tag{14}$$

and

$$\square\phi = \frac{1}{\sqrt{(D-2)(D-1)}}e^{(-D/\sqrt{(D-2)(D-1)})\phi}\left(\frac{D}{2}f - f'\bar{R}\right), \tag{15}$$

where

$$f' = \frac{df}{d\bar{R}} := e^{A\phi} > 0,\ \ A := \sqrt{\frac{D-2}{D-1}}. \tag{16}$$

Equation (16) can be used to express $\bar{R}$ as a function of the dimensionless field ϕ: $\bar{R} = \bar{R}(\phi)$. It is easily seen that Eqs. (14) and (15) are the equations of motion for the action

$$S = \frac{1}{2\kappa_D^2}\int_M d^Dx\sqrt{|g|}[R[g] - g^{ab}\phi_{,a}\phi_{,b} - 2U(\phi)], \tag{17}$$

where

$$U(\phi) = \frac{1}{2}e^{-B\phi}[\bar{R}(\phi)e^{A\phi} - f(\bar{R}(\phi))],$$

$$B := \frac{D}{\sqrt{(D-2)(D-1)}} \tag{18}$$

and they can be written as follows:

$$G_{ab} = T_{ab}[\phi, g], \tag{19}$$

$$\square\phi = \frac{\partial U(\phi)}{\partial\phi}. \tag{20}$$

Here, $T_{ab}[\phi, g]$ is the standard expression of the energy-momentum tensor for the minimally coupled scalar field with potential (18). Equation (20) can be considered as a constraint equation following from the reduction of the nonlinear theory (3) to the linear one (17).

Let us consider what will happen if, in some way, the scalar field ϕ tends asymptotically to a constant: $\phi\to\phi_0$. From Eq. (16) we see that in this limit the nonlinearity disappears and Eq. (3) becomes a linear theory $f(\bar{R}) \sim c_1\bar{R} + c_2$ with $c_1 = f' = \exp(A\phi_0)$ and a cosmological constant $-c_2/(2c_1)$. In the case of homogeneous and isotropic space-time manifolds, linear purely geometrical theories with a constant Λ term necessarily imply an (A)dS geometry. Thus in the limit $\phi\to\phi_0$ the D-dimensional theory (3) can asymptotically lead to an (A)dS with scalar curvature:

$$\bar{R} \to -\frac{D}{D-2}\frac{c_2}{c_1}. \tag{21}$$

Clearly, the linear theory (17) would reproduce this asymptotic (A)dS limit for $\phi\to\phi_0$:

$$R \to 2\frac{D}{D-2}U(\phi_0) = -\frac{D}{D-2}c_2\, c_1^{-[D/(D-2)]}. \tag{22}$$

Hence, in this limit $\bar{R}/R \to c_1^{D/(D-2)}$ in accordance with Eq. (13) and $f' = c_1$. In Sec. IV we shall show that the stabilization of the extra dimensions automatically results in the condition $\phi\to\phi_0$ with $U(\phi_0) < 0$. Thus the D-dimensional space-time (bulk) can become asymptotically AdS_D.

In the rest of the paper we consider the quadratic theory:

$$f(\bar{R}) = \bar{R} + \alpha\bar{R}^2 - 2\Lambda_D\,, \tag{23}$$

where the parameter α has dimensions $\mathcal{O}(m^{-2})$. For this theory we obtain

$$1 + 2\alpha\bar{R} = e^{A\phi} \Leftrightarrow \bar{R} = \frac{1}{2\alpha}(e^{A\phi} - 1) \tag{24}$$

and

$$U(\phi) = \frac{1}{2}e^{-B\phi}\left[\frac{1}{4\alpha}(e^{A\phi} - 1)^2 + 2\Lambda_D\right]. \tag{25}$$

The condition $f' > 0$ implies $1 + 2\alpha\bar{R} > 0$.

