SIAM J. MATH. ANAL.
Vol. 42, No. 3, pp. 1413–1447

ON THE SPECTRUM OF THE MAGNETOHYDRODYNAMIC MEAN-FIELD α^2-DYNAMO OPERATOR*

UWE GÜNTHER†, HEINZ LANGER‡, AND CHRISTIANE TRETTER§

Abstract. The existence of magnetohydrodynamic mean-field α^2-dynamos with spherically symmetric, isotropic helical turbulence function α is related to a non-self-adjoint spectral problem for a coupled system of two singular second order ordinary differential equations. We establish global estimates for the eigenvalues of this system in terms of the turbulence function α and its derivative α'. They allow us to formulate an antidynamo theorem and a nonoscillation theorem. The conditions of these theorems, which again involve α and α', must be violated in order to reach supercritical or oscillatory regimes.

Key words. eigenvalue estimates, magnetohydrodynamic dynamo problem, non-self-adjoint operator, block operator matrix, system of Bessel operators

AMS subject classifications. Primary, 47A10, 76W05; Secondary, 47A55, 34L15, 35Q86

DOI. 10.1137/100782024

1. Introduction. One of the simplest models of a magnetohydrodynamic dynamo is a mean-field α^2-dynamo with spherically symmetric, isotropic turbulence function (α-profile) $\alpha(r)$. Models of this type were among the first capable of explaining the turbulence-based inverse energy cascade leading to nondecaying dynamo regimes with self-sustaining magnetic fields (see [15]). Recently, α^2-dynamo models were used for detailed studies of polarity reversal processes of magnetic fields (see, e.g., [23], [24]), as suggested by paleomagnetic data of the Earth's magnetic field. Deep insight into the reversal dynamics could be gained by analyzing the highly nonlinear back-reaction-based induction processes by a time foliation method built over a series of instantaneously linearized (kinematic) auxiliary setups. For the latter, the α-profiles were assumed to be fixed radial functions so that the well-known decomposition of the magnetic field into poloidal and toroidal components with subsequent expansion in spherical harmonics could be used. As a result, one arrives at a set of eigenvalue problems, indexed by the degree $l \in \mathbb{N} = \{1, 2, \dots\}$ of the spherical harmonics, for pairs of coupled linear ordinary differential equations:

$$\begin{pmatrix} \partial_r^2 - \dfrac{l(l+1)}{r^2} & \alpha(r) \\ -\partial_r \alpha(r) \partial_r + \alpha(r)\dfrac{l(l+1)}{r^2} & \partial_r^2 - \dfrac{l(l+1)}{r^2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad r \in (0,1], \tag{1.1}$$

*Received by the editors January 8, 2010; accepted for publication March 22, 2010; published electronically June 9, 2010. The first and the last authors were supported by the Deutsche Forschungsgemeinschaft (DFG) within the Collaborative Research Center (Sonderforschungsbereich) 609 and under grant TR368/6-1, respectively.
http://www.siam.org/journals/sima/42-3/78202.html
†Research Center Dresden-Rossendorf, P.O. Box 510119, D-01314 Dresden, Germany (u.guenther@fzd.de).
‡Institut für Analysis und Scientific Computing, Technische Universität Wien, Wiedner Hauptstr. 8–10, A-1040 Wien, Austria (hlanger@email.tuwien.ac.at).
§Mathematisches Institut, Universität Bern, Sidlerstr. 5, 3012 Bern, Switzerland (tretter@math.unibe.ch).

in $L_2(0,1) \oplus L_2(0,1)$ subject to the boundary condition

$$\begin{pmatrix} (\partial_r + l)y_1 \\ y_2 \end{pmatrix}(1) = 0 \tag{1.2}$$

(see [15, section 14.2], [22, equations (6)–(8)]). In the following, for brevity, we call the eigenvalue problem (1.1), (1.2) the dynamo problem.

In view of the time separation ansatz $\bar{\mathbf{B}}(x,t) = \exp(\lambda t)\bar{\mathbf{b}}(x)$ of the corresponding magnetic field modes, one naturally distinguishes between decaying or subcritical modes ($\operatorname{Re}\lambda < 0$) and amplifying or supercritical modes ($\operatorname{Re}\lambda > 0$), as well as between oscillatory modes ($\operatorname{Im}\lambda \neq 0$) and nonoscillatory modes ($\operatorname{Im}\lambda = 0$). The physically relevant self-sustaining dynamo configurations are mainly defined by a few supercritical modes, whereas possible polarity reversals of the magnetic fields are closely related to the existence of oscillatory modes close to criticality ($\operatorname{Re}\lambda \approx 0$, $\operatorname{Im}\lambda \neq 0$) (see [23], [24]). For a deeper understanding of the physical interplay between the plasma- (or conducting fluid-)based α-profile and the dynamo dynamics, knowledge of the spectral properties of the dynamo problem (1.1), (1.2) is of utmost interest.

In contrast to the large number of general results for kinematic fast dynamo problems (see, e.g., [4, Chapter V] as well as [6]) and despite the simplicity of the physical model, no proven analytic information for the spectrum of the kinematic mean-field dynamo problem (1.1), (1.2) seems to be available, except for the case of constant α where the spectrum is real. For nonconstant functions α, only numerical calculations for eigenvalues were performed (see, e.g., [22]). However, for a nonsymmetric spectral problem like (1.1), (1.2), numerical computations are prone to be unreliable; a convincing example for this is a nonnormal 7×7 matrix due to Godunov for which numerical algorithms yield results far away from the true eigenvalues (see, e.g., [9], [12, Example 5.2.5]).

Other attempts to attack the above dynamo problem include replacing the physical boundary conditions by the so-called idealized boundary condition

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(1) = 0, \tag{1.3}$$

assuming that the helical turbulence function α is strictly positive, and/or analyzing the simpler monopole modes, that is, the case $l = 0$ (see, e.g., [11], [10], [14]); all of these restrictions are not desirable from the physical point of view.

The aim of this paper is to establish analytic enclosures for the spectrum of the dynamo problem (1.1), (1.2) in terms of the function α, not making any of the above simplifying assumptions. We show, in particular, that the eigenvalues lie in a horizontal strip around the real axis, and we establish an upper bound for their real parts. More precisely, every eigenvalue λ of (1.1), (1.2) satisfies

$$|\operatorname{Im}\lambda| \leq \|\alpha'\|, \quad \operatorname{Re}\lambda \leq \rho_\theta, \tag{1.4}$$

where the constant ρ_θ depends on $\|\alpha\|$ and $\|\alpha'\|$, as well as on the smallest eigenvalues $\lambda_1(l)$ and $\lambda_1(\infty)$ of the Bessel operator $-\partial_r^2 + l(l+1)/r^2$ with boundary condition $y'(1) + ly(1) = 0$ and $y(1) = 0$, respectively (compare (1.2)). Here, for a continuous function q on $[0,1]$, we denote by $\|q\| := \max_{r\in[0,1]} |q(r)|$ the maximum norm of q. The estimate for the real part in (1.4) yields, in particular, a so-called antidynamo theorem: if

$$\|\alpha\|^2 + \frac{\|\alpha\|\|\alpha'\|}{\sqrt{\lambda_1(l)}} < \lambda_1(\infty),$$

then the dynamo operator has no eigenvalues in the (supercritical) right half-plane (compare the more geometrical antidynamo theorems for kinematic fast dynamos listed, e.g., in [4, Chapter V, section 3]). In addition, we establish conditions ensuring that a particular eigenvalue remains on the real axis when $\alpha \not\equiv 0$, which could be called a local nonoscillation theorem. Our results also cover other boundary conditions such as the idealized case (1.3), and they do not require α to be sign definite. The methods we employ stem from the perturbation theory of linear operators and from the spectral theory of block operator matrices (see, e.g., [13], [26]).

In the following we give a brief outline of the paper. In section 2 we introduce a family of linear operators $\mathcal{A}_\theta$ in $L_2(0,1) \oplus L_2(0,1)$ so that we can write the boundary eigenvalue problems (1.1), (1.2) and (1.1), (1.3) as spectral problems $(\mathcal{A}_\theta - \lambda)y = 0$ with $\theta = l$ and $\theta = \infty$, respectively. The operators $\mathcal{A}_\theta$ are block operator matrices of the form

$$\mathcal{A}_\theta := \begin{pmatrix} -A_\theta & \alpha \\ A_{\theta,\alpha} & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_\theta) := \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty).$$

Here A_θ and $A_{\theta,\alpha}$ are linear operators in $L_2(0,1)$ given by the Bessel(-type) differential expressions $-\partial_r^2 + l(l+1)/r^2$ and $-\partial_r \alpha \partial_r + \alpha l(l+1)/r^2$, respectively, and the boundary condition $x'(1) + \theta x(1) = 0$ for $\theta \in [0,\infty]$ (note that $\theta = \infty$ corresponds to $x(1) = 0$). In section 3 we investigate the entries of the block operator matrices $\mathcal{A}_\theta$ in detail and collect all the properties needed in the subsequent sections, e.g., to show that $\mathcal{A}_\theta$ defines a closed linear operator and that its spectrum consists only of eigenvalues of finite algebraic multiplicities with no finite accumulation point.

Section 4 contains our first main result, the eigenvalue enclosure in Theorem 4.6. This first estimate is based on decomposing $\mathcal{A}_\theta$ into a lower triangular block operator matrix plus a bounded part containing only the right upper corner α:

$$\mathcal{A}_\theta := \begin{pmatrix} -A_\theta & 0 \\ A_{\theta,\alpha} & -A_\infty \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = \mathcal{Q}_\theta + \mathcal{R}.$$

This decomposition, together with a Neumann series argument, enables us to identify a region in the complex plane in which the eigenvalues must lie. In addition, we establish bounds on $\|\alpha\|$ and $\|\alpha'\|$ such that an eigenvalue of one of the diagonal elements of $\mathcal{A}_\theta$, that is, an eigenvalue when $\alpha \equiv 0$, remains real for $\alpha \not\equiv 0$ (see Propositions 4.10 and 5.12). Physically speaking, this means that a particular mode remains nonoscillating.

Section 5 contains our second main result, the eigenvalue enclosure in Theorem 5.10. Here we use a quasi-similarity transformation of $\mathcal{A}_\theta$ (with the unbounded operator $\mathcal{W}_\theta = \operatorname{diag}(A_\theta^{1/2}, I)$) such that the transformed operator $\mathcal{B}_\theta$ is a bounded perturbation of a self-adjoint operator:

$$\mathcal{B}_\theta = \begin{pmatrix} -A_\theta & A_\theta^{1/2}\alpha \\ \alpha A_\theta^{1/2} & -A_\infty \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\alpha' D A_\theta^{-1/2} & 0 \end{pmatrix} = \mathcal{S}_\theta + \mathcal{T}_\theta.$$

This allows us to conclude that the eigenvalues of $\mathcal{A}_\theta$ lie in discs of radius $\|\alpha'\|$ (the norm of the perturbation $\mathcal{T}_\theta$) around the eigenvalues of the self-adjoint operator $\mathcal{S}_\theta$; in particular, the imaginary parts of the eigenvalues of $\mathcal{A}_\theta$ are bounded by $\|\alpha'\|$ and their real parts are bounded by $\max \sigma(\mathcal{S}_\theta) + \|\alpha'\|$.

In section 6 we compare the two different eigenvalue enclosures of sections 4 and 5. It turns out that, generically, a combination of both estimates yields the best result:

it consists of the uniform estimate for the imaginary parts of the eigenvalues from section 5 and of the upper bound for the real parts of the eigenvalues from section 4 (see (1.4) and Figure 4). At the end of section 6 we summarize the physically most relevant result in the form of an antidynamo theorem (see (6.3)).

In section 7 we illustrate our results with some examples. They include the case of constant α where the eigenvalues of the physical dynamo problem are given only implicitly as solutions of an equation involving four different Bessel functions. We also consider the particular nonconstant function α for which dipole-dominated oscillatory criticality was first found numerically by Stefani and Gerbeth (see [22]); in their computations they obtained special α-profiles such that the rightmost eigenvalues pass from $\operatorname{Re}\lambda < 0$ to $\operatorname{Re}\lambda > 0$ with $\operatorname{Im}\lambda \neq 0$ first for the dipole modes ($l = 1$) and only afterwards for quadrupole and higher-degree modes ($l > 1$).

The following notation is used throughout the paper. In the Hilbert space $L_2(0,1)$ the scalar product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. By $\mathrm{AC}_{\mathrm{loc}}(I)$ we denote the space of locally absolutely continuous functions on a (sub)interval $I \subset [0,1]$ (that is, the space of functions that are differentiable (Lebesgue-)almost everywhere on I and have locally Lebesgue-integrable derivative on I). For a linear operator T we denote by $\mathcal{D}(T)$ its domain, and by $\sigma(T)$, $\sigma_{\mathrm{p}}(T)$, $\rho(T)$ its spectrum, the set of its eigenvalues, and its resolvent set, respectively; if T is bounded, we denote by $\|T\|$ its operator norm. For a continuous function $q \in C([0,1])$, we use the same symbol q to denote the (bounded) multiplication operator by the function q in $L_2(0,1)$; in this case, the operator norm is given by the maximum norm $\|q\| = \max_{r\in[0,1]} |q(r)|$. For details on linear operators in Hilbert spaces we refer the reader to [13, Chapter III, sections 2, 5, 6], [21, Chapters VI, VIII], and [20].

2. Operator model.

In this section we introduce a family of linear operators $\mathcal{A}_\theta$ in the product Hilbert space $L_2(0,1)\oplus L_2(0,1)$ which describe the system of differential equations (1.1) with physical boundary conditions (1.2) and idealized boundary conditions (1.3).

To this end, we first associate linear operators in the Hilbert space $L_2(0,1)$ with the two differential expressions

$$(2.1)\qquad \tau := -\partial_r^2 + \frac{l(l+1)}{r^2}, \quad \tau_\alpha := -\partial_r \alpha(r) \partial_r + \alpha(r)\frac{l(l+1)}{r^2}, \quad \partial_r := \frac{\mathrm{d}}{\mathrm{d}r},$$

occurring in (1.1). Here and in the following, we always suppose that $l \in \mathbb{N} = \{1,2,\dots\}$ is fixed and that $\alpha\colon [0,1] \to \mathbb{R}$ is a continuously differentiable real-valued function, $\alpha \in C^1([0,1])$.

The classical Bessel differential expression τ appears twice on the diagonal in (1.1), but it is subject to two different boundary conditions at $r = 1$ in (1.2). Therefore we define a family of operators A_θ, $\theta \in [0,\infty]$, by means of the boundary condition $y'(1) + \theta\, y(1) = 0$, so that $\theta = l$ yields the first order boundary condition for y_1 and $\theta = \infty$ yields the Dirichlet boundary condition for y_2 in (1.2).

DEFINITION 2.1. *Let $\theta \in [0,\infty]$. Define the linear operator A_θ in $L_2(0,1)$ by*

$$\mathcal{D}(A_\theta) := \bigl\{x \in L_2(0,1) : x, x' \in \mathrm{AC}_{\mathrm{loc}}((0,1]),\ \tau x \in L_2(0,1),\ x'(1) + \theta\, x(1) = 0\bigr\},$$
$$(A_\theta x)(r) := (\tau x)(r) = -x''(r) + \frac{l(l+1)}{r^2} x(r) \quad \textit{for almost every } r \in (0,1].$$

Remark 2.2. Every $x \in \mathcal{D}(A_\theta)$ automatically satisfies $\lim_{r\searrow 0} x(r) = 0$. This follows easily if one calculates the inverse A_θ^{-1} and checks that $\lim_{r\searrow 0}(A_\theta^{-1} f)(r) = 0$

for $f \in L_2(0,1)$ (see, e.g., [3, Lemma 2.6] and its proof); note that a fundamental system of $\tau y = 0$ is given by the simple functions $x_1(r) = r^{-l}$, $x_2(r) = r^{l+1}$, $r \in (0,1]$.

The differential expression τ_α in (2.1) is not classical since its leading coefficient α may change sign and may hence also have singularities in the interior of the interval $[0,1]$. Since τ_α is only located in an off-diagonal corner in (1.1), the following definition is sufficient for us.