III. DIMENSIONAL REDUCTION

In this section we assume that the D-dimensional bulk space-time M undergoes a spontaneous compactification to a warped product manifold

$$M = M_0 \times M_1 \times \cdots \times M_n \tag{26}$$

with metric

$$\bar{g}=\bar{g}_{ab}(X)dX^a\otimes dX^b=\overline{g^{(0)}}+\sum_{i=1}^{n} e^{2\bar{\beta}^i(x)}g^{(i)}. \tag{27}$$

The coordinates on the $(D_0=d_0+1)$-dimensional manifold M_0 [usually interpreted as our $(D_0=4)$-dimensional Universe] are denoted by x and the corresponding metric by

$$\overline{g^{(0)}}=\bar{g}^{(0)}_{\mu\nu}(x)dx^\mu\otimes dx^\nu. \tag{28}$$

Let the internal factor manifolds M_i be d_i-dimensional Einstein spaces with metric $g^{(i)}=g^{(i)}_{m_in_i}(y_i)dy_i^{m_i}\otimes dy_i^{n_i}$, i.e.,

$$R_{m_in_i}[g^{(i)}]=\lambda^i g^{(i)}_{m_in_i}, \quad m_i,n_i=1,\ldots,d_i \tag{29}$$

and

$$R[g^{(i)}]=\lambda^i d_i\equiv R_i\sim r_i^{-2}, \tag{30}$$

where $r_i=(\int d^{d_i}y\sqrt{|g^{(i)}|})^{1/d_i}$ is a characteristic size of M_i. For the metric ansatz (27) the scalar curvature $\bar{R}$ depends only on x: $\bar{R}[\bar{g}]=\bar{R}(x)$. Thus ϕ is also a function of x: $\phi=\phi(x)$.

The conformally transformed metric (9) reads

$$g=\Omega^2\bar{g}=(e^{A\phi})^{2/(D-2)}\bar{g}:=g^{(0)}+\sum_{i=1}^{n} e^{2\beta^i(x)}g^{(i)} \tag{31}$$

with

$$g^{(0)}_{\mu\nu}=(e^{A\phi})^{2/(D-2)}\bar{g}^{(0)}_{\mu\nu}, \tag{32}$$

$$\beta^i=\bar{\beta}^i+\frac{A}{D-2}\phi. \tag{33}$$

The fact that the fields ϕ and β^i depend only on x allows us to perform the dimensional reduction of action (17). Without loss of generality we set the compactification scales of the internal spaces at the present time at $\beta^i=0$ $(i=1,\ldots,n)$. The corresponding total volume of the internal spaces is given by

$$V_{D'}\equiv\prod_{i=1}^{n}\int_{M_i}d^{d_i}y\sqrt{|g^{(i)}|}=\prod_{i=1}^{n} r_i^{d_i}, \tag{34}$$

where $V_{D'}$ has dimensions $\mathcal{O}(m^{-D'})$, and $D'=D-D_0=\sum_{i=1}^n d_i$ is the number of the extra dimensions. After dimensional reduction action (17) reads

$$S=\frac{1}{2\kappa_0^2}\int_{M_0}d^{D_0}x\sqrt{|g^{(0)}|}\times\prod_{i=1}^{n} e^{d_i\beta^i}\bigg\{R[g^{(0)}]-G_{ij}g^{(0)\mu\nu}\partial_\mu\beta^i\,\partial_\nu\beta^j -g^{(0)\mu\nu}\partial_\mu\phi\partial_\nu\phi+\sum_{i=1}^{n} R[g^{(i)}]e^{-2\beta^i}-2U(\phi)\bigg\}, \tag{35}$$

where $G_{ij}=d_i\delta_{ij}-d_id_j$ $(i,j=1,\ldots,n)$ is the midisuperspace metric [25,26] and

$$\kappa_0^2:=\frac{\kappa_D^2}{V_{D'}} \tag{36}$$

is the D_0-dimensional (four-dimensional) gravitational constant. If we take the electroweak scale M_{EW} and the Planck scale M_{Pl} as fundamental ones for D-dimensional [see Eq. (4)] and four-dimensional space-times $(\kappa_0^2=8\pi/M_{Pl}^2)$, respectively, then we reproduce Eqs. (1) and (2).

Action (35) is written in the Brans-Dicke frame. Conformal transformation to the Einstein frame [15,16],

$$\tilde{g}^{(0)}_{\mu\nu}=\left(\prod_{i=1}^{n} e^{d_i\beta^i}\right)^{2/(D_0-2)} g^{(0)}_{\mu\nu}, \tag{37}$$

yields

$$S=\frac{1}{2\kappa_0^2}\int_{M_0}d^{D_0}x\sqrt{|\tilde{g}^{(0)}|}\{R[\tilde{g}^{(0)}]-\bar{G}_{ij}\tilde{g}^{(0)\mu\nu}\partial_\mu\beta^i\,\partial_\nu\beta^j -\tilde{g}^{(0)\mu\nu}\partial_\mu\phi\partial_\nu\phi-2U_{eff}(\beta,\phi)\}. \tag{38}$$

The tensor components of the midisuperspace metric (target space metric on $\mathbb{R}_T^n$) $\bar{G}_{ij}$ $(i,j=1,\ldots,n)$, its inverse metric $\bar{G}^{ij}$, and the effective potential are, respectively,

$$\bar{G}_{ij}=d_i\delta_{ij}+\frac{1}{D_0-2}d_id_j, \tag{39}$$

$$\bar{G}^{ij}=\frac{\delta^{ij}}{d_i}+\frac{1}{2-D}, \tag{40}$$

and

$$U_{eff}(\beta,\phi)=\left(\prod_{i=1}^{n} e^{d_i\beta^i}\right)^{-[2/(D_0-2)]}\times\left[-\frac{1}{2}\sum_{i=1}^{n} R_ie^{-2\beta^i}+U(\phi)\right]. \tag{41}$$

IV. STABILIZATION OF THE INTERNAL SPACE

Without loss of generality,[3] we consider in the present section a model with only one d_1-dimensional internal space. The corresponding action (38) reads

$$S=\frac{1}{2\kappa_0^2}\int_{M_0} d^{D_0}x\sqrt{|\tilde{g}^{(0)}|}\{R[\tilde{g}^{(0)}]-\tilde{g}^{(0)\mu\nu}\partial_\mu\varphi\partial_\nu\varphi -\tilde{g}^{(0)\mu\nu}\partial_\mu\phi\partial_\nu\phi-2U_{eff}(\varphi,\phi)\}, \qquad (42)$$

where

$$\varphi:=-\sqrt{\frac{d_1(D-2)}{D_0-2}}\beta^1 \qquad (43)$$

and

$$U_{eff}(\varphi,\phi)=e^{2\varphi\sqrt{d_1/[(D-2)(D_0-2)]}}\times\left[-\frac{1}{2}R_1 e^{2\varphi\sqrt{(D_0-2)/[d_1(D-2)]}}+U(\phi)\right]. \qquad (44)$$

For simplicity we continue to work with dimensionless scalar fields φ,ϕ instead of passing to canonical ones (modulo 8π): $\tilde{\varphi}=\varphi M_{Pl}$, $\tilde{\phi}=\phi M_{Pl}$, and $\tilde{U}_{eff}=M_{Pl}^2 U_{eff}$. The restoration of the correct dimensionality is obvious.

The equations of motion for φ and ϕ are, respectively,

$$\tilde{\Box}\varphi=\frac{\partial U_{eff}}{\partial\varphi}, \qquad (45)$$

$$\tilde{\Box}\phi=\frac{\partial U_{eff}}{\partial\phi}, \qquad (46)$$

where

$$\frac{\partial U_{eff}}{\partial\varphi}=2\sqrt{\frac{d_1}{(D-2)(D_0-2)}}U_{eff} -R_1\sqrt{\frac{D_0-2}{d_1(D-2)}}e^{2\varphi\sqrt{(D-2)/[d_1(D_0-2)]}} \qquad (47)$$

and

$$\frac{\partial U_{eff}}{\partial\phi}=e^{2\varphi\sqrt{d_1/[(D-2)(D_0-2)]}}\ \frac{\partial U(\phi)}{\partial\phi}. \qquad (48)$$

In order to obtain a stable compactification of the internal space, the potential $U_{eff}(\varphi,\phi)$ should have a minimum with respect to φ and ϕ. This is obvious with respect to the field φ because it is precisely the stabilization of this field that we aim to achieve. It is also clear that potential $U_{eff}(\varphi,\phi)$ should have a minimum with respect to ϕ because without stabilization of ϕ the effective potential remains a dynamical function and the extremum condition $\partial U_{eff}/\partial\varphi|_{\varphi=0}=0$ is not satisfied [see Eq. (47)]. Furthermore, Eq. (48) shows that the extrema of the potentials $U_{eff}(\varphi,\phi)$ and $U(\phi)$ with respect to the field ϕ coincide with each other. Thus the stabilization of the extra dimension takes place if the field ϕ goes to the minimum of the potential $U(\phi)$. According to the discussion in Sec. II [see Eqs. (21) and (22)] this results in an asymptotically constant curvature space-time [for a nonzero minimum of $U(\phi)$].