DEFINITION 2.3. *Let $\theta \in [0,\infty]$. Introduce the linear operator $A_{\theta,\alpha}$ in $L_2(0,1)$ by*

$$\mathcal{D}(A_{\theta,\alpha}) := \mathcal{D}(A_\theta),$$

$$(A_{\theta,\alpha}x)(r) := (\tau_\alpha x)(r) = -(\alpha x')'(r) + \alpha(r)\frac{l(l+1)}{r^2}x(r) \text{ for almost every } r \in (0,1].$$

Remark 2.4. In Proposition 3.6 below we show that, in fact, $A_{\theta,\alpha}x \in L_2(0,1)$ for $x \in \mathcal{D}(A_\theta)$, that is, $A_{\theta,\alpha}$ is well-defined.

Now we are able to formulate the dynamo problems (1.1), (1.2) and (1.1), (1.3) as spectral problems for a linear operator $\mathcal{A}_\theta$ acting in the product Hilbert space $L_2(0,1) \oplus L_2(0,1)$. Here the case $\theta = l$ corresponds to the physical boundary condition (1.2), while $\theta = \infty$ corresponds to the idealized boundary condition (1.3).

PROPOSITION 2.5. *Let $\theta \in [0,\infty]$. Define a linear operator $\mathcal{A}_\theta$ in the product space $L_2(0,1) \oplus L_2(0,1)$ by the block operator matrix*

$$\mathcal{A}_\theta := \begin{pmatrix} -A_\theta & \alpha \\ A_{\theta,\alpha} & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_\theta) := \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty). \tag{2.2}$$

Then the boundary eigenvalue problems (1.1), (1.2), *and* (1.1), (1.3) *can be written equivalently as*

$$(\mathcal{A}_\theta - \lambda)y = 0, \quad y \in \mathcal{D}(\mathcal{A}_\theta),$$

for $\theta = l$ and $\theta = \infty$, respectively.

Proof. The claim is evident from the definitions of the operator $\mathcal{A}_\theta$ and of its entries A_θ and $A_{\theta,\alpha}$ for $\theta = l$ and $\theta = \infty$. □

3. Auxiliary results. In this section we study the properties of the entries A_θ and $A_{\theta,\alpha}$ of the block operator matrix $\mathcal{A}_\theta$ introduced in the previous section. They are used in the next two sections to derive estimates for the eigenvalues of the dynamo problem (1.1), (1.2).

PROPOSITION 3.1. *Let $\theta \in [0,\infty]$. The linear operator A_θ in Definition* 2.1 *has the following properties:*

(i) *A_θ is self-adjoint and positive; the domain of its square root is given by*

$$\mathcal{D}\big(A_\theta^{1/2}\big) = \Big\{x \in L_2(0,1) \colon x \in \mathrm{AC_{loc}}((0,1]),\ x', \frac{x}{r} \in L_2(0,1)\Big\}, \quad \theta \in [0,\infty), \tag{3.1}$$

$$\mathcal{D}\big(A_\infty^{1/2}\big) = \Big\{x \in L_2(0,1) \colon x \in \mathrm{AC_{loc}}((0,1]),\ x', \frac{x}{r} \in L_2(0,1),\ x(1) = 0\Big\}. \tag{3.2}$$

(ii) *A_θ has compact resolvent, and the spectrum $\sigma(A_\theta)$ consists of a sequence of simple eigenvalues $0 < \lambda_1(\theta) < \lambda_2(\theta) < \cdots$ tending to ∞.*

(iii) *$\lambda_k(\theta)$ is strictly increasing in θ for every $k = 1,2,\dots$, and for different values θ_1, θ_2 the sequences $\big(\lambda_k(\theta_1)\big)_{k=1}^\infty$, $\big(\lambda_k(\theta_2)\big)_{k=1}^\infty$ interlace; in particular,*

$$0 < \lambda_1(\theta) < \lambda_1(\infty) < \lambda_2(\theta) < \lambda_2(\infty) < \cdots, \quad \theta \in [0,\infty). \tag{3.3}$$

(iv) *For* $0 \le \theta_1 \le \theta_2 \le \infty$ *and* $\lambda \ge 0$, *we have*

$$(A_{\theta_2} + \lambda)^{-1} \le (A_{\theta_1} + \lambda)^{-1}.$$

Proof. All claims of the proposition are well known; for the convenience of the reader, we repeat the main arguments.

(i) The Bessel differential expression τ (with $l \ge 1$) is in limit point case at the singular end-point 0 (see, e.g., [2, Appendix II, section 9.IV]); in fact, the $L_2(0,1)$-solutions of the differential equation $(\tau - \lambda)x = 0$ are spanned by the Riccati–Bessel function (see [1, equation 10.3.1])

$$f_l(r,\lambda) := r\sqrt{\lambda}\, j_l(r\sqrt{\lambda}) = \sqrt{\frac{\pi}{2}}\sqrt{r\sqrt{\lambda}}\, J_{l+1/2}(r\sqrt{\lambda}), \quad r \in [0,1]. \tag{3.4}$$

This implies that the operator A_θ as defined above is self-adjoint (see [27, Satz 13.21 a)]); moreover, the set

$$\mathcal{D}_0(A_\theta) := \{x \in \mathcal{D}(A_\theta) : \operatorname{supp} x \text{ compact in } (0,1]\} \tag{3.5}$$

forms a core of A_θ; that is, the closure of the restriction $A_\theta|_{\mathcal{D}_0(A_\theta)}$ coincides with A_θ (see [19, section 17.4]).

The positivity of A_θ follows from the identity

$$\mathfrak{a}_\theta[x] := \big(A_\theta x, x\big) =: \langle x, x\rangle_\theta, \quad x \in \mathcal{D}(A_\theta), \tag{3.6}$$

where

$$\langle x, y\rangle_\theta := c_\theta\, x(1)\overline{y(1)} + \int_0^1 x'(r)\overline{y'(r)}\,\mathrm{d}r + l(l+1)\int_0^1 \frac{x(r)\overline{y(r)}}{r^2}\,\mathrm{d}r \tag{3.7}$$

with $c_\theta = \theta$ for $\theta \in [0,\infty)$ and $c_\infty = 0$. The inner product $\langle x, y\rangle_\theta$ is defined for all $x, y \in L_2(0,1)$ such that x', y', x/r, $y/r \in L_2(0,1)$; in particular, it is defined on the sets on the right-hand sides of (3.1) and of (3.2).

The domain $\mathcal{D}(A_\theta^{1/2})$ is the domain $\mathcal{D}(\overline{\mathfrak{a}_\theta})$ of the form closure $\overline{\mathfrak{a}_\theta}$ of the quadratic form $\mathfrak{a}_\theta$ given by (3.6) (see [13, Theorems VI.2.23, VI.2.1, and Corollary VI.2.2]). Hence for every $x \in \mathcal{D}(A_\theta^{1/2})$ there exists a sequence $(x_n)_0^\infty \subset \mathcal{D}(A_\theta)$ such that $x_n \to x$ in $L^2(0,1)$, $n \to \infty$, and

$$\big(A_\theta(x_m - x_n), x_m - x_n\big) = \langle x_m - x_n, x_m - x_n\rangle_\theta \longrightarrow 0, \quad m, n \to \infty. \tag{3.8}$$

The relations (3.8), (3.7) yield that $x_m' - x_n' \to 0$, $x_m/r - x_n/r \to 0$ in $L_2(0,1)$ and, if $c_\theta \ne 0$, also $x_m(1) - x_n(1) \to 0$ for $m, n \to \infty$. Since convergence in $L_2(0,1)$ implies convergence almost everywhere, we can choose $r_0 \in (0,1]$ such that $x_n(r_0) \to x(r_0)$, $n \to \infty$; if $c_\theta \ne 0$, we can always choose $r_0 = 1$. Together with

$$x_n(r) - x_n(r_0) = \int_{r_0}^r x'(t)\,\mathrm{d}t, \quad r \in [0,1],$$

it readily follows that the sequence $(x_n)_0^\infty$ converges uniformly in $[0,1]$ to x, that x is absolutely continuous with $x' \in L_2(0,1)$, and that $x_n' \to x'$ in $L_2(0,1)$, $n \to \infty$. The relation (3.8) also shows that the functions x_n/r form a Cauchy sequence in $L_2(0,1)$ and hence also $x/r \in L_2(0,1)$. If $\theta = \infty$, then $x_n \in \mathcal{D}(A_\infty)$ implies that $x_n(1) = 0$,

$n \in \mathbb{N}$, and due to the uniform convergence of (x_n) it follows that $x(1) = 0$. Thus we have proved the inclusions "$\subset$" in (3.1) and (3.2).

In order to prove the converse inclusions "$\supset$" in (3.1), (3.2), we equip the set $\mathcal{H}_\theta$ on the right-hand side of (3.1) or (3.2), respectively, with the inner product $\langle\cdot,\cdot\rangle_\theta$ defined in (3.7). Then $\mathcal{H}_\theta$ becomes a Hilbert space and, by its definition, the subset $\mathcal{D}(\overline{\mathfrak{a}_\theta})$ is a closed subspace of $\mathcal{H}_\theta$. Now assume that $y_0 \in \mathcal{H}_\theta$ is orthogonal to $\mathcal{D}(\overline{\mathfrak{a}_\theta})$. Then, in particular, for every $x \in \mathcal{D}_0(A_\theta) \subset \mathcal{D}(A_\theta) = \mathcal{D}(\mathfrak{a}_\theta)$ with $\mathcal{D}_0(A_\theta)$ given by (3.5),

$$0 = \langle x, y_0\rangle_\theta = (A_\theta x, y_0). \tag{3.9}$$

Since $\mathcal{D}_0(A_\theta)$ is a core of A_θ, we conclude that $y_0 \in \mathcal{D}(A_\theta^*) = \mathcal{D}(A_\theta)$ and $A_\theta y_0 = 0$. The positivity of A_θ now implies $y_0 = 0$. This proves that $\mathcal{D}\big(A_\theta^{1/2}\big) = \mathcal{D}(\overline{\mathfrak{a}_\theta}) = \mathcal{H}_\theta$.

(ii) In [2, Appendix II, section 9.IV] it was proved that the spectrum of A_∞, and hence of every other operator A_θ (see [2, Appendix II, Theorem 6.2] or [27, Abschnitt 14.2]), is discrete or, equivalently, A_θ has compact resolvent; moreover, all eigenvalues are simple. In fact, if we choose a fundamental system $\{\varphi(\cdot,\lambda), \psi(\cdot,\lambda)\}$ of $(\tau - \lambda)x = 0$ such that

$$\begin{aligned} \varphi(1,\lambda) &= 0, & \psi(1,\lambda) &= 1,\\ \varphi'(1,\lambda) &= -1, & \psi'(1,\lambda) &= 0, \end{aligned}$$

where $'$ denotes the derivative with respect to the first variable r, then the corresponding Weyl–Titchmarsh function m_∞ is given by (see [25, equation (4.8.2)])

$$m_\infty(\lambda) = -\sqrt{\lambda}\,\frac{J'_{l+1/2}(\sqrt{\lambda})}{J_{l+1/2}(\sqrt{\lambda})} - \frac{1}{2} = -\sqrt{\lambda}\,\frac{f_l'(1,\lambda)}{f_l(1,\lambda)}, \quad \lambda \in \mathbb{C}.^1$$

The Weyl–Titchmarsh function m_θ for $\theta \in [0,\infty)$ can be written as

$$m_\theta(\lambda) = \frac{1 + \theta\, m_\infty(\lambda)}{\theta - m_\infty(\lambda)}, \quad \lambda \in \mathbb{C}$$

(see [27, Lemma 14.10]). Since the eigenvalues $\lambda_k(\theta)$ of A_θ are the poles of the Weyl–Titchmarsh function m_θ, $\lambda_k(\infty)$ is the square of the kth nonzero zero of the Bessel function $J_{l+1/2}$, while $\lambda_k(\theta)$ is the kth nonzero zero of the function $m_\infty - \theta$ for $\theta \in [0,\infty)$. For all $\theta \in [0,\infty]$, the eigenspace of A_θ at an eigenvalue $\lambda_k(\theta)$ is spanned by the function

$$x_{k,\theta}(r) = f_l\big(r\sqrt{\lambda_k(\theta)}\big) = \sqrt{\frac{\pi}{2}}\sqrt{r\sqrt{\lambda_k(\theta)}}\,J_{l+1/2}\big(r\sqrt{\lambda_k(\theta)}\big), \quad r \in (0,1].$$

(iii) The function m_∞ is a Nevanlinna function; that is, it is analytic in the upper half-plane $\mathbb{C}^+$, maps $\mathbb{C}^+$ into $\mathbb{C}^+ \cup \mathbb{R}$, and is complex symmetric with respect to $\mathbb{R}$ (see, e.g., [2, Appendix II, sections 9.4 and VI.59, Theorem 2]); in particular, m_∞ is real-valued on $\mathbb{R}$. Moreover, m_∞ is meromorphic, its singularities are the simple poles $\lambda_k(\infty)$, and it is strictly increasing between two neighboring poles. This and the property that $\lambda_k(\theta)$ is the kth zero of $m_\infty - \theta$ for $\theta \in [0,\infty)$ imply both claims of (iii).

[1] Note that, in the first edition of [25] from 1946, the term $-1/2$ was missing.

(iv) The claimed inequality is an immediate consequence of Kreĭn's resolvent formula (see [16], [2, section VIII.106], or [8, section 9]). ∎

Remark 3.2. By Proposition 3.1 (i), for $\theta=\infty$ the boundary condition $x(1)=0$ carries over from $\mathcal{D}(A_\infty)$ to $\mathcal{D}(A_\infty^{1/2})$; it is called an *essential boundary condition* (cf. [17, section 7]). For $\theta\in[0,\infty)$ the boundary condition $x'(1)+\theta\, x(1)=0$ in $\mathcal{D}(A_\theta)$ disappears in $\mathcal{D}(A_\theta^{1/2})$; it is called a *nonessential* or *natural boundary condition* (see [5, section 10.5, p. 234])). As a consequence, $\mathcal{D}(A_\theta^{1/2})$ does not depend on θ for $\theta\in[0,\infty)$, whereas the action of $A_\theta^{1/2}$ does.

In order to study the problems (1.1), (1.2) and (1.1), (1.3), we shall need, in particular, the eigenvalues of the operators A_∞ and A_l.

LEMMA 3.3. *The eigenvalues $\lambda_k(\theta)$, $k=1,2,\ldots$, for $\theta=l$ and $\theta=\infty$ are as follows:*

(i) $\lambda_k(\infty)$ *is the kth nonzero zero of the function $\lambda\mapsto J_{l+1/2}(\sqrt{\lambda})$;*

(ii) $\lambda_k(l)$ *is the kth nonzero zero of the function $\lambda\mapsto J_{l-1/2}(\sqrt{\lambda})$*

for $k=1,2,\ldots$.

Proof. For $\theta=\infty$, the boundary condition $x_l(1)=0$ implies that $\lambda\in\mathbb{C}$ is an eigenvalue of A_∞ if and only if $J_{l+1/2}(\sqrt{\lambda})=0$. For $\theta=l$, the boundary condition $x_l'(1)+lx_l(1)=0$ implies that $\lambda\in\mathbb{C}$ is an eigenvalue of A_l if and only if

$$0=\frac{\mathrm{d}}{\mathrm{d}r}\big(\sqrt{r}\,J_{l+1/2}(r\sqrt{\lambda})\big)+l\,\sqrt{r}\,J_{l+1/2}(r\sqrt{\lambda})\Big|_{r=1}=\sqrt{\lambda}\,J_{l-1/2}(\sqrt{\lambda}); \tag{3.10}$$

here we have used the differentiation formula $\big(z^l\cdot z\,j_l(z)\big)'=z^{l+1}j_{l-1}(z)$ for the spherical Bessel functions j_l (see [1, equation 10.1.23]). ∎

Remark 3.4. Note that the index θ in A_θ and its eigenvalues $\lambda_k(\theta)$ refers only to the boundary condition $x'(1)+\theta x(1)=0$. Since the differential expression $\tau=-\partial_r^2+l(l+1)/r^2$ defining A_θ always depends on l, so do all eigenvalues $\lambda_k(\theta)$ (compare Lemma 3.3). As $l\in\mathbb{N}$ is assumed to be fixed throughout the paper, we do not indicate this dependence in general.

For the differential expression τ_α in (2.1), we use the relation

$$\tau_\alpha=\alpha\,\tau-\alpha'\partial_r,$$

thus relating the operator $A_{\theta,\alpha}$ induced by τ_α to the operator A_θ induced by τ.