Let us now present a detailed analysis of the quadratic gravitational theory (23) with potential $U(\phi)$ (25). First, we shall investigate the range of parameters which ensures a minimum of $U(\phi)$. The extremum condition gives $\partial_\phi U=0$ so that

$$(2A-B)x^2+2(B-A)x-(q+1)B=0, \qquad (49)$$

where $x:=e^{A\phi}>0$ and $q:=8\alpha\Lambda_D$. The non-negative solution of this equation defines the position of the extremum:

$$x_0=e^{A\phi_0} =\frac{-(B-A)+\sqrt{(B-A)^2+(2A-B)(q+1)B}}{2A-B} =\frac{-(B-A)+\sqrt{A^2+(2A-B)Bq}}{2A-B}. \qquad (50)$$

From the inequalities

$$B-A=\frac{2}{\sqrt{(D-2)(D-1)}}>0 \qquad (51)$$

and

$$2A-B=\frac{D-4}{\sqrt{(D-2)(D-1)}}>0 \quad \text{for} \quad D>4 \qquad (52)$$

it follows that the parameter q should be restricted to the half-line

$$q=8\alpha\Lambda_D>-1. \qquad (53)$$

The case $q=-1$ corresponds to $\phi_0\to-\infty$ and is not considered in the following.

The necessary condition for the existence of a minimum of the potential $U(\phi)$

$$\partial^2_{\phi\phi}U(\phi)|_{\text{extr}}=\frac{A}{4\alpha}e^{(A-B)\phi_0}[(2A-B)e^{A\phi_0}+(B-A)] =\frac{1}{4\alpha}\frac{1}{D-1}x_0^{-[2/(D-2)]}[(D-4)x_0+2]>0 \qquad (54)$$

requires positive values of the parameter $\alpha>0$. From the explicit expression of $U(\phi)$ at the extremum

[3]The only difference between a general model with $n>1$ internal spaces and the particular one with $n=1$ consists of an additional diagonalization of the geometrical moduli excitations.

$$U(\phi)|_{\rm extr}=\frac{1}{8\alpha}x_0^{-[D/(D-2)]}[(x_0-1)^2+q], \tag{55}$$

it is easy to see that $U|_{min}\geq 0$ for $\Lambda_D\geq 0$ and $U|_{min}<0$ for $\Lambda_D<0$. In the latter case we have $-1<8\alpha\Lambda_D<0$.

Let us show now that the total potential $U_{eff}(\varphi,\phi)$ also has a global minimum in the case when $U(\phi)$ has a negative minimum. To prove it, it is convenient to rewrite potential (44) as

$$U_{eff}(\varphi,\phi)=F(\varphi)G(\varphi,\phi)$$

with

$$F(\varphi)=e^{2\varphi\sqrt{d_1/[(D-2)(D_0-2)]}},$$

$$G(\varphi,\phi)=-\frac{1}{2}R_1e^{2\varphi\sqrt{(D_0-2)-[d_1(D-2)]}}+U(\phi)\,. \tag{56}$$

The extremum condition gives

$$\partial_\varphi U_{eff}=\left(2\sqrt{\frac{d_1}{(D-2)(D_0-2)}}G+\partial_\varphi G\right)F=0,$$

$$\partial_\varphi G=-2\sqrt{\frac{d_1}{(D-2)(D_0-2)}}G, \tag{57}$$

$$\partial_\phi U_{eff}=F(\partial_\phi U)=0\ \Rightarrow\ \partial_\phi U=0, \tag{58}$$

whereas the eigenvalues of the Hessian at the minimum should be non-negative,

$$\partial^2_{\varphi\varphi}U_{eff}=\left[\partial^2_{\varphi\varphi}G-4\frac{d_1}{(D-2)(D_0-2)}G\right]F>0, \tag{59}$$

$$\partial^2_{\phi\phi}U_{eff}=F\partial^2_{\phi\phi}U>0\ \Rightarrow\ \partial^2_{\phi\phi}U>0, \tag{60}$$

$$\partial^2_{\varphi\phi}U_{eff}=2\sqrt{\frac{d_1}{(D-2)(D_0-2)}}F\partial_\phi U=0. \tag{61}$$

Choosing the compactification scale of the extra dimension at $\beta^1_{min}=\varphi_{min}=0$, we find the following relations at the extremum:

$$R_1=\frac{2d_1}{D-2}U(\phi)|_{\rm extr}, \tag{62}$$

$$G\Big|_{\rm extr}=\frac{D_0-2}{D-2}U(\phi)\Big|_{\rm extr}, \tag{63}$$

and hence

$$\mathrm{sign}\,(R_1)=\mathrm{sign}\,[U(\phi)|_{\rm extr}]=\mathrm{sign}\,(G|_{\rm extr}). \tag{64}$$

Using the obvious relation

$$\partial^2_{\varphi\varphi}G=-2\frac{D_0-2}{d_1(D-2)}R_1e^{2\varphi\sqrt{(D_0-2)/[d_1(D-2)]}} \tag{65}$$

and Eqs. (59), (62), and (63) we see that

$$-\frac{4}{D-2}U(\phi)|_{min}>0\Rightarrow U(\phi)|_{min}<0. \tag{66}$$

This inequality sets strong restrictions on the considered nonlinear model.

(1) According to Eq. (22) it implies that the stabilization of the extra dimension leads asymptotically to a negative constant curvature bulk space-time.

(2) Only models with parameters from the range $\alpha>0$ and $-1<8\alpha\Lambda_D<0$ will stabilize [see Eqs. (54) and (55)]. All other configurations are excluded.

(3) The global minimum of the whole effective potential U_{eff} is also negative:

$$U_{eff}|_{min}=\frac{D_0-2}{D-2}U(\phi)|_{min}=\frac{D_0-2}{2d_1}R_1<0. \tag{67}$$

Its value plays the role of a D_0-dimensional effective cosmological constant $\Lambda_{eff}=U_{eff}|_{min}$.

(4) From Eqs. (64) and (66) it follows that the compactified internal space should have negative curvature.

The latter restriction agrees with the results of [10,13] because the negative value of the effective potential in the minimum violates the null energy condition so that the stabilized internal space should be (compact) hyperbolic (see also [12,13]). We note that adding to our nonlinear model some kind of matter, satisfying the null energy condition, can shift the effective D_0-dimensional cosmological constant to non-negative values and the internal space can acquire positive curvature.

A further restriction on the model follows from Eqs. (2), (30), and (62). According to these equations the free parameters α and Λ_D, or α and q, are strongly connected with the compactification radius r_1 of the extra dimensional factor space M_1, as well as with the fundamental mass scale $M_{*(4+d_1)}$ and the four-dimensional Planck scale $M_{Pl(4)}$:

$$\frac{2d_1}{D-2}U[\phi_0(q),\alpha]|_{\rm extr}=R_1=-\frac{d_1(d_1-1)}{r_1^2}\sim-\left(\frac{M_{*(4+d_1)}}{M_{Pl(4)}}\right)^{4/d_1}M^2_{*(4+d_1)}. \tag{68}$$

For fixed compactification radius $r_1<\infty$ the constraint (68) forbids the limit $\Lambda_D\to-0$, whereas $\alpha\to 0$ is allowed. This behavior is easily understood. According to Eq. (23) the limit $\alpha\to 0$ describes the transition to a linear Einstein gravity model with D-dimensional cosmological constant Λ_D. For $\alpha\to 0$ the mass of the ϕ-field excitations tends to infinity $m^2_\phi\to\infty$ [see Eq. (74) below] and the field itself becomes frozen at the minimum position $\phi_0(\alpha\to 0)\to 0$ of the potential $U(\phi)$

$$U[\alpha\to 0]|_{\rm extr}\to\Lambda_D,\ \ \partial^2_{\phi\phi}U|_{\rm extr}\to\infty. \tag{69}$$

The resulting D-dimensional space-time has constant scalar curvature $\overline{R}=R=2D\Lambda_D/(D-2)$ and a stabilization of internal spaces in such models is possible [15] for $\Lambda_D<0$ and $R_i<0$.