LEMMA 3.5. *Let $\theta\in[0,\infty]$ and let D be the operator of differentiation in $L_2(0,1)$,*

$$\mathcal{D}(D):=W_2^1(0,1),\quad Dx:=x'.$$

Then the operators $DA_\theta^{-1/2}$ and DA_θ^{-1} are defined on $L_2(0,1)$ and bounded with

$$\|DA_\theta^{-1/2}\|\le 1,\quad \big\|DA_\theta^{-1}\big\|\le\frac{1}{\sqrt{\lambda_1(\theta)}}. \tag{3.11}$$

Proof. Proposition 3.1(i) and its proof yield that $\mathcal{D}(A_\theta^{1/2})=\mathcal{D}\big(\overline{\mathfrak{a}_\theta}\big)\subset\mathcal{D}(D)$ and

$$\|A_\theta^{1/2}x\|^2=\overline{\mathfrak{a}_\theta}[x]\ge\int_0^1\big|x'(r)\big|^2\,\mathrm{d}r=\|Dx\|^2,\quad x\in\mathcal{D}(A_\theta^{1/2}),$$

that is, $\|DA_\theta^{-1/2}\|\le 1$. The claims for DA_θ^{-1} now follow from the identity $DA_\theta^{-1}=DA_\theta^{-1/2}A_\theta^{-1/2}$ and from the estimate $\|A_\theta^{-1/2}\|\le 1/\sqrt{\lambda_1(\theta)}$. ∎

PROPOSITION 3.6. *Let $\theta \in [0,\infty]$. The linear operator $A_{\theta,\alpha}$ in Definition 2.3 is densely defined, symmetric and hence closable, and it satisfies*

$$A_{\theta,\alpha} = \alpha A_\theta - \alpha' D. \tag{3.12}$$

Proof. First we have to show that $A_{\theta,\alpha}$ is well-defined, that is, $\alpha x' \in \mathrm{AC}_{\mathrm{loc}}((0,1])$ and $\tau_\alpha x \in L_2(0,1)$ for $x \in \mathcal{D}(A_\theta)$. To this end, let $x \in \mathcal{D}(A_\theta)$. By definition, this implies $x' \in \mathrm{AC}_{\mathrm{loc}}((0,1])$ and $\tau x \in L_2(0,1)$. Since $\alpha \in C^1([0,1])$, it follows that $\alpha x' \in \mathrm{AC}_{\mathrm{loc}}((0,1])$. Further, Lemma 3.5 shows that $\mathcal{D}(A_\theta) \subset \mathcal{D}(D)$ and hence $x' \in L_2(0,1)$. Together with $\alpha, \alpha' \in C([0,1])$, we obtain

$$\tau_\alpha x = \alpha \tau x - \alpha' x' \in L_2(0,1),$$

and also the operator identity (3.12). For the symmetry of $A_{\theta,\alpha}$, it suffices to show that $(A_{\theta,\alpha}x, x) \in \mathbb{R}$ for all $x \in \mathcal{D}(A_{\theta,\alpha}) = \mathcal{D}(A_\theta)$. Using the boundary condition at $r = 1$, we see that

$$\begin{aligned}(A_{\theta,\alpha}x, x) = \alpha(0) \lim_{\varepsilon\to 0} \big(x'(\varepsilon)\,\overline{x(\varepsilon)}\big) + c_\theta\, \alpha(1)\, |x(1)|^2 \\ + \int_0^1 \alpha(r)\big|x'(r)\big|^2\,\mathrm{d}r + \int_0^1 \alpha(r)\frac{l(l+1)}{r^2}|x(r)|^2\,\mathrm{d}r,\end{aligned}$$

where $c_\theta = \theta$ for $\theta \in [0,\infty)$ and $c_\infty = 0$. Since $x \in \mathcal{D}(A_\theta)$ and since the differential expression τ defining A_θ is in the limit point case at 0, the limit in the first term on the right-hand side is real (see [27, Satz 13.19]). Because α is real-valued, it follows that $(A_{\theta,\alpha}x, x) \in \mathbb{R}$. ∎

We close this section with some resolvent estimates for the self-adjoint operators A_θ, which will be used in the following sections.

LEMMA 3.7. *For $\theta \in [0,\infty]$ and $\lambda \notin \big(-\infty, -\lambda_1(\theta)\big]$, we have the norm estimates*

$$\big\|(A_\theta+\lambda)^{-1}\big\| \begin{cases} = \dfrac{1}{|\lambda_1(\theta)+\lambda|} & \textit{if } \operatorname{Re}\lambda \ge -\lambda_1(\theta), \quad \text{(i)} \\ \le \dfrac{1}{|\operatorname{Im}\lambda|} & \textit{if } \operatorname{Re}\lambda \le -\lambda_1(\theta), \quad \text{(ii)} \end{cases}$$

$$\big\|A_\theta^{1/2}(A_\theta+\lambda)^{-1}\big\| \begin{cases} = \dfrac{\sqrt{\lambda_1(\theta)}}{|\lambda_1(\theta)+\lambda|} & \textit{if } |\lambda| \le \lambda_1(\theta), \quad \text{(a)} \\ \le \dfrac{\sqrt{|\lambda|}}{\big||\lambda|+\lambda\big|} & \textit{if } |\lambda| \ge \lambda_1(\theta), \quad \text{(b)} \end{cases}$$

$$\big\|A_\theta(A_\theta+\lambda)^{-1}\big\| \begin{cases} = 1 & \textit{if } \operatorname{Re}\lambda \ge 0, \quad (1) \\ \le \dfrac{|\lambda|}{|\operatorname{Im}\lambda|} & \textit{if } \operatorname{Re}\lambda \le 0,\ |\lambda+\lambda_1(\theta)/2| \ge \lambda_1(\theta)/2, \quad (2) \\ = \dfrac{\lambda_1(\theta)}{|\lambda_1(\theta)+\lambda|} & \textit{if } |\lambda+\lambda_1(\theta)/2| \le \lambda_1(\theta)/2; \quad (3) \end{cases}$$

in each of the three cases, the bounds on the right-hand side define a continuous function of λ on $\mathbb{C} \setminus \big(-\infty, -\lambda_1(\theta)\big]$.

Proof. The resolvent of the self-adjoint operator A_θ satisfies the well-known relation (see, e.g., [13, section V.3.5])

$$\big\|A_\theta^s(A_\theta+\lambda)^{-1}\big\| = \sup_{t\in\sigma(A_\theta)} \frac{|t|^s}{|t+\lambda|}, \quad s \in [0,1], \quad \lambda \notin \sigma(A_\theta). \tag{3.13}$$

For $s = 0$, the right-hand side of (3.13) equals $1/\operatorname{dist}(-\lambda, \sigma(A_\theta))$ and, together with the inclusion $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$, the equality (i) and the estimate (ii) follow.

For $s = 1/2$, it is not difficult to see that the function $f(t) := \sqrt{t}/|t+\lambda|$, $t \in [0,\infty)$, has a local maximum at $t_\lambda = |\lambda|$. Hence, if $|\lambda| \le \lambda_1(\theta)$, then the function f restricted to $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$ attains its maximum at $t = \lambda_1(\theta) \in \sigma(A_\theta)$ and the equality (a) follows; if $|\lambda| > \lambda_1(\theta)$, then f restricted to $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$ is estimated by its maximum on $[\lambda_1(\theta), \infty)$ attained at $t = |\lambda|$ and (b) follows.

For $s = 1$, a short calculation yields that the function $g(t) := t/|t+\lambda|$, $t \in [0,\infty)$, is monotonically increasing if $\operatorname{Re}\lambda \ge 0$ and has a local maximum at $t_\lambda = |\lambda|^2/|\operatorname{Re}\lambda|$ if $\operatorname{Re}\lambda < 0$. Thus, if $\operatorname{Re}\lambda \ge 0$, then the function g restricted to $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$ attains its maximum at ∞ and (1) follows. If $\operatorname{Re}\lambda < 0$, we note that

$$t_\lambda = \frac{|\lambda|^2}{|\operatorname{Re}\lambda|} \le \lambda_1(\theta) \iff |\lambda|^2 + \operatorname{Re}\lambda \cdot \lambda_1(\theta) \le 0 \iff |\lambda + \lambda_1(\theta)/2| \le \frac{\lambda_1(\theta)}{2},$$

and therefore in this case the function g restricted to $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$ attains its maximum at $t = \lambda_1(\theta) \in \sigma(A_\theta)$ and the equality (3) follows; if $|\lambda + \lambda_1(\theta)/2| > \lambda_1(\theta)/2$, then g restricted to $\sigma(A_\theta) \subset [\lambda_1(\theta), \infty)$ is estimated by its maximum on $[\lambda_1(\theta), \infty)$ attained at $t_\lambda = |\lambda|^2/|\operatorname{Re}\lambda|$. Now another short calculation shows that $g(t_\lambda)$ coincides with the upper bound in (2). ∎

4. The spectrum of $\mathcal{A}_\theta$ and a first eigenvalue estimate. In the following we present a first perturbational approach to studying the spectral properties of the operator $\mathcal{A}_\theta$ associated with the dynamo problem. To this end, we regard the block operator matrix $\mathcal{A}_\theta$ as a bounded perturbation of its lower triangular part. We use this decomposition to show that $\mathcal{A}_\theta$ is a closed operator with compact resolvent and to establish estimates for its eigenvalues.

THEOREM 4.1. *Let $\theta \in [0, \infty]$ and define*

$$\mathcal{Q}_\theta := \begin{pmatrix} -A_\theta & 0 \\ A_{\theta,\alpha} & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{Q}_\theta) = \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty), \tag{4.1}$$

$$\mathcal{R} := \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{D}(\mathcal{R}) = L_2(0,1) \oplus L_2(0,1). \tag{4.2}$$

Then the block operator matrix

$$\mathcal{A}_\theta = \mathcal{Q}_\theta + \mathcal{R} \tag{4.3}$$

is closed and has compact resolvent; its spectrum is symmetric to $\mathbb{R}$ and consists of isolated eigenvalues of finite algebraic multiplicities with no finite accumulation point.

Proof. Since $\alpha \in C([0,1])$, the operator $\mathcal{R}$ is everywhere defined and bounded. Moreover, $\mathcal{D}(\mathcal{Q}_\theta) = \mathcal{D}(\mathcal{A}_\theta)$ and hence the operator identity (4.3) holds.

Next we show that

$$\rho(\mathcal{Q}_\theta) = \rho(-A_\theta) \cap \rho(-A_\infty) \ne \emptyset, \tag{4.4}$$

which implies the closedness of $\mathcal{Q}_\theta$, and that $(\mathcal{Q}_\theta - \lambda)^{-1}$ is compact for all $\lambda \in \rho(Q_\theta)$. By (3.12), we have

$$\begin{aligned} A_{\theta,\alpha}(A_\theta + \lambda)^{-1} &= \big(\alpha(A_\theta + \lambda) - \alpha\lambda - \alpha' D\big)(A_\theta + \lambda)^{-1} \\ &= \alpha - \big(\alpha\lambda + \alpha' D\big)(A_\theta + \lambda)^{-1}. \end{aligned} \tag{4.5}$$

Since α, $\alpha' \in C([0,1])$ and $D(A_\theta+\lambda)^{-1}$ is bounded for $\lambda \in \rho(-A_\theta)$ by Lemma 3.5, the operator $A_{\theta,\alpha}(A_\theta+\lambda)^{-1}$ is bounded for every $\lambda \in \rho(-A_\theta)$. Therefore, for $\lambda \in \rho(-A_\theta) \cap \rho(-A_\infty)$, the inverse

$$(4.6)\qquad \big(\mathcal{Q}_\theta - \lambda\big)^{-1} = \begin{pmatrix} -(A_\theta+\lambda)^{-1} & 0 \\ -(A_\infty+\lambda)^{-1}A_{\theta,\alpha}(A_\theta+\lambda)^{-1} & -(A_\infty+\lambda)^{-1} \end{pmatrix}$$

is bounded and everywhere defined. This proves the inclusion "$\supset$" in (4.4). Conversely, assume that $\lambda \in \sigma(-A_\theta) \cup \sigma(-A_\infty) = \sigma_{\rm p}(-A_\theta) \cup \sigma_{\rm p}(-A_\infty)$. If $\lambda \in \sigma_{\rm p}(-A_\infty)$, then there is an element $y_2 \in \ker(-A_\infty - \lambda) \setminus \{0\}$. Then $(0, y_2)^{\rm t} \in \ker(\mathcal{Q}_\theta - \lambda) \setminus \{0\}$ and hence $\lambda \in \sigma_{\rm p}(\mathcal{Q}_\theta) \subset \sigma(\mathcal{Q}_\theta)$. If $\theta \in [0,\infty)$ and $\lambda \in \sigma_{\rm p}(-A_\theta)$, then there is an element $y_1 \in \ker(-A_\theta - \lambda) \setminus \{0\}$. Since $\theta \neq \infty$, we have $\lambda \notin \sigma(-A_\infty)$ by Proposition 3.1(iii) and so $(y_1, (A_\infty+\lambda)^{-1}A_{\theta,\alpha}y_1)^{\rm t} \in \ker(\mathcal{Q}_\theta - \lambda) \setminus \{0\}$, that is, $\lambda \in \sigma_{\rm p}(\mathcal{Q}_\theta) \subset \sigma(\mathcal{Q}_\theta)$. This proves (4.4).

By Proposition 3.1(ii) the inverses $(A_\theta+\lambda)^{-1}$ are compact and we have shown that $A_{\theta,\alpha}(A_\theta+\lambda)^{-1}$ is bounded for all $\lambda \in \rho(-A_\theta)$ and $\theta \in [0,\infty]$. Thus, by (4.6), the inverses $(\mathcal{Q}_\theta - \lambda)^{-1}$ are compact as well.

Since $\mathcal{A}_\theta$ is a bounded perturbation of $\mathcal{Q}_\theta$, it is immediate that $\mathcal{A}_\theta$ is closed. In order to show that $\mathcal{A}_\theta$ has compact resolvent, it suffices to show that $(\mathcal{A}_\theta - \lambda)^{-1}$ exists and is compact for some $\lambda \in \mathbb{C}$ (see, e.g., [13, Theorems IV.1.1 and III.6.29]). For $\lambda \geq 0$, we can write

$$(4.7)\qquad \mathcal{A}_\theta - \lambda = \big(I + \mathcal{R}\big(\mathcal{Q}_\theta - \lambda\big)^{-1}\big)\big(\mathcal{Q}_\theta - \lambda\big).$$

By Lemmas 3.5 and 3.7(i), (1), we have the uniform estimates

$$\begin{aligned} \big\|\lambda(A_\theta+\lambda)^{-1}\big\| &= \frac{\lambda}{\lambda+\lambda_1(\theta)} \leq 1, \\ \big\|D(A_\theta+\lambda)^{-1}\big\| &= \big\|DA_\theta^{-1}\big\|\big\|A_\theta(A_\theta+\lambda)^{-1}\big\| \leq \frac{1}{\sqrt{\lambda_1(\theta)}}, \end{aligned} \qquad \lambda \geq 0.$$

Together with (4.5), it follows that $A_{\theta,\alpha}(A_\theta+\lambda)^{-1}$ is uniformly bounded for $\lambda \geq 0$. Thus, (4.6) and the fact that $\|(A_\theta+\lambda)^{-1}\| \to 0$ for $\lambda \to \infty$ and $\theta \in [0,\infty]$ show that $\|(\mathcal{Q}_\theta - \lambda)^{-1}\| \to 0$ for $\lambda \to \infty$ and $\theta \in [0,\infty]$. As a consequence, we can choose $\lambda_0 > 0$ sufficiently large such that $\|\mathcal{R}(\mathcal{Q}_\theta - \lambda)^{-1}\| < 1$ for $\lambda \geq \lambda_0$ and so

$$\big(\mathcal{A}_\theta - \lambda\big)^{-1} = \big(\mathcal{Q}_\theta + \mathcal{R} - \lambda\big)^{-1} = \big(\mathcal{Q}_\theta - \lambda\big)^{-1}\big(I + \mathcal{R}\big(\mathcal{Q}_\theta - \lambda\big)^{-1}\big)^{-1}$$

is compact for $\lambda \geq \lambda_0$.

The symmetry of the (point) spectrum of $\mathcal{A}_\theta$ is evident since the entries of the block operator matrix $\mathcal{A}_\theta$ are differential operators with real coefficients. □

Although $\mathcal{A}_\theta$ is a bounded perturbation of $\mathcal{Q}_\theta$, the norm of the perturbation being $\|\alpha\|$, we cannot conclude that $\sigma(\mathcal{A}_\theta)$ lies in an $\|\alpha\|$-neighborhood of $\sigma(\mathcal{Q}_\theta) = \sigma(-A_\theta) \cup \sigma(-A_\infty)$ since $\mathcal{Q}_\theta$ is neither self-adjoint nor normal. Therefore, in the next proposition, we use a Neumann series argument to exclude points $\lambda \in \mathbb{C}$ from the (point) spectrum of $\mathcal{A}_\theta$.

PROPOSITION 4.2. *Let $\theta \in [0,\infty]$. If $\lambda \notin \sigma(-A_\theta) \cup \sigma(-A_\infty)$ and*

$$(4.8)\qquad \big\|\alpha(A_\infty+\lambda)^{-1}(\alpha A_\theta - \alpha' D)(A_\theta+\lambda)^{-1}\big\| < 1,$$

then $\lambda \in \rho(\mathcal{A}_\theta)$.

Proof. By (4.7), we have $\lambda \in \rho(\mathcal{A}_\theta)$ if and only if $\lambda \in \rho(\mathcal{Q}_\theta) = \rho(-A_\theta) \cap \rho(-A_\infty)$ and $I + \mathcal{R}(\mathcal{Q}_\theta - \lambda)^{-1}$ is boundedly invertible. By (4.6), the operator

$$I + \mathcal{R}(\mathcal{Q}_\theta - \lambda)^{-1} = I + \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -(A_\theta + \lambda)^{-1} & 0 \\ -(A_\infty + \lambda)^{-1} A_{\theta,\alpha} (A_\theta + \lambda)^{-1} & -(A_\infty + \lambda)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} I - \alpha (A_\infty + \lambda)^{-1} A_{\theta,\alpha} (A_\theta + \lambda)^{-1} & -\alpha (A_\infty + \lambda)^{-1} \\ 0 & I \end{pmatrix}$$

is boundedly invertible if and only if the operator $I - \alpha(A_\infty + \lambda)^{-1} A_{\theta,\alpha} (A_\theta + \lambda)^{-1}$ is boundedly invertible. The latter holds if condition (4.8) is satisfied. ∎

Remark 4.3. The block operator matrix $\mathcal{A}_\theta$ can also be decomposed as

$$\mathcal{A}_\theta := \begin{pmatrix} -A_\theta & 0 \\ 0 & -A_\infty \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ A_{\theta,\alpha} & 0 \end{pmatrix}.$$

Here the first term is self-adjoint, while the second term is only relatively bounded with respect to the first one. A corresponding Neumann series argument leads to the same condition (4.8) for a point to be in the resolvent set.

In the following we use Proposition 4.2 to obtain an enclosure for the eigenvalues of $\mathcal{A}_\theta$. To this end, we estimate the norm on the left-hand side of (4.8) by means of Lemma 3.7. According to the different resolvent estimates therein, we decompose the complex plane $\mathbb{C}$ as

$$\mathbb{C} = Z_0 \cup Z_1 \cup Z_2 \cup Z_3 \cup Z_4 \cup Z_5 \cup Z_6 \tag{4.9}$$

into the pairwise disjoint sets

$$Z_0 := \Big\{ z \in \mathbb{C} : \operatorname{Re} z \le -\lambda_1(\theta),\ \operatorname{Im} z = 0 \Big\} = \big(-\infty, -\lambda_1(\theta) \big],$$
$$Z_1 := \Big\{ z \in \mathbb{C} : \operatorname{Re} z \le -\lambda_1(\infty),\ \operatorname{Im} z \ne 0 \Big\},$$
$$Z_2 := \Big\{ z \in \mathbb{C} :\ -\lambda_1(\infty) < \operatorname{Re} z \le 0,\ \operatorname{Im} z \ne 0,\ |z| > \lambda_1(\theta) \Big\},$$
$$Z_3 := \Big\{ z \in \mathbb{C} :\ -\lambda_1(\theta) < \operatorname{Re} z \le 0,\ |z + \lambda_1(\theta)/2| > \lambda_1(\theta)/2,\ |z| \le \lambda_1(\theta) \Big\},$$
$$Z_4 := \Big\{ z \in \mathbb{C} :\ -\lambda_1(\theta) < \operatorname{Re} z \le 0,\ |z + \lambda_1(\theta)/2| \le \lambda_1(\theta)/2 \Big\},$$
$$Z_5 := \Big\{ z \in \mathbb{C} : \operatorname{Re} z > 0,\ |z| \le \lambda_1(\theta) \Big\},$$
$$Z_6 := \Big\{ z \in \mathbb{C} : \operatorname{Re} z > 0,\ |z| > \lambda_1(\theta) \Big\}.$$

The sets Z_i are shown in Figure 1 for $\theta = l = 1$, where $\lambda_1(\theta) = \pi^2 \approx 9.87$, $\lambda_1(\infty) \approx 20.19$.

PROPOSITION 4.4. *Let* $\theta \in [0, \infty]$. *Then*

$$\sigma(\mathcal{A}_\theta) \subset \Sigma_\theta := \big(-\infty, -\lambda_1(\theta) \big] \cup \big\{ \lambda \in \mathbb{C} : f(\lambda) \ge 1 \big\}, \tag{4.10}$$

where the function $f : \mathbb{C} \setminus (-\infty, -\lambda_1(\theta)] \to [0, \infty)$ *is defined as*

$$f(\lambda) := f_j(\lambda), \quad \lambda \in Z_j,\ j = 1, 2, \ldots, 6, \tag{4.11}$$

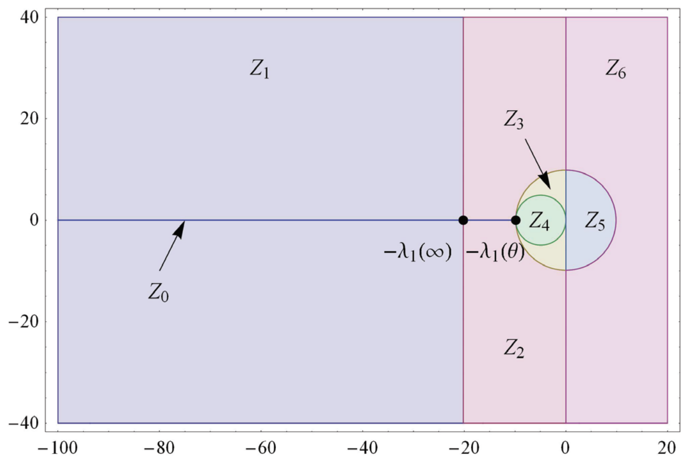

FIG. 1. *Decomposition* (4.9) *of the complex plane for* $l = 1$.

with

$$f_1(\lambda) := \left(\|\alpha\|^2 \frac{|\lambda|}{|\operatorname{Im}\lambda|} + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{|\lambda|}}{|\lambda + |\lambda||} \right) \frac{1}{|\operatorname{Im}\lambda|}, \qquad \lambda \in Z_1,$$

$$f_2(\lambda) := \left(\|\alpha\|^2 \frac{|\lambda|}{|\operatorname{Im}\lambda|} + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{|\lambda|}}{|\lambda + |\lambda||} \right) \frac{1}{|\lambda + \lambda_1(\infty)|}, \qquad \lambda \in Z_2,$$

$$f_3(\lambda) := \left(\|\alpha\|^2 \frac{|\lambda|}{|\operatorname{Im}\lambda|} + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{\lambda_1(\theta)}}{|\lambda + \lambda_1(\theta)|} \right) \frac{1}{|\lambda + \lambda_1(\infty)|}, \qquad \lambda \in Z_3,$$

$$f_4(\lambda) := \left(\|\alpha\|^2 \frac{\lambda_1(\theta)}{|\lambda + \lambda_1(\theta)|} + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{\lambda_1(\theta)}}{|\lambda + \lambda_1(\theta)|} \right) \frac{1}{|\lambda + \lambda_1(\infty)|}, \qquad \lambda \in Z_4,$$

$$f_5(\lambda) := \left(\|\alpha\|^2 + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{\lambda_1(\theta)}}{|\lambda + \lambda_1(\theta)|} \right) \frac{1}{|\lambda + \lambda_1(\infty)|}, \qquad \lambda \in Z_5,$$

$$f_6(\lambda) := \left(\|\alpha\|^2 + \|\alpha\|\,\|\alpha'\| \frac{\sqrt{|\lambda|}}{|\lambda + |\lambda||} \right) \frac{1}{|\lambda + \lambda_1(\infty)|}, \qquad \lambda \in Z_6.$$

The set Σ_θ *is symmetric to* $\mathbb{R}$ *and bounded from the right with*

$$a_\theta := \max \operatorname{Re} \Sigma_\theta \geq -\lambda_1(\theta), \tag{4.12}$$

but it is neither bounded from above nor from below in the complex plane $\mathbb{C}$.

Proof. By Lemma 3.5, we have $\left\| D A_\theta^{-1/2} \right\| \leq 1$. Hence condition (4.8) in Proposition 4.2 is clearly satisfied for $\lambda \notin (-\infty, -\lambda_1(\theta)]$ if

$$\left\|(A_\infty + \lambda)^{-1}\right\| \left(\|\alpha\|^2 \left\|A_\theta (A_\theta + \lambda)^{-1}\right\| + \|\alpha\|\,\|\alpha'\| \left\|A_\theta^{1/2} (A_\theta + \lambda)^{-1}\right\| \right) < 1. \tag{4.13}$$

Now we obtain the spectral inclusion (4.10) from Proposition 4.2 by combining the various resolvent estimates in Lemma 3.7, e.g., the estimates (ii), (b), and (2) for Z_1.

The symmetry of Σ_θ follows since $\lambda_1(\theta)$, $\lambda_1(\infty) \in \mathbb{R}$ and hence $f_j(\lambda) = f_j(\overline{\lambda})$, $\lambda \notin \mathbb{C} \setminus (-\infty, -\lambda_1(\theta)]$. Because $f_6(\lambda) \to 0$ for $\operatorname{Re}\lambda \to \infty$, the set Σ_θ is bounded from the right. Inequality (4.12) is immediate from the inclusion $\big(-\infty, -\lambda_1(\theta)\big] \subset \Sigma_\theta$. If Σ_θ were bounded from above or from below, there would exist an $M_0 \geq 0$ such that $f_1(\lambda) < 1$ for all $\lambda \in \mathbb{C}$ with $|\operatorname{Im}\lambda| > M_0$ and $\operatorname{Re}\lambda \leq -\lambda_1(\infty)$; on the other hand,

$$f_1(\lambda) \geq \|\alpha\|^2 \frac{|\lambda|}{|\operatorname{Im}\lambda|^2} \geq \|\alpha\|^2 \frac{|\operatorname{Re}\lambda|}{M_0^2} \longrightarrow \infty, \quad \operatorname{Re}\lambda \to -\infty,$$

a contradiction. ∎

In what follows we describe the shape and the location of the set Σ_θ in dependence of $\|\alpha\|$ and $\|\alpha'\|$ for $\alpha \not\equiv 0$; if $\alpha \equiv 0$, then clearly $\Sigma_\theta = \big(-\infty, -\lambda_1(\theta)\big]$.

Lemma 4.5. *Let f be given by (4.11). Then $\varphi : \mathbb{R}^2 \setminus \big((-\infty, -\lambda_1(\theta)] \times \{0\}\big) \to [0,\infty)$ defined by*

$$\varphi(\xi,\eta) := f(\lambda), \quad \xi + \mathrm{i}\eta = \lambda \in \mathbb{C} \setminus (-\infty, -\lambda_1(\theta)],$$

is continuous and continuously differentiable on $\mathbb{R}^2 \setminus \big((-\infty, -\lambda_1(\theta)] \times \{0\}\big)$ with

$$\frac{\partial\varphi}{\partial\xi}(\xi,\eta) < 0, \quad \frac{\partial\varphi}{\partial\eta}(\xi,\eta) < 0, \quad \xi \in \mathbb{R},\ \eta > 0, \tag{4.14}$$

and

$$\frac{\partial\varphi}{\partial\xi}(\xi,0) < 0, \quad \frac{\partial\varphi}{\partial\eta}(\xi,0) = 0, \quad \xi \in \mathbb{R} \setminus (-\infty, -\lambda_1(\theta)]. \tag{4.15}$$

Proof. According to the definition of f in (4.11), the function φ is given by

$$\varphi(\xi,\eta) = \varphi_j(\xi,\eta) := f_j(\lambda), \quad \lambda = \xi + \mathrm{i}\eta \in Z_j, \quad j = 1,2,\dots,6.$$

Hence φ is continuous since the functions φ_j are continuous on their domains Z_j and, by the last claim in Lemma 3.7, the continuous extensions of two functions φ_i and φ_j coincide on common boundary points of Z_i and Z_j.

It is elementary, but very tedious, to calculate the partial derivatives of the functions φ_j on Z_j and to show that the first partial derivatives of φ_i and φ_j coincide on common boundary points of Z_i and Z_j. This proves that φ has continuous first partial derivatives on $\mathbb{R}^2 \setminus \big((-\infty, -\lambda_1(\theta)] \times \{0\}\big)$ and is hence totally differentiable there. These calculations also show that (4.14) and (4.15) hold. ∎

In the next theorem the boundary of the set Σ_θ is described by a function h_θ; in particular, we derive formulas for the rightmost point a_θ of Σ_θ (see (4.12)).

Theorem 4.6. *Let $\theta \in [0,\infty]$ and $\alpha \not\equiv 0$. Then there exists a continuous strictly decreasing function $h_\theta \colon (-\infty, a_\theta] \to [0,\infty)$ with*

$$\lim_{t\to-\infty} h_\theta(t) = \infty, \quad h_\theta(a_\theta) = 0, \quad \lim_{t\nearrow a_\theta} h'_\theta(a_\theta) = -\infty \tag{4.16}$$

such that

$$\sigma(\mathcal{A}_\theta) \subset \Sigma_\theta = \big\{\lambda \in \mathbb{C} \colon \operatorname{Re}\lambda \leq a_\theta,\ |\operatorname{Im}\lambda| \leq h_\theta(\operatorname{Re}\lambda)\big\}. \tag{4.17}$$

Depending on $\|\alpha\|$ and $\|\alpha'\|$, the following cases occur:

(i) $0 < \|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\theta)}} \leq \lambda_1(\infty)$: *Then*

$$-\lambda_1(\theta) < a_\theta \leq 0$$

and a_θ is given by

$$a_\theta = -\frac{\lambda_1(\infty)+\lambda_1(\theta)}{2} + \sqrt{\left(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\left(\|\alpha\|^2 + \frac{\|\alpha\|\|\alpha'\|}{\sqrt{\lambda_1(\theta)}}\right)}.$$

(ii) $\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\theta)}} > \lambda_1(\infty)$ *and* $\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{2\sqrt{\lambda_1(\theta)}} \le \lambda_1(\infty) + \lambda_1(\theta)$: *Then*

$$0 < a_\theta \le \lambda_1(\theta)$$

and a_θ is given by

$$a_\theta = -\frac{\lambda_1(\infty)+\lambda_1(\theta)-\|\alpha\|^2}{2} + \sqrt{\left(\frac{\lambda_1(\infty)-\lambda_1(\theta)-\|\alpha\|^2}{2}\right)^2 + \|\alpha\|\|\alpha'\|\sqrt{\lambda_1(\theta)}};$$

moreover, Σ_θ contains the disc $Z_4 = \{z \in \mathbb{C} : |z + \lambda_1(\theta)/2| \le \lambda_1(\theta)/2\}$.

(iii) $\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{2\sqrt{\lambda_1(\theta)}} > \lambda_1(\infty) + \lambda_1(\theta)$: *Then*

$$\lambda_1(\theta) < a_\theta$$

and a_θ is the (unique) solution of the equation

$$2\sqrt{\lambda}\left(\lambda + \lambda_1(\infty) - \|\alpha\|^2\right) = \|\alpha\|\,\|\alpha'\| \quad in \quad (\lambda_1(\theta), \infty); \tag{4.18}$$

moreover, Σ_θ contains the disc $Z_3 \cup Z_4 \cup Z_5 = \{z \in \mathbb{C} : |z| \le \lambda_1(\theta)\}$.

In Figure 2 the boundary of the set Σ_θ containing the spectrum of the dynamo operator is displayed for the three cases above and $l = 1$, keeping the color scheme for the sets Z_i from Figure 1.