In contrast, the transition $\Lambda_D \to -0$ necessarily implies $U(\phi)|_{\rm extr} \to -0$, $R_1 \to -0$ which is connected with a decompactification $r_1 \to \infty$ of the extra dimensions according to Eq. (68). From the derivatives (59)–(61) of the effective potential at the extremum position $(\varphi_{\rm extr}=0,\phi_0)$ and

$$\partial_\varphi^n U_{eff}|_{\rm extr} = -2^{n-1}\left[\frac{D-2}{d_1(D_0-2)}\right]^{n/2} R_1 + 2^n \left[\frac{d_1}{(D-2)(D_0-2)}\right]^{n/2} U|_{\rm extr} \quad (70)$$

we read off that in the limit $\Lambda_D \to -0$ the potential becomes flat with respect to φ: $\partial_\varphi^n U_{eff} \to 0$, whereas it remains well-behaved with respect to ϕ:

$$\partial_{\phi\phi}^2 U_{eff}|_{\rm extr} \to \frac{D-2}{4\alpha(D-1)} > 0. \quad (71)$$

This is due to $x_0(\Lambda_D \to 0) \to 1$ and Eq. (54). The potential $U_{eff}(\varphi,\phi)$ itself coincides in this case with the effective potential of a model with Ricci-flat factor space M_1

$$U_{eff}(\varphi,\phi) = e^{2\varphi\sqrt{d_1/[(D-2)(D_0-2)]}} U(\phi), \quad (72)$$

what is known to have no stabilized extra dimensions. A stabilization could be achieved, e.g., by accounting for additional matter fields [15–17,27].

Finally, let us turn to the masses of the excitation fields φ and ϕ near the minimum of U_{eff}. These masses are defined by the relations

$$m_\varphi^2 = \partial_{\varphi\varphi}^2 U_{eff}|_{min} = -\frac{4}{D-2} U(\phi)|_{min} = -\frac{2}{d_1} R_1, \quad (73)$$

$$m_\phi^2 = \partial_{\phi\phi}^2 U_{eff}|_{min} = \partial_{\phi\phi}^2 U(\phi)|_{min} = \frac{1}{4\alpha}\frac{1}{D-1} x_0^{-[2/(D-2)]}[(D-4)x_0+2]. \quad (74)$$

In the decompactification limit $r_1 \to \infty$, $\Lambda_D \to -0$, $R_1 \to -0$ the mass of the gravexciton vanishes $m_\varphi^2 \to 0$, whereas the mass of the ϕ field remains nonzero $m_\phi^2 \to (D-2)/[4\alpha(D-1)] > 0$. For fixed compactification scale r_1 the constraint (62) and its implication

$$\frac{1}{4\alpha} = \frac{D-2}{d_1} x_0^{D/(D-2)}[(x_0-1)^2+q]^{-1} R_1 \quad (75)$$

can be used to express Eq. (74) in terms of x_0 and R_1,

$$m_\phi^2 = \frac{D-2}{(D-1)d_1} \frac{x_0[(D-4)x_0+2]}{(x_0-1)^2+q} R_1. \quad (76)$$

This means that in an ADD scenario, where relation (68) necessarily holds, the basic mass scale of the excitations φ and ϕ is defined by the fundamental mass scale $M_{*(4+d_1)}$ and the four-dimensional Planck scale $M_{Pl(4)}$

$$m_{\varphi,\phi}^2 \sim R_1 \sim r_1^{-2} \sim \left(\frac{M_{*(4+d_1)}}{M_{Pl(4)}}\right)^{4/d_1} M_{*(4+d_1)}^2. \quad (77)$$

V. CONCLUSIONS AND DISCUSSION

In the present paper we investigated multidimensional gravitational models with a non-Einsteinian form of the action. In particular, we assumed that the action is an arbitrary smooth function of the scalar curvature $f(R)$. For such models, we concentrated on the problem of extra dimension stabilization in the case of factorizable geometry. To perform such an analysis, we reduced the pure nonlinear gravitational model to a linear one with an additional self-interacting scalar field. The factorization of the geometry allowed for a dimensional reduction of the considered models and to obtain an effective four-dimensional model with additional minimally coupled scalar fields in the Einstein frame. These fields describe conformal excitations of the internal space scale factors. A detailed stability analysis was carried out for a model with quadratic curvature term: $f(R)=R+\alpha R^2 - 2\Lambda_D$. It was shown that a stabilization is only possible for the parameter range $-1 < 8\alpha\Lambda_D < 0$.

This necessarily implies that the extra dimensions are stabilized if the compact internal spaces M_i, $i=1,\ldots,n$, have negative constant curvatures. More precisely, these spaces have a quotient structure $M_i = H^{d_i}/\Gamma_i$, where H^{d_i} and Γ_i are hyperbolic spaces and their discrete isometry groups, respectively. In this case, the four-dimensional cosmological constant (which corresponds to the minimum of the effective four-dimensional potential) is also negative. As a consequence, the homogeneous and isotropic external $(D_0=4)$-dimensional space is asymptotically AdS_{D_0}. As the extra dimensional scale factors approach their stability position the bulk space-time curvature asymptotically (dynamically) tends to a negative constant value. Let us note that this would allow, e.g., for a spontaneous compactification scenario along the lines[4]

$$\mathrm{AdS}_D \to \mathrm{AdS}_{D_0} \times H^{d_1}/\Gamma_1 \times \cdots \times H^{d_n}/\Gamma_n. \quad (78)$$

We further found that the compactification scale completely defines the effective cosmological constant and the mass of the internal scale factor excitations (gravexcitons) near the minimum position. It is also strongly connected with the parameters α and Λ_D of the nonlinear model [see Eq. (75)]. For example, in the limit $\Lambda_D \to 0$ the extra dimensions necessarily decompactify $(r_1 \to \infty)$ and the effective potential U_{eff} becomes indistinguishable from an effective potential of a model with Ricci-flat factor space M_1. The correspond-

[4]An explicit generalized de Sitter solution for a similar stabilized warped product space was obtained in [28]. The warped product consisted of a Ricci-flat or $R \times S^3$ external space-time and Einstein spaces with positive constant scalar curvatures as internal spaces.

ing scale factor is then not stabilized and the gravexciton becomes massless. In contrast to models with possible decompactification, ADD scenarios are characterized by a compactification scale which is fixed by relations (1) and (2) between the fundamental mass scales $M_{Pl(4)}$ and $M_{*(4+D')}$. The same relations enforce in this case a constraint on the parameters of the nonlinear model. In contrast with the masses of gravexcitons m_φ which are completely fixed by the compactification scale, the mass m_ϕ of the scalar field ϕ (which originates from the nonlinearity of the starting model) can still depend in a specific way on the parameters α and Λ_D.

From a cosmological perspective, it is of interest to consider the possibility of inflation for the four-dimensional external space-time within our nonlinear model. For a linear multidimensional model with an arbitrary scalar field (inflaton), such an analysis was carried out in [17]. As described in Sec. II, our pure gravitational quadratic curvature action (23) can be mapped to a scenario linear in the curvature with a rather specific self-interaction potential (25) for the scalar field ϕ. This allows us to extend some of the techniques of [17] to our model.

It can be shown that there is a possibility for inflation to occur if the scalar fields start to roll down from the region:

$$|U(\phi)|\geqslant|U(\phi)|_{min}|\gg|R_1|e^{2\varphi\sqrt{(D_0-2)/[d_1(D-2)]}}, \quad (79)$$

where the effective potential (44) reads

$$U_{eff}\approx e^{2\varphi\sqrt{d_1/[(D-2)(D_0-2)]}}U(\phi). \quad (80)$$

If

$$e^{\sqrt{(D-2)/(D-1)}\phi}\gg 1, \quad (81)$$

and hence $U(\phi)\approx[1/(8\alpha)]e^{(2A-B)\phi}=[1/(8\alpha)]\times e^{[(D-4)\phi/\sqrt{(D-2)(D-1)}]}$, the slow-roll parameters ϵ and $\eta_{1,2}$ (see paper [17]) read

$$\epsilon\approx\eta_1\approx\eta_2\approx\frac{2d_1}{(D-2)(D_0-2)}+\frac{(D-4)^2}{2(D-2)(D-1)}. \quad (82)$$

For the dimensionality of our observable Universe $D_0=4$, these parameters are restricted to the range

$$\frac{3}{5}\leqslant\epsilon,\eta_1,\eta_2\leqslant 1 \quad \text{for} \quad 6\leqslant D\leqslant 10. \quad (83)$$

Thus, generally speaking, the slow-roll conditions for inflation are satisfied in this region. The scalar field ϕ can act as an inflaton and drive the inflation of the external space. It is clear that estimates (83) point only to the possibility for inflation to occur. For the considered model with negative effective cosmological constant inflation is not successfully completed [29] if the reheating due to the decay of the ϕ-field excitations and gravexcitons is not sufficient for a transition to the radiation dominated era. In any case, for scenarios with successful transition or without, the external space has a turning point at its maximal scale factor where the evolution changes from expansion to contraction.[5] Obviously, for such models the negative effective cosmological constant forbids a late time acceleration of the Universe as indicated by recent observational data. In order to cure this problem, the model should be generalized, e.g., by inclusion of additional form fields [27] or other matter fields.

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