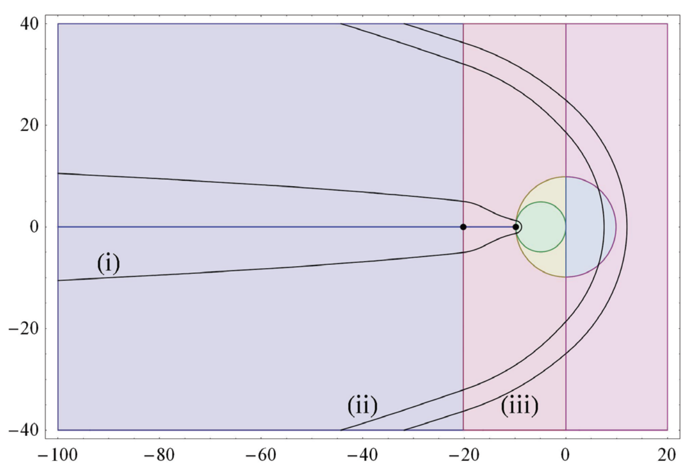

FIG. 2. *Boundary of Σ_θ in Theorem* 4.6(i)–(iii) *for $l = 1$.*

Remark 4.7. The second inequality in case (ii) can also be written as

$$\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\theta)}} \le \lambda_1(\infty) + \lambda_1(\theta) + \frac{\|\alpha\|\,\|\alpha'\|}{2\sqrt{\lambda_1(\theta)}}.$$

This shows that case (ii) indeed appears and that (i), (ii), and (iii) exhaust all possible cases for $\|\alpha\|$ and $\|\alpha'\|$.

COROLLARY 4.8. *The operator $\mathcal{A}_\theta$ has no spectrum in the closed right half-plane if*

$$\|\alpha\|^2 + \frac{\|\alpha\|\|\alpha'\|}{\sqrt{\lambda_1(\theta)}} < \lambda_1(\infty).$$

Proof of Theorem 4.6. The existence of the function h_θ with the claimed properties follows from Proposition 4.4, Lemma 4.5, and the implicit function theorem, applied to the restriction of φ to the open upper half-plane.

The monotonicity properties of φ in Lemma 4.5 induce corresponding monotonicity properties for the function f; in particular, f is strictly decreasing on $(-\lambda_1(\theta), \infty)$. Therefore, since

$$\lim_{t\searrow-\lambda_1(\theta)} f(t) = \lim_{t\searrow-\lambda_1(\theta)} f_4(t) = \infty, \quad \lim_{t\to\infty} f(t) = \lim_{t\to\infty} f_6(t) = 0,$$

the equation $f(\lambda) = 1$ has exactly one solution in $(-\lambda_1(\theta), \infty)$; this solution is the unique zero a_θ of h_θ and hence equal to $\max \operatorname{Re} \Sigma_\theta$. The location of a_θ is classified by the three cases in Theorem 4.6:

(i) The condition in (i) is equivalent to $f(0) = f_4(0) \leq 1$ and thus $a_\theta \in (-\lambda_1(\theta), 0]$. Now the formula for a_θ is obtained by solving the quadratic equation

$$\big(a_\theta + \lambda_1(\infty)\big)\big(a_\theta + \lambda_1(\theta)\big) = \|\alpha\|^2\lambda_1(\theta) + \|\alpha\|\,\|\alpha'\|\sqrt{\lambda_1(\theta)}$$

for a_θ in the interval $(-\lambda_1(\theta), 0]$, which is equivalent to $f_5(a_\theta) = 1$.

(ii) The first condition in (ii) is equivalent to $f(0) = f_5(0) > 1$, while the second condition in (ii) is equivalent to $f_5(\lambda_1(\theta)) \leq 1$. In this case $a_\theta \in (0, \lambda_1(\theta)]$. The formula for a_θ is obtained by solving the quadratic equation

$$\big(a_\theta + \lambda_1(\infty) - \|\alpha\|^2\big)\big(a_\theta + \lambda_1(\theta)\big) = \|\alpha\|\,\|\alpha'\|\sqrt{\lambda_1(\theta)}$$

for a_θ in the interval $(0, \lambda_1(\theta)]$, which is equivalent to $f_4(a_\theta) = 1$.
Moreover, for $\lambda \in Z_4 = \{z \in \mathbb{C} : -\lambda_1(\theta) < \operatorname{Re} z \leq 0,\ |z + \lambda_1(\theta)/2)| \leq \lambda_1(\theta)/2\}$, we have the estimates $|\lambda + \lambda_1(\theta)| \leq \lambda_1(\theta)$, $|\lambda + \lambda_1(\theta)| \leq \lambda_1(\infty)$, and hence

$$f_4(\lambda) \geq \left(\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\theta)}}\right)\frac{1}{\lambda_1(\infty)} > 1, \quad \lambda \in Z_4,$$

by the first condition in case (ii). This proves that $Z_4 \subset \Sigma_\theta$.

(iii) The condition in (iii) is equivalent to $f(\lambda_1(\theta)) = f_6(\lambda_1(\theta)) > 1$. Therefore $a_\theta \in (\lambda_1(\theta), \infty)$ and the equation $f(\lambda) = 1$ is equivalent to $f_6(\lambda) = 1$, which is (4.18).
Furthermore, for $\lambda \in Z_5 = \{z \in \mathbb{C} : \operatorname{Re} z > 0, |z| \leq \lambda_1(\theta)\}$, we have the estimates $|\lambda + \lambda_1(\theta)| \leq 2\lambda_1(\theta)$, $|\lambda + \lambda_1(\theta)| \leq \lambda_1(\infty) + \lambda_1(\theta)$, and hence

$$f_5(\lambda) \geq \left(\|\alpha\|^2 + \|\alpha\|\,\|\alpha'\|\frac{1}{2\sqrt{\lambda_1(\theta)}}\right)\frac{1}{\lambda_1(\infty) + \lambda_1(\theta)} > 1, \quad \lambda \in Z_5,$$

by the condition in (iii). This shows that $Z_5 \subset \Sigma_\theta$. From the monotonicity properties of f it follows that also $Z_3 \cup Z_4 \subset \Sigma_\theta$. ∎

Remark 4.9. The spectral inclusion in Theorem 4.6 does not provide a uniform bound for the imaginary parts of the eigenvalues in the left half-plane. Such a bound is obtained in the next section, where we show that the imaginary parts of all eigenvalues are bounded by $\|\alpha'\|$ (see Theorem 5.10).

While Proposition 4.2 was applied in the proof of Theorem 4.6 to establish a global inclusion for all eigenvalues of $\mathcal{A}_\theta$, it can also be used to study the local behavior of eigenvalues.

The eigenvalues of $\mathcal{A}_\theta$ for $\alpha \equiv 0$ coincide with the eigenvalues of the diagonal elements $-A_\theta$ and $-A_\infty$ and are hence real; their multiplicity is 1 for $\theta \in [0,\infty)$ and 2 for $\theta = \infty$. The following proposition provides a condition guaranteeing that, e.g., if $\theta \in [0,\infty)$, for $\alpha \not\equiv 0$ an eigenvalue remains on the real axis. With regard to the dynamo problem (1.1), (1.2), such a result could be called a local nonoscillation theorem.

PROPOSITION 4.10. *Let $\theta \in [0,\infty]$ and $\lambda_0 \in \sigma(-A_\theta) \cup \sigma(-A_\infty)$. Set*

$$\delta_0 := \frac{1}{2}\operatorname{dist}\Big(\lambda_0, \big(\sigma(-A_\theta)\cup\sigma(-A_\infty)\big)\setminus\{\lambda_0\}\Big)$$

and denote by Γ_0 the circle centered at λ_0 with radius δ_0. If

$$\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{|\lambda_0|+2\delta_0}} < \frac{\delta_0^2}{|\lambda_0|+2\delta_0}, \tag{4.19}$$

then, for $\theta \in [0,\infty)$, the operator $\mathcal{A}_\theta$ has exactly one eigenvalue within the circle Γ_0, and this eigenvalue is simple and real; the operator $\mathcal{A}_\infty$ has exactly two eigenvalues within the circle Γ_0 (counted with multiplicities), and these eigenvalues are either real or form a complex conjugate pair.

Proof. For $\lambda \in \Gamma_0$, we have (see (3.13))

$$\begin{aligned}
\max_{\lambda\in\Gamma_0}\|(A_\theta+\lambda)^{-1}\| &= \frac{1}{\delta_0},\\
\max_{\lambda\in\Gamma_0}\|A_\theta(A_\theta+\lambda)^{-1}\| &= \max_{\lambda\in\Gamma_0}\|I-\lambda(A_\theta+\lambda)^{-1}\| \le 1+\frac{|\lambda_0|+\delta_0}{\delta_0},\\
\max_{\lambda\in\Gamma_0}\|A_\theta^{1/2}(A_\theta+\lambda)^{-1}\| &= \max_{\lambda\in\Gamma_0}\|A_\theta(A_\theta+\lambda)^{-2}\|^{1/2}\\
&\le \max_{\lambda\in\Gamma_0}\big(\|(A_\theta+\lambda)^{-1}\| + |\lambda|\,\|(A_\theta+\lambda)^{-2}\|\big)^{1/2}\\
&\le \frac{\sqrt{|\lambda_0|+2\delta_0}}{\delta_0}.
\end{aligned}$$

Using these estimates, it is not difficult to see that condition (4.19) guarantees that condition (4.13), and hence condition (4.8) in Proposition 4.2, holds for all $\lambda \in \Gamma_0$. Now Proposition 4.2 yields that $\Gamma_0 \subset \rho(\mathcal{A}_\theta)$.

If we introduce the operators $\mathcal{A}_\theta^{(\varepsilon)} := \mathcal{Q}_\theta + \varepsilon\mathcal{R}$ for $0 \le \varepsilon \le 1$, then the same arguments as above yield the inclusion $\Gamma_0 \subset \rho(\mathcal{A}_\theta^{(\varepsilon)})$ for all $0 \le \varepsilon \le 1$. Since the eigenprojection

$$P(\varepsilon) = -\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0}\big(\mathcal{A}_\theta^{(\varepsilon)} - z\big)^{-1}\,\mathrm{d}z$$

onto the part of the spectrum of $\mathcal{A}_\theta^{(\varepsilon)}$ inside Γ_0 depends continuously on ε, the dimension of its range is constant for $0 \le \varepsilon \le 1$ (see [13, Lemma I.4.10, Theorem IV.3.16

and its proof]). For $\theta \in [0,\infty)$, the operator $\mathcal{A}_\theta^{(0)}$, and hence every operator $\mathcal{A}_\theta^{(\varepsilon)}$ for $0 \le \varepsilon \le 1$, has exactly one eigenvalue of multiplicity 1 inside Γ_0. The operator $\mathcal{A}_\infty^{(0)}$ has exactly one eigenvalue of multiplicity 2 inside Γ_0, and hence every operator $\mathcal{A}_\infty^{(\varepsilon)}$ for $0 \le \varepsilon \le 1$ has exactly two eigenvalues in Γ_0 counted with multiplicities.

Since all entries of the operator matrices $\mathcal{A}_\theta^{(\varepsilon)}$ are differential operators with real coefficients, the spectra $\sigma(\mathcal{A}_\theta^\varepsilon)$ are symmetric to $\mathbb{R}$ for all $0 \le \varepsilon \le 1$ (cf. Theorem 4.1). Thus, for $\theta \in [0,\infty)$, the single eigenvalue of $\mathcal{A}_\theta^{(1)} = \mathcal{A}_\theta$ inside Γ_0 must be real, while $\mathcal{A}_\infty^{(1)} = \mathcal{A}_\infty$ has two eigenvalues inside Γ_0 counted with multiplicities which are either real or form a complex conjugate pair. □

Corollary 4.11. *Let $\theta \in [0,\infty)$ and let Γ_0 be the circle around the largest eigenvalue $-\lambda_1(\theta)$ of the diagonal elements of $\mathcal{A}_\theta$ with radius $\delta_0 = \big(\lambda_1(\infty)-\lambda_1(\theta)\big)/2$. If*

$$\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\infty)}} < \frac{\big(\lambda_1(\infty)-\lambda_1(\theta)\big)^2}{4\lambda_1(\infty)}, \tag{4.20}$$

then the operator $\mathcal{A}_\theta$ has exactly one real eigenvalue in Γ_0.

5. A similarity transformation and a second eigenvalue estimate. Our second approach to studying the spectral properties of the operator $\mathcal{A}_\theta$ is based on a quasi-similarity transformation of $\mathcal{A}_\theta$. The transformed operator $\mathcal{B}_\theta$ turns out to be a bounded perturbation of a self-adjoint operator $\mathcal{S}_\theta$. This allows us to prove another estimate for the eigenvalues of $\mathcal{A}_\theta$ which shows, in particular, that the imaginary parts of all eigenvalues have modulus at most $\|\alpha'\|$. In addition, we investigate the number of positive eigenvalues of the unperturbed operator $\mathcal{S}_\theta$.

Proposition 5.1. *Let $\theta \in [0,\infty]$ and let the linear operator $\mathcal{W}_\theta$ in the product space $L_2(0,1) \oplus L_2(0,1)$ be given by*

$$\mathcal{W}_\theta := \begin{pmatrix} A_\theta^{1/2} & 0 \\ 0 & I \end{pmatrix}, \quad \mathcal{D}(\mathcal{W}_\theta) := \mathcal{D}\big(A_\theta^{1/2}\big) \oplus L_2(0,1).$$

Then the linear operator $\mathcal{B}_\theta$ in $L_2(0,1) \oplus L_2(0,1)$ defined as

$$\mathcal{B}_\theta := \mathcal{W}_\theta \mathcal{A}_\theta \mathcal{W}_\theta^{-1} \tag{5.1}$$

is closed and the eigenvalues of $\mathcal{B}_\theta$ coincide with those of $\mathcal{A}_\theta$,

$$\sigma_{\rm p}(\mathcal{B}_\theta) = \sigma_{\rm p}\big(\mathcal{A}_\theta\big);$$

moreover, $(x_j)_{j=0}^k$ is a Jordan chain of $\mathcal{A}_\theta$ at $\lambda \in \sigma_{\rm p}\big(\mathcal{A}_\theta\big)$ if and only if $(\mathcal{W}_\theta x_j)_{j=0}^k$ is a Jordan chain of $\mathcal{B}_\theta$ at λ.

Proof. Since $\mathcal{W}_\theta$ is boundedly invertible and $\mathcal{A}_\theta$ is closed, the product $\mathcal{B}_\theta$ is a closed operator (see [13, section III.5.2]).

In order to prove the equality of the point spectra and the claim about the Jordan chains, we observe that, by definition, the domain of $\mathcal{B}_\theta$ is given by

$$\mathcal{D}(\mathcal{B}_\theta) = \big\{ y \in L_2(0,1) \oplus L_2(0,1) : \mathcal{W}_\theta^{-1} y \in \mathcal{D}\big(\mathcal{A}_\theta\big),\ \mathcal{A}_\theta \mathcal{W}_\theta^{-1} y \in \mathcal{D}(\mathcal{W}_\theta) \big\}. \tag{5.2}$$

Let $\lambda \in \sigma_{\rm p}(\mathcal{A}_\theta)$ and let $(x_j)_{j=0}^k \subset \mathcal{D}(\mathcal{A}_\theta)$ be a corresponding Jordan chain, that is, $(\mathcal{A}_\theta - \lambda)x_j = x_{j-1}$ for $j = 0, 1, \ldots, k$, where $x_{-1} := 0$. Since $\mathcal{D}(\mathcal{A}_\theta) \subset \mathcal{D}(\mathcal{W}_\theta)$, it follows that $\mathcal{A}_\theta x_j = \lambda_0 x_j + x_{j-1} \in \mathcal{D}(\mathcal{W}_\theta)$. Hence $\mathcal{W}_\theta x_j \in \mathcal{D}(\mathcal{B}_\theta)$ by (5.2) and

$$\big(\mathcal{B}_\theta - \lambda\big)\mathcal{W}_\theta x_j = \big(\mathcal{W}_\theta \mathcal{A}_\theta \mathcal{W}_\theta^{-1} - \lambda\big)\mathcal{W}_\theta x_j = \big(\mathcal{W}_\theta \mathcal{A}_\theta - \lambda \mathcal{W}_\theta\big)x_j = \mathcal{W}_\theta x_{j-1}$$

for $j = 0, 1, \ldots, k$. On the contrary, let $\lambda \in \sigma_{\rm p}(\mathcal{B}_\theta)$ and let $(y_j)_{j=0}^k \subset \mathcal{D}(\mathcal{B}_\theta)$ be a corresponding Jordan chain, that is, $(\mathcal{B}_\theta - \lambda)y_j = y_{j-1}$ for $j = 0, 1, \ldots, k$, where $y_{-1} := 0$. Since $\mathcal{W}_\theta^{-1} y_j \in \mathcal{D}(\mathcal{A}_\theta)$ by (5.2), we obtain

$$(\mathcal{A}_\theta - \lambda)\mathcal{W}_\theta^{-1} y_j = (\mathcal{W}_\theta^{-1}\mathcal{B}_\theta\mathcal{W}_\theta - \lambda)\mathcal{W}_\theta^{-1} y_j = (\mathcal{W}_\theta^{-1}\mathcal{B}_\theta - \lambda\mathcal{W}_\theta^{-1})y_j = \mathcal{W}_\theta^{-1} y_{j-1}$$

for $j = 0, 1, \ldots, k$. ◻

In the next theorem we show that the operator $\mathcal{B}_\theta$ is a bounded perturbation of a self-adjoint semibounded block operator matrix. First we need an auxiliary lemma.

LEMMA 5.2. *Let $\theta \in [0, \infty]$. Then*

(i) *$\alpha A_\theta^{1/2}$ is A_θ-compact,*

(ii) *$A_\theta^{1/2}\alpha$ is A_∞-compact.*

Proof. (i) Since A_θ has compact resolvent by Proposition 3.1(ii), we conclude that $\alpha A_\theta^{1/2} A_\theta^{-1} = \alpha A_\theta^{-1/2}$ is compact.

(ii) First we show that

$$\mathcal{D}(A_\infty^{1/2}) \subset \mathcal{D}\big(A_\theta^{1/2}\alpha\big). \tag{5.3}$$

The description of $\mathcal{D}\big(A_\theta^{1/2}\big)$ in Proposition 3.1(i) and the assumption $\alpha \in C^1([0,1])$ show that $x \in \mathcal{D}\big(A_\theta^{1/2}\big)$ implies $\alpha x \in \mathcal{D}\big(A_\theta^{1/2}\big)$, that is, $\mathcal{D}\big(A_\theta^{1/2}\big) \subset \mathcal{D}\big(A_\theta^{1/2}\alpha\big)$. Since $\mathcal{D}(A_\infty^{1/2}) = \big\{x \in \mathcal{D}(A_\theta^{1/2}) : x(1) = 0\big\} \subset \mathcal{D}(A_\theta^{1/2})$ by Proposition 3.1(i), the inclusion (5.3) follows.

Because $A_\infty^{1/2}$ and $A_\theta^{1/2}\alpha$ are closed, (5.3) implies that $A_\theta^{1/2}\alpha A_\infty^{-1/2}$ is a bounded operator (see [13, Remark IV.1.5]). Since A_∞ has compact resolvent by Proposition 3.1(ii), it follows that $A_\theta^{1/2}\alpha A_\infty^{-1} = A_\theta^{1/2}\alpha A_\infty^{-1/2} A_\infty^{-1/2}$ is compact. ◻

THEOREM 5.3. *Let $\theta \in [0, \infty]$. Then the operator $\mathcal{B}_\theta$ defined in* (5.1) *has the form*

$$\mathcal{B}_\theta = \begin{pmatrix} -A_\theta & A_\theta^{1/2}\alpha \\ A_{\theta,\alpha}A_\theta^{-1/2} & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{B}_\theta) = \mathcal{D}\big(A_\theta\big) \oplus \mathcal{D}\big(A_\infty\big). \tag{5.4}$$

If we define

$$\mathcal{S}_\theta := \begin{pmatrix} -A_\theta & A_\theta^{1/2}\alpha \\ \alpha A_\theta^{1/2} & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{S}_\theta) := \mathcal{D}\big(A_\theta\big) \oplus \mathcal{D}\big(A_\infty\big), \tag{5.5}$$

$$\mathcal{T}_\theta := \begin{pmatrix} 0 & 0 \\ -\alpha' D A_\theta^{-1/2} & 0 \end{pmatrix}, \quad \mathcal{D}(\mathcal{T}_\theta) := L_2(0,1) \oplus L_2(0,1), \tag{5.6}$$

then $\mathcal{S}_\theta$ is self-adjoint and bounded from above with compact resolvent, $\mathcal{T}_\theta$ is bounded with $\|\mathcal{T}_\theta\| \le \|\alpha'\|$, and

$$\mathcal{B}_\theta = \mathcal{S}_\theta + \mathcal{T}_\theta. \tag{5.7}$$

Proof. First we prove the identity (5.4). By the definition of $A_{\theta,\alpha}$ in Proposition 3.6, we have $\mathcal{D}(A_{\theta,\alpha}) = \mathcal{D}(A_\theta)$ and thus $\mathcal{D}(A_\theta) \subset \mathcal{D}(A_\theta^{1/2}) = \mathcal{D}(A_{\theta,\alpha}A_\theta^{-1/2})$; by Lemma 5.2(ii) (see also (5.3)), we have $\mathcal{D}(A_\infty) \subset \mathcal{D}\big(A_\theta^{1/2}\alpha\big)$. This shows that $\mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty)$ is the domain of the block operator matrix in (5.4). Formally, the relation (5.4) follows from the definition of $\mathcal{B}_\theta$ in (5.1); it remains to be shown that $\mathcal{D}(\mathcal{B}_\theta) = \mathcal{D}\big(A_\theta\big) \oplus \mathcal{D}\big(A_\infty\big)$.

According to (5.2), we have $y \in \mathcal{D}(\mathcal{B}_\theta)$ if and only if $\mathcal{W}_\theta^{-1} y \in \mathcal{D}(\mathcal{A}_\theta)$ and $\mathcal{A}_\theta \mathcal{W}_\theta^{-1} y \in \mathcal{D}(\mathcal{W}_\theta)$. Since $\mathcal{D}(\mathcal{A}_\theta) = \mathcal{D}\big(A_\theta\big) \oplus \mathcal{D}\big(A_\infty\big)$ by (2.2) and

$$A_{\theta,\alpha} A_\theta^{-1/2} = \alpha A_\theta^{1/2} - \alpha' D A_\theta^{-1/2} \tag{5.8}$$

by (3.12), it follows that $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{B}_\theta)$ if and only if

$$\begin{cases} \text{(a)} \quad A_\theta^{-1/2} y_1 \in \mathcal{D}(A_\theta), \quad y_2 \in \mathcal{D}(A_\infty), \\ \text{(b)} \quad -A_\theta^{1/2} y_1 + \alpha\, y_2 \in \mathcal{D}\big(A_\theta^{1/2}\big), \quad \big(\alpha A_\theta^{1/2} - \alpha' D A_\theta^{-1/2}\big) y_1 - A_\infty y_2 \in L_2(0,1). \end{cases}$$

The first condition in (a) is equivalent to $y_1 \in \mathcal{D}(A_\theta^{1/2})$. Thus the second condition in (b) always holds as $D A_\theta^{-1/2}$ is bounded by Lemma 3.5. By Lemma 5.2(ii) (see also (5.3)), we have $\mathcal{D}(A_\infty) \subset \mathcal{D}(A_\theta^{1/2}\alpha)$, and hence the first condition in (b) reduces to $A_\theta^{1/2} y_1 \in \mathcal{D}(A_\theta^{1/2})$. Altogether, $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{B}_\theta)$ if and only if $y_1 \in \mathcal{D}(A_\theta)$ and $y_2 \in \mathcal{D}(A_\infty)$. This completes the proof of (5.4).

To see that $\mathcal{S}_\theta$ is symmetric, we observe that $\alpha \in C([0,1])$ and α is real-valued. This implies that the corresponding multiplication operator is bounded and self-adjoint, and hence

$$\big(\alpha A_\theta^{1/2}\big)^* = \big(A_\theta^{1/2}\big)^* \alpha^* = A_\theta^{1/2} \alpha.$$

Furthermore, Lemma 5.2 implies that

$$\mathcal{S}_\theta = \begin{pmatrix} -A_\theta & 0 \\ 0 & -A_\infty \end{pmatrix} + \begin{pmatrix} 0 & A_\theta^{1/2}\alpha \\ \alpha A_\theta^{1/2} & 0 \end{pmatrix}$$

is a relatively compact perturbation of its block diagonal part $\operatorname{diag}(-A_\theta, -A_\infty)$. The latter is self-adjoint and semibounded (in fact, negative) with compact resolvent by Proposition 3.1. Hence $\mathcal{S}_\theta$ has the same properties by the Rellich–Kato theorem, the stability theorem for semiboundedness, and Weyl's essential spectrum theorem (see [13, Theorems V.4.3 and V.4.11], [20, Corollary XIII.4.2]).

Finally, because $\alpha' \in C([0,1])$ and $D A_\theta^{-1/2}$ is bounded with $\|D A_\theta^{-1/2}\| \le 1$ (see Lemma 3.5), the operator $\mathcal{T}_\theta$ is bounded with $\|\mathcal{T}_\theta\| \le \|\alpha'\|$.

Since $\mathcal{D}(\mathcal{S}_\theta) = \mathcal{D}(\mathcal{B}_\theta)$, the identity (5.7) is immediate from (5.4) and (5.8). □

Remark 5.4. The boundedness from above of $\mathcal{S}_\theta$ is proved independently in Proposition 5.8, where also a concrete upper bound for $\mathcal{S}_\theta$ is established.

PROPOSITION 5.5. *Let $\theta \in [0,\infty]$. The spectrum $\sigma(\mathcal{S}_\theta) = \sigma_{\mathrm{p}}(\mathcal{S}_\theta)$ of the self-adjoint operator $\mathcal{S}_\theta$ satisfies*

$$\sigma_{\mathrm{p}}(\mathcal{S}_\theta) \setminus \sigma_{\mathrm{p}}(-A_\theta) = \sigma_{\mathrm{p}}(S_{2,\theta}), \tag{5.9}$$

where the so-called Schur complement $S_{2,\theta}$ is the operator function given by

$$S_{2,\theta}(\lambda) := -A_\infty - \lambda + \alpha A_\theta^{1/2} (A_\theta + \lambda)^{-1} A_\theta^{1/2} \alpha, \quad \mathcal{D}\big(S_{2,\theta}(\lambda)\big) := \mathcal{D}(A_\infty),$$

for all $\lambda \in \mathbb{C} \setminus \sigma(-A_\theta)$. Moreover, the multiplicities of the eigenvalues of $\mathcal{S}_\theta$ and of $S_{2,\theta}$ coincide.

Proof. Let $\lambda \notin \sigma(-A_\theta) = \sigma_{\mathrm{p}}(-A_\theta)$. It is not difficult to check that the block operator matrix $\mathcal{S}_\theta$ can be factorized as (see, e.g., [26, Theorem 2.2.14])

$$\mathcal{S}_\theta - \lambda = \begin{pmatrix} I & 0 \\ -\alpha A_\theta^{1/2} (A_\theta + \lambda)^{-1} & I \end{pmatrix} \begin{pmatrix} -A_\theta - \lambda & 0 \\ 0 & S_{2,\theta}(\lambda) \end{pmatrix} \begin{pmatrix} I & -(A_\theta + \lambda)^{-1} A_\theta^{1/2} \alpha \\ 0 & I \end{pmatrix}. \tag{5.10}$$

Since the outer two factors are bounded and boundedly invertible, as is $-A_\theta-\lambda$ by the assumption on λ, it follows that $\mathcal{S}_\theta - \lambda$ is boundedly invertible or injective if and only if the operator $S_{2,\theta}(\lambda)$ is as well (see, e.g., [26, Theorem 2.3.3]). This proves (5.9).

It is easy to see that $y_0 = (y_{0,1}, y_{0,2})^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}_\theta)$ is an eigenvector of $\mathcal{S}_\theta$ at $\lambda_0 \in \sigma_{\mathrm{p}}(\mathcal{S}_\theta)$ if and only if $y_{0,2} \in \mathcal{D}(A_\infty)$ is an eigenvector of $S_{2,\theta}$ at λ_0, that is, $S_{2,\theta}(\lambda_0)y_{0,2} = 0$, and $y_{0,1} = (A_\theta + \lambda)^{-1} A_\theta^{1/2} \alpha\, y_{0,2}$. Clearly, the self-adjoint operator $\mathcal{S}_\theta$ has no associated vectors. Since $S_{2,\theta}$ is a self-adjoint operator function, that is, $S_{2,\theta}(\lambda) = S_{2,\theta}(\overline{\lambda})^*$ for $\lambda \in \mathbb{C} \setminus \sigma(-A_\theta)$, and its derivative

$$S'_{2,\theta}(\lambda) = -I - \alpha A_\theta^{1/2} (A_\theta + \lambda)^{-2} A_\theta^{1/2} \alpha \le -I$$

is uniformly negative for all $\lambda \in \mathbb{C} \setminus \sigma(-A_\theta)$, $S_{2,\theta}$ has no associated vectors either (cf. [18, Lemma 30.13]). ☐

Next we establish an explicit upper bound for the self-adjoint operator $\mathcal{S}_\theta$. To this end, we first show that $\mathcal{S}_\theta$ satisfies the following two-sided estimate, which might be of independent interest.

LEMMA 5.6. *Let $\theta \in [0, \infty]$ and let $\gamma \in (0, 1]$ be arbitrary. Then the operator $\mathcal{S}_\theta$ admits the two-sided estimate*

$$\begin{pmatrix} -\big(1+\gamma\big) A_\theta & 0 \\ 0 & -A_\infty - \dfrac{1}{\gamma}\alpha^2 \end{pmatrix} \le \mathcal{S}_\theta \le \begin{pmatrix} -\big(1-\gamma\big) A_\theta & 0 \\ 0 & -A_\infty + \dfrac{1}{\gamma}\alpha^2 \end{pmatrix}; \tag{5.11}$$

in particular, for $\gamma = 1$,

$$\begin{pmatrix} -2A_\theta & 0 \\ 0 & -A_\infty - \alpha^2 \end{pmatrix} \le \mathcal{S}_\theta \le \begin{pmatrix} 0 & 0 \\ 0 & -A_\infty + \alpha^2 \end{pmatrix}. \tag{5.12}$$

Proof. Let $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}_\theta) = \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty)$. Then the right inequality in (5.11) follows from the estimate

$$\begin{aligned} (\mathcal{S}_\theta y, y) &= -(A_\theta y_1, y_1) - (A_\infty y_2, y_2) + 2\,\mathrm{Re}\,\big(A_\theta^{1/2} \alpha\, y_2, y_1\big) \\ &\le -(A_\theta y_1, y_1) - (A_\infty y_2, y_2) + 2\,\gamma^{1/2} \big\|A_\theta^{1/2} y_1\big\| \frac{1}{\gamma^{1/2}} \|\alpha y_2\| \\ &\le -(A_\theta y_1, y_1) - (A_\infty y_2, y_2) + \gamma (A_\theta y_1, y_1) + \frac{1}{\gamma}\big(\alpha^2 y_2, y_2\big); \end{aligned}$$

the left inequality in (5.11) is obtained analogously. ☐

PROPOSITION 5.7. *If α is constant, $\alpha \equiv \alpha_0$, we also have the estimate*

$$\max \sigma(\mathcal{A}_\theta) = \max \sigma(\mathcal{S}_\theta) \le \frac{|\alpha_0|^2}{4}. \tag{5.13}$$

Proof. Let $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}_\theta) = \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty)$; then $y_2 \in \mathcal{D}(A_\infty) \subset \mathcal{D}(A_\infty^{1/2}) \subset \mathcal{D}(A_\theta^{1/2})$ by Proposition 3.1(i). Since α is constant, it commutes with $A_\theta^{1/2}$.

Hence

$$\begin{aligned}(\mathcal{S}_\theta y, y) &= -(A_\theta y_1, y_1) - (A_\infty y_2, y_2) + \big(\alpha_0 A_\theta^{1/2} y_2, y_1\big) + \big(\alpha_0 A_\theta^{1/2} y_1, y_2\big)\\ &\le -(A_\theta y_1, y_1) + |\alpha_0|\sqrt{(A_\theta y_1, y_1)}\|y_2\| - (A_\infty y_2, y_2) + |\alpha_0|\,\|A_\theta^{1/2} y_2\|\,\|y_1\|\\ &\le -\left(\sqrt{(A_\theta y_1, y_1)} - \frac{|\alpha_0|}{2}\|y_2\|\right)^2 + \frac{|\alpha_0|^2}{4}\|y_2\|^2\\ &\quad -\left(\sqrt{(A_\infty y_2, y_2)} - \frac{|\alpha_0|}{2}\frac{\|A_\theta^{1/2} y_2\|}{\|A_\infty^{1/2} y_2\|}\|y_1\|\right)^2 + \frac{|\alpha_0|^2}{4}\frac{\|A_\theta^{1/2} y_2\|^2}{\|A_\infty^{1/2} y_2\|^2}\|y_1\|^2\\ &\le \frac{|\alpha_0|^2}{4}\big(\|y_1\|^2 + \|y_2\|^2\big) = \frac{|\alpha_0|^2}{4}\|y\|^2\end{aligned}$$

if we show that

$$\|A_\theta^{1/2} y_2\| \le \|A_\infty^{1/2} y_2\|, \quad y_2 \in \mathcal{D}(A_\infty),$$

or, equivalently, $\|A_\theta^{1/2} A_\infty^{-1/2}\| \le 1$. This follows from the relation $A_\theta^{1/2} A_\infty^{-1/2} = (A_\infty^{-1/2} A_\theta^{1/2})^*$ (note that $A_\theta^{1/2} A_\infty^{-1/2}$ is bounded as $\mathcal{D}(A_\infty^{1/2}) \subset \mathcal{D}(A_\theta^{1/2})$) and from the estimate

$$\|A_\infty^{-1/2} A_\theta^{1/2} x\|^2 = (A_\infty^{-1} A_\theta^{1/2} x, A_\theta^{1/2} x) \le (A_\theta^{-1} A_\theta^{1/2} x, A_\theta^{1/2} x) = \|x\|^2, \quad x \in \mathcal{D}(A_\theta^{1/2}),$$

where we have used the inequality $A_\infty^{-1} \le A_\theta^{-1}$ from Proposition 3.1(iv).

Since α is constant and hence $\alpha' \equiv 0$, we have $\mathcal{B}_\theta = \mathcal{S}_\theta$, and so (5.13) follows from Proposition 5.1. □

The following proposition provides an upper bound for the unperturbed operator $\mathcal{S}_\theta$ for arbitrary functions α.

PROPOSITION 5.8. *Let $\theta \in [0, \infty]$. Then the self-adjoint operator $\mathcal{S}_\theta$ defined in* (5.6) *satisfies the estimate*

$$\max \sigma(\mathcal{S}_\theta) \le s_\theta,$$

where $s_\theta \in [-\lambda_1(\theta), -\lambda_1(\infty) + \|\alpha\|^2]$ is given by

$$s_\theta := \begin{cases} -\dfrac{\lambda_1(\infty) + \lambda_1(\theta)}{2} + \sqrt{\left(\dfrac{\lambda_1(\infty) - \lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\|\alpha\|^2} & \text{if } \|\alpha\|^2 \le \lambda_1(\infty),\\ -\lambda_1(\infty) + \|\alpha\|^2 & \text{if } \|\alpha\|^2 > \lambda_1(\infty).\end{cases}$$

Remark 5.9.

(i) The bound s_θ satisfies

$$\begin{aligned} -\lambda_1(\theta) \le s_\theta \le 0 \quad &\iff \quad \|\alpha\|^2 \le \lambda_1(\infty),\\ 0 < s_\theta \le -\lambda_1(\infty) + \|\alpha\|^2 \quad &\iff \quad \|\alpha\|^2 > \lambda_1(\infty).\end{aligned}$$

(ii) If $\|\alpha\|^2 < \lambda_1(\infty)$, then s_θ can be written equivalently as

$$s_\theta = -\lambda_1(\theta) + \|\alpha\|\sqrt{\lambda_1(\theta)}\,\tan\left(\frac{1}{2}\arctan\frac{2\|\alpha\|\sqrt{\lambda_1(\theta)}}{\lambda_1(\infty) - \lambda_1(\theta)}\right).$$

(iii) In the particular case $\theta = \infty$, the expression for s_∞ simplifies to

$$s_\infty = \begin{cases} -\lambda_1(\infty) + \|\alpha\|\sqrt{\lambda_1(\infty)} & \text{if } \|\alpha\|^2 \le \lambda_1(\infty), \\ -\lambda_1(\infty) + \|\alpha\|^2 & \text{if } \|\alpha\|^2 > \lambda_1(\infty) \end{cases}$$
$$= -\lambda_1(\infty) + \|\alpha\| \max\bigl\{\|\alpha\|, \sqrt{\lambda_1(\infty)}\,\bigr\}.$$

Proof of Proposition 5.8. Since $-A_\theta \le -\lambda_1(\theta)$ and $-A_\infty \le -\lambda_1(\infty)$, the right inequality in (5.11) yields that, for arbitrary $\gamma \in (0,1]$,

$$\bigl(\mathcal{S}_\theta y, y\bigr) \le -(1-\gamma)\,\lambda_1(\theta)\,\|y_1\|^2 + \left(-\lambda_1(\infty) + \frac{1}{\gamma}\|\alpha\|^2\right)\|y_2\|^2$$
$$\le \max\bigl\{h_1(\gamma), h_2(\gamma)\bigr\}\,\|y\|^2,$$

where we have set

$$h_1(\gamma) := -(1-\gamma)\,\lambda_1(\theta), \quad h_2(\gamma) := -\lambda_1(\infty) + \frac{1}{\gamma}\|\alpha\|^2, \quad \gamma \in (0,1].$$

If $\|\alpha\|^2 \ge \lambda_1(\infty)$, it is not difficult to see that $h_2(\gamma) \ge 0 \ge h_1(\gamma)$ for all $\gamma \in (0,1]$; in this case, the optimal estimate is obtained for $\gamma = 1$, that is,

$$\bigl(\mathcal{S}_\theta y, y\bigr) \le \bigl(-\lambda_1(\infty) + \|\alpha\|^2\bigr)\,\|y\|^2.$$

If $\|\alpha\|^2 < \lambda_1(\infty)$, a short calculation shows that the function $\max\{h_1(\gamma), h_2(\gamma)\}$ attains its minimum at the point $\gamma_0 \in (0,1]$ where h_1 and h_2 intersect and which is given by the relation

$$\gamma_0 = \frac{1}{\lambda_1(\theta)}\left(-\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2} + \sqrt{\left(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\|\alpha\|^2}\right);$$

in this case, the optimal estimate becomes

$$\bigl(\mathcal{S}_\theta y, y\bigr) \le \left(-\frac{\lambda_1(\infty)+\lambda_1(\theta)}{2} + \sqrt{\left(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\|\alpha\|^2}\right)\|y\|^2,$$

where we have used that $\lambda_1(\theta) \le \lambda_1(\infty)$ for all $\theta \in [0,\infty]$ by Proposition 3.1(iii). □

While the first spectral enclosure in Theorem 4.6 only provides some region in the complex plane containing the eigenvalues, the next theorem gives more detailed information in terms of the eigenvalues of the self-adjoint operator $\mathcal{S}_\theta$.

THEOREM 5.10. *Let $\theta \in [0,\infty]$. The eigenvalues of $\mathcal{A}_\theta$ lie in discs of radius $\|\alpha'\|$ around the eigenvalues of the self-adjoint operator $\mathcal{S}_\theta$:*

$$\sigma(\mathcal{A}_\theta) \subset \bigl\{\lambda \in \mathbb{C} : \operatorname{dist}\bigl(\lambda, \sigma_{\mathrm{p}}(\mathcal{S}_\theta)\bigr) \le \|\alpha'\|\bigr\}. \tag{5.14}$$

As a consequence,

$$\sigma(\mathcal{A}_\theta) \subset \bigl\{\lambda\in\mathbb{C} : \operatorname{Re}\lambda \le s_\theta,\ |\operatorname{Im}\lambda| \le \|\alpha'\|\bigr\} \cup \bigl\{\lambda\in\mathbb{C} : |\lambda - s_\theta| \le \|\alpha'\|\bigr\} \tag{5.15}$$

and

$$\lambda \in \sigma(\mathcal{A}_\theta) \implies |\operatorname{Im}\lambda| \le \|\alpha'\|, \quad \operatorname{Re}\lambda \le s_\theta + \|\alpha'\| =: b_\theta. \tag{5.16}$$

Proof. All claims follow from Theorem 5.3 and Proposition 5.8 by means of classical perturbation theorems for self-adjoint operators (see, e.g., [13, Theorem V.4.5]) if we observe that $\mathcal{B}_\theta = \mathcal{S}_\theta + \mathcal{T}_\theta$ and $\|\mathcal{T}_\theta\| \le \|\alpha'\|$. ◻

The following corollary guarantees that $\mathcal{A}_\theta$ has no eigenvalues in the closed right half-plane; it is an immediate consequence of Theorem 5.10 and of the definition of s_θ in Proposition 5.8.

COROLLARY 5.11. *Let $\theta \in [0,\infty]$. Then $\mathcal{A}_\theta$ has no spectrum in the closed right half-plane if*

$$\|\alpha\| < \sqrt{\lambda_1(\infty)}, \quad \|\alpha'\| < \frac{\lambda_1(\infty)+\lambda_1(\theta)}{2} - \sqrt{\left(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\|\alpha\|^2}.$$

In a way similar to Proposition 4.10, the following local result can be proved.

PROPOSITION 5.12. *Let $\theta \in [0,\infty]$ and let $\lambda_0 \in \sigma(\mathcal{S}_\theta)$ be an eigenvalue of $\mathcal{S}_\theta$ with multiplicity m_0. Set $\delta_0 := \operatorname{dist}\big(\lambda_0, \sigma(\mathcal{S}_\theta)\setminus\{\lambda_0\}\big)/2$ and denote by Γ_0 the circle around λ_0 with radius δ_0. If $\|\alpha'\| < \delta_0$, then the operator $\mathcal{A}_\theta$ has m_0 eigenvalues inside Γ_0* (*counted with multiplicities*)*; if $m_0 = 1$, then the unique eigenvalue of $\mathcal{A}_\theta$ in Γ_0 is real.*

While in section 5 the spectrum of the unperturbed operator $\mathcal{Q}_\theta$ is known, this is not true for the unperturbed operator $\mathcal{S}_\theta$. Therefore, in the remainder of this section, we investigate the spectrum of the semibounded self-adjoint operator $\mathcal{S}_\theta$ by means of variational principles.

The quadratic form version of the min-max characterization of the eigenvalues of self-adjoint operators bounded from below (see [28, Theorem 1, section 2.2], [20, Theorem XIII.2]) shows that the eigenvalues $\lambda_k(\mathcal{S}_\theta)$, $k \in \mathbb{N}$, of $\mathcal{S}_\theta$, enumerated such that $\lambda_1(\mathcal{S}_\theta) \ge \lambda_2(\mathcal{S}_\theta) \ge \cdots$ and counted with multiplicities, are given by

$$\lambda_j(\mathcal{S}_\theta) = \max_{\substack{\mathcal{L}\subset\mathcal{Q}(\mathcal{S}_\theta)\\ \dim\mathcal{L}=j}} \min_{\substack{y\in\mathcal{L}\\ \|y\|=1}} \mathfrak{s}_\theta[y], \quad j = 1,2,\dots. \tag{5.17}$$

Here $\mathcal{Q}(\mathcal{S}_\theta) = \mathcal{D}(A_\theta^{1/2}) \oplus \mathcal{D}(A_\infty^{1/2})$ is the form domain of $\mathcal{S}_\theta$ and $\mathfrak{s}_\theta$ is the corresponding quadratic form given by

$$\mathfrak{s}_\theta[y] := -\big(A_\theta^{1/2}y_1, A_\theta^{1/2}y_1\big) + \big(\alpha y_2, A_\theta^{1/2}y_1\big) + \big(A_\theta^{1/2}y_1, \alpha y_2\big) - \big(A_\infty^{1/2}y_2, A_\infty^{1/2}y_2\big) \tag{5.18}$$

for $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{Q}(\mathcal{S}_\theta)$ (see (5.6) and [21, section VIII.6]).

PROPOSITION 5.13. *The number k_0 of positive eigenvalues of the operator $\mathcal{S}_\theta$ in $L_2(0,1) \oplus L_2(0,1)$ coincides with the number of positive eigenvalues of the shifted Bessel differential operator $-A_\infty + \alpha^2$ in $L_2(0,1)$, and we have*

$$0 < \lambda_j(\mathcal{S}_\theta) \le \lambda_j(-A_\infty + \alpha^2), \quad j = 1,2,\dots,k_0. \tag{5.19}$$

Proof. Denote by κ_0 the number of positive eigenvalues of $-A_\infty + \alpha^2$. The inequality $k_0 \le \kappa_0$ and the inequalities (5.19) are immediate consequences of the right estimate in (5.12), which implies that

$$\mathfrak{s}_\theta[y] = (\mathcal{S}_\theta y, y) \le \big((-A_\infty + \alpha^2)y_2, y_2\big)$$

for $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{S}_\theta) = \mathcal{D}(A_\theta) \oplus \mathcal{D}(A_\infty)$ and of the variational principle (5.17).

It remains to be shown that $k_0 \ge \kappa_0$. By the definition of κ_0 and by the variational principle (5.17) applied to the semibounded operator $-A_\infty + \alpha^2$, there exists a subspace $\mathcal{L} \subset \mathcal{D}(A_\infty^{1/2}) = \mathcal{Q}(A_\infty)$ (the form domain of the positive operator A_∞) such that $\dim \mathcal{L} = \kappa_0$ and

$$\min_{\substack{x\in\mathcal{L}\\ \|x\|=1}} \big(-\big(A_\infty^{1/2}x, A_\infty^{1/2}x\big) + (\alpha x, \alpha x)\big) > 0.$$

Using (5.18), it is easy to see that the subspace $\big\{\big(A_\theta^{-1/2}\alpha\, y_2, y_2\big)^{\mathrm{t}} : y_2 \in \mathcal{L}\big\} \subset \mathcal{Q}(\mathcal{S}_\theta)$ has dimension κ_0 and

$$\mathfrak{s}_\theta\left[\begin{pmatrix} A_\theta^{-1/2}\alpha y_2 \\ y_2 \end{pmatrix}\right] = \big((-A_\infty + \alpha^2)y_2, y_2\big) > 0, \quad y_2 \in \mathcal{L} \subset \mathcal{D}(A_\infty^{1/2}). \tag{5.20}$$

This and the variational principle (5.17) show that $\lambda_{\kappa_0}(\mathcal{S}_\theta) > 0$, and hence $\mathcal{S}_\theta$ has at least κ_0 positive eigenvalues. ∎

The eigenvalues of the block operator matrix $\mathcal{S}_\theta$ coincide with the eigenvalues of its Schur complements,

$$\begin{aligned} S_{1,\theta}(\lambda) &:= -A_\theta - \lambda + A_\theta^{1/2}\alpha(A_\infty+\lambda)^{-1}\alpha A_\theta^{1/2}, \quad \mathcal{D}(S_{1,\theta}(\lambda)) = \mathcal{D}(A_\theta),\\ S_{2,\theta}(\lambda) &:= -A_\infty - \lambda + \alpha A_\theta^{1/2}(A_\theta+\lambda)^{-1}A_\theta^{1/2}\alpha, \quad \mathcal{D}(S_{2,\theta}(\lambda)) = \mathcal{D}(A_\infty), \end{aligned}$$

which are defined for $\lambda \in \mathbb{C}\setminus\sigma(-A_\infty)$ and $\lambda \in \mathbb{C}\setminus\sigma(-A_\theta)$, respectively:

$$\sigma_{\mathrm{p}}(\mathcal{S}_\theta)\setminus\sigma(-A_\infty) = \sigma_{\mathrm{p}}(S_{1,\theta}), \quad \sigma_{\mathrm{p}}(\mathcal{S}_\theta)\setminus\sigma(-A_\theta) = \sigma_{\mathrm{p}}(S_{2,\theta})$$

(cf. Proposition 5.5). This allows us to characterize and estimate the eigenvalues of $\mathcal{S}_\theta$ in the intervals $\big(-\lambda_1(\infty),\infty\big)$ and $\big(-\lambda_1(\theta),\infty\big)$ by variational principles for $S_{1,\theta}$ and $S_{2,\theta}$, respectively (see [7] and, e.g., [26, section 2.10]). As an example, we consider the eigenvalues of $\mathcal{S}_\theta$ in $(-\lambda_1(\theta),\infty)$.

LEMMA 5.14. *Let $\theta \in [0,\infty]$. The Schur complement $S_{2,\theta}$ satisfies the estimates*

$$\begin{aligned} S_{2,\theta}(\lambda) &\le -A_\infty + \frac{\lambda_1(\theta)}{\lambda_1(\theta)+\lambda}\,\alpha^2 - \lambda, \quad && \lambda \in \big(-\lambda_1(\theta), 0\big],\\ S_{2,\theta}(\lambda) &\le -A_\infty + \alpha^2 - \lambda, && \lambda \in [0,\infty). \end{aligned}$$

Moreover, the derivative of $S_{2,\theta}$ is strictly negative and satisfies

$$S_{2,\theta}'(\lambda) \le -I, \quad \lambda \in \big(-\lambda_1(\theta),\infty\big).$$

Proof. Let $\lambda \in \big(-\lambda_1(\theta),\infty\big)$. The Schur complement $S_{2,\theta}$ can be rewritten as

$$\begin{aligned} S_{2,\theta}(\lambda) &= -A_\infty - \lambda + \alpha A_\theta (A_\theta+\lambda)^{-1}\alpha\\ &= -A_\infty - \lambda + \alpha^2 - \lambda\,\alpha(A_\theta+\lambda)^{-1}\alpha. \end{aligned}$$

Since $A_\theta \ge \lambda_1(\theta) > 0$ (see Proposition 3.1), the resolvent of A_θ satisfies the two-sided inequality $0 \le (A_\theta+\lambda)^{-1} \le 1/(\lambda_1(\theta)+\lambda)$. For $\lambda \in \big(-\lambda_1(\theta),0\big]$, the right inequality yields the first estimate claimed for $S_{2,\theta}(\lambda)$, while for $\lambda \in [0,\infty)$ the left inequality yields the second one.

The inequality for the derivative of $S_{2,\theta}(\lambda)$ follows from the identity

$$S_{2,\theta}'(\lambda) = -I - \alpha A_\theta^{1/2}(A_\theta+\lambda)^{-2}A_\theta^{1/2}\alpha, \quad \lambda \in \big(-\lambda_1(\theta),\infty\big). \qquad ∎$$

PROPOSITION 5.15. *Let $\theta \in [0, \infty]$. For every subinterval $[a, \infty) \subset \big(-\lambda_1(\theta), \infty\big)$ let $\lambda_{k_a}(\mathcal{S}_\theta) \leq \cdots \leq \lambda_1(\mathcal{S}_\theta)$ be the eigenvalues of $\mathcal{S}_\theta$ in $[a, \infty)$ (counted with multiplicities). Then*

$$\lambda_j(\mathcal{S}_\theta) = \min_{\substack{\mathcal{L} \subset \mathcal{D}(A_\infty) \\ \dim \mathcal{L} = j}} \max_{\substack{y_2 \in \mathcal{L} \\ \|y_2\| = 1}} p_{2,\theta}(y_2), \quad j = 1, 2, \dots, k_a, \tag{5.21}$$

where $p_{2,\theta}(y_2)$ is the (unique) zero of $(S_{2,\theta}(\cdot)y_2, y_2)$ on $[a, \infty)$ if a zero exists and $p_{2,\theta}(y_2) := -\infty$ otherwise for $y_2 \in \mathcal{D}(A_\infty)$. Moreover, the number k_a of eigenvalues of $\mathcal{S}_\theta$ in $[a, \infty)$ is given by

$$k_a = \dim \mathcal{L}_{[0,\infty)}\big(S_{2,\theta}(a)\big);$$

here $\mathcal{L}_I(S_{2,\theta}(\lambda))$ denotes the spectral subspace of the self-adjoint operator $S_{2,\theta}(\lambda)$ corresponding to an interval $I \subset \big(-\lambda_1(\theta), \infty\big)$.

Proof. The Schur complement $S_{2,\theta}$ satisfies the assumptions of [7, Theorem 2.1] on every subinterval of $\big(-\lambda_1(\theta), \infty\big)$: $\mathcal{D}(S_{2,\theta}(\lambda)) = \mathcal{D}(A_\infty)$ is independent of λ, $S_{2,\theta}$ is strictly decreasing with $S'_{2,\theta} \leq -I$, and $(S_{2,\theta}(\lambda)y_2, y_2) \to -\infty$, $\lambda \to \infty$, for all $y_2 \in \mathcal{D}(A_\infty)$ by Lemma 5.14. Now all claims follow from the fact that the eigenvalues of $\mathcal{S}_\theta$ in $\big(-\lambda_1(\theta), \infty\big)$ coincide with those of $S_{2,\theta}$ by Proposition 5.5 and from the variational principle in [7, Theorem 2.1] applied to $S_{2,\theta}$. □

Remark 5.16. Setting $a = 0$ in Proposition 5.15, we obtain another proof of Proposition 5.13. In fact, the estimate (5.19) follows from the variational principle (5.21) if we observe that $S_{2,\theta}(\lambda) \leq -A_\infty + \alpha^2 - \lambda$ for $\lambda \in [0, \infty)$ by Lemma 5.14, and hence

$$p_{2,\theta}(y_2) \leq \frac{\big((-A_\infty + \alpha^2)y_2, y_2\big)}{(y_2, y_2)}, \quad y_2 \in \mathcal{D}(A_\infty),$$

the right-hand side being the zero of the function $\lambda \mapsto \big((-A_\infty + \alpha^2 - \lambda)y_2, y_2\big)$. Further, since $S_{2,\theta}(0) = -A_\infty + \alpha^2$, Proposition 5.15 shows that the number k_0 of positive eigenvalues of $\mathcal{S}_\theta$ is given by

$$k_0 = \dim \mathcal{L}_{[0,\infty)}\big(S_{2,\theta}(0)\big) = \dim \mathcal{L}_{[0,\infty)}(-A_\infty + \alpha^2).$$

6. Comparison of the two eigenvalue estimates. The two eigenvalue estimates obtained in Theorems 4.6 and 5.10 show, in particular, that every eigenvalue of the dynamo operator $\mathcal{A}_\theta$ satisfies the inequalities

$$|\mathrm{Im}\,\lambda| \leq \|\alpha'\|, \quad \mathrm{Re}\,\lambda \leq \min\{a_\theta, b_\theta\};$$

here the uniform bound for the imaginary parts was proved in Theorem 5.10 and a_θ and $b_\theta = s_\theta + \|\alpha'\|$ are the right bounds for $\sigma(\mathcal{A}_\theta)$ derived in Theorems 4.6 and 5.10, respectively. In this section we prove that, apart from a small bounded set of values of $\|\alpha\|$ and $\|\alpha'\|$, we always have $a_\theta < b_\theta$. Hence, in general, a combination of the two eigenvalue estimates from Theorems 4.6 and 5.10 yields the best result.

First we present two auxiliary technical lemmas. The graphs of the functions introduced therein are displayed in Figure 3 below.

LEMMA 6.1. *The functions*

$$k_1 : [0, \sqrt{\lambda_1(\infty)}] \to [0,\infty), \quad k_1(t) := \frac{\lambda_1(\infty)+\lambda_1(\theta)}{2} - \sqrt{\Big(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\Big)^2 + \lambda_1(\theta)\, t^2},$$

$$k_2 : (0, \sqrt{\lambda_1(\infty)}] \to [0,\infty), \qquad k_2(t) := \frac{\lambda_1(\infty) - t^2}{t}\sqrt{\lambda_1(\theta)},$$

$$k_3 : (0, \sqrt{\lambda_1(\infty)+\lambda_1(\theta)}] \to [0,\infty), \quad k_3(t) := \frac{\lambda_1(\infty) - t^2 + \lambda_1(\theta)}{t} 2\sqrt{\lambda_1(\theta)}$$

are continuous and strictly decreasing with

$$k_1(\sqrt{\lambda_1(\infty)}) = k_2(\sqrt{\lambda_1(\infty)}) = k_3(\sqrt{\lambda_1(\infty)+\lambda_1(\theta)}) = 0.$$

They satisfy the inequalities

$$\begin{aligned} &k_1 < k_2 \quad \text{on} \quad (0, \sqrt{\lambda_1(\infty)}),\\ &k_2 < k_3 \quad \text{on} \quad (0, \sqrt{\lambda_1(\infty)+\lambda_1(\theta)}),\end{aligned} \tag{6.1}$$

where we have set $k_2(t) := 0$ *for* $t \in (\sqrt{\lambda_1(\infty)}, \sqrt{\lambda_1(\infty)+\lambda_1(\theta)}]$. *Moreover,*

$$\begin{aligned} b_\theta \le 0 \quad &\iff \quad \|\alpha\| \le \sqrt{\lambda_1(\infty)}, \qquad \|\alpha'\| \le k_1(\|\alpha\|),\\ a_\theta \le 0 \quad &\iff \quad \|\alpha\| \le \sqrt{\lambda_1(\infty)}, \qquad \|\alpha'\| \le k_2(\|\alpha\|),\\ 0 < a_\theta \le \lambda_1(\theta) \quad &\iff \quad \|\alpha\| \le \sqrt{\lambda_1(\infty)+\lambda_1(\theta)}, \quad \|\alpha'\| \le k_3(\|\alpha\|).\end{aligned}$$

Proof. The claims for the functions k_1, k_2, and k_3 are easy to check. The last two equivalences follow if we use the formulas for a_θ in Theorem 4.6 and solve the corresponding inequalities on the left-hand sides for $\|\alpha'\|$. Further, the condition $b_\theta = s_\theta + \|\alpha'\| \le 0$ is satisfied if and only if $s_\theta \le 0$ and $\|\alpha'\| < -s_\theta$. It remains to use the respective formula for s_θ from Proposition 5.8 to obtain the first equivalence. ∎

LEMMA 6.2. *Define two functions* $k_4^\pm$ *implicitly by the equation*

$$\sqrt{\Big(\frac{\lambda_1(\infty)-\lambda_1(\theta)}{2}\Big)^2 + \lambda_1(\theta)t^2} + k_4^\pm(t) = \frac{t^2}{2} + \sqrt{\Big(\frac{\lambda_1(\infty)-\lambda_1(\theta)-t^2}{2}\Big)^2 + \sqrt{\lambda_1(\theta)}\, t\, k_4^\pm(t)}$$

for $t \in [0, \sqrt{\lambda_1(\infty)}]$ *and let*

$$k_5 : \left[\sqrt{\lambda_1(\infty)}, \frac{\sqrt{\lambda_1(\theta)}}{2} + \sqrt{\lambda_1(\infty) - \frac{3\lambda_1(\theta)}{4}}\right] \to [0,\infty),$$
$$k_5(t) := \lambda_1(\infty) - \lambda_1(\theta) + t\sqrt{\lambda_1(\theta)} - t^2.$$

Then the graphs of k_4^-, k_4^+, *and* k_5 *form a continuous curve* Γ^{ex} *connecting the points*

$$C_1 := \Big(\sqrt{\lambda_1(\infty)}, 0\Big), \quad C_2 := \Big(\sqrt{\lambda_1(\infty)}, \sqrt{\lambda_1(\infty)\lambda_1(\theta)} - \lambda_1(\theta)\Big),$$
$$C_3 := \left(\frac{\sqrt{\lambda_1(\theta)}}{2} + \sqrt{\lambda_1(\infty) - \frac{3\lambda_1(\theta)}{4}}, 0\right).$$

Let Δ^{ex} *be the open bounded set surrounded by the curve* Γ^{ex} *and the segment* $\overline{C_1C_3}$ *on the ordinate axis. Then*

$$\Delta^{\text{ex}} \subset \big\{ \big(\|\alpha\|, \|\alpha'\|\big) : k_2(\|\alpha\|) < \|\alpha'\| < k_3(\|\alpha\|), \ \|\alpha'\| < \sqrt{\lambda_1(\infty)\lambda_1(\theta)} - \lambda_1(\theta) \big\}$$

and, for $0 < a_\theta \le \lambda_1(\theta)$*, we have*

$$\begin{aligned} b_\theta = a_\theta &\iff \big(\|\alpha\|, \|\alpha'\|\big) \in \Gamma^{\text{ex}},\\ b_\theta < a_\theta &\iff \big(\|\alpha\|, \|\alpha'\|\big) \in \Delta^{\text{ex}}. \end{aligned}$$

Proof. It is easy to see that the graph of the function k_5, which lies to the right of the vertical line $\|\alpha\| = \sqrt{\lambda_1(\infty)}$, is strictly decreasing from the point C_2 on this line to C_3, the zero of k_5. Moreover, if $0 < a_\theta \le \lambda_1(\theta)$ and $\|\alpha\| \ge \sqrt{\lambda_1(\infty)}$, the formulas for a_θ in Theorem 4.6(ii) and $b_\theta = s_\theta + \|\alpha'\| = -\lambda_1(\infty) + \|\alpha\|^2 + \|\alpha'\|$ (see Proposition 5.8) show that $k_5 \ge 0$ on the interval where it is defined,

$$b_\theta \le a_\theta \iff \|\alpha'\| \le k_5(\|\alpha\|),$$

and that equality holds on the graph of k_5.

By elementary calculations (see Remark 6.3 below), one can show that the graphs of the implicitly defined functions k_4^- and k_4^+ form an arc from C_1 to C_2 lying to the left of the vertical line $\|\alpha\| = \sqrt{\lambda_1(\infty)}$. Furthermore, if $0 < a_\theta \le \lambda_1(\theta)$ and $\|\alpha\| \le \sqrt{\lambda_1(\infty)}$, the formulas for a_θ in Theorem 4.6(ii) and for $b_\theta = s_\theta + \|\alpha'\|$ with s_θ as in Proposition 5.8 show that

$$b_\theta \le a_\theta \iff k_4^-(\|\alpha\|) \le \|\alpha'\| \le k_4^+(\|\alpha\|)$$

and that equality holds on the graphs of $k_4^\pm$. □

Remark 6.3. The two functions $k_4^\pm$ in Lemma 6.2 can be calculated explicitly:

$$\begin{aligned} k_4^\pm(t) &= \frac{t^2}{2} + \frac{t}{2}\sqrt{\lambda_1(\theta)} - \sqrt{\Big(\frac{\lambda_1(\infty) - \lambda_1(\theta)}{2}\Big)^2 + \lambda_1(\theta)t^2}\\ &\pm \sqrt{\Big(\frac{\lambda_1(\infty) - \lambda_1(\theta) - t^2}{2}\Big)^2 + \frac{t^2}{4}\lambda_1(\theta) - t\sqrt{\lambda_1(\theta)}\left(\sqrt{\Big(\frac{\lambda_1(\infty) - \lambda_1(\theta)}{2}\Big)^2 + \lambda_1(\theta)t^2} - \frac{t^2}{2}\right)} \end{aligned}$$

for $t \in [0, \sqrt{\lambda_1(\infty)}]$ such that the term under the last square root is positive. The set of these t is a (small) interval of the form $[\mu, \sqrt{\lambda_1(\infty)}]$. A formula for μ may be found, e.g., with Maple, but it is extremely involved; a lower bound for μ is the point where k_2 attains the value $\sqrt{\lambda_1(\theta)\lambda_1(\infty)} - \lambda_1(\theta)$ (the height of Δ^{ex}):

$$\mu \ge -\frac{\sqrt{\lambda_1(\infty)} + \sqrt{\lambda_1(\theta)}}{2} + \sqrt{\Big(\frac{\sqrt{\lambda_1(\infty)} + \sqrt{\lambda_1(\theta)}}{2}\Big)^2 + \lambda_1(\infty)}. \tag{6.2}$$

PROPOSITION 6.4. *Let* $\theta \in [0, \infty]$. *Then the right bounds* a_θ *and* b_θ *for* $\sigma(\mathcal{A}_\theta)$ *established in Theorems* 4.6 *and* 5.10*, respectively, satisfy*

$$\begin{aligned} a_\theta < b_\theta &\iff \big(\|\alpha\|, \|\alpha'\|\big) \in \big([0,\infty) \times [0,\infty)\big) \setminus \Delta^{\text{ex}},\\ a_\theta = b_\theta &\iff \big(\|\alpha\|, \|\alpha'\|\big) \in \Gamma^{\text{ex}}, \end{aligned}$$

where the bounded set Δ^{ex} *and the curve* Γ^{ex} *are as in Lemma* 6.2*. Moreover,*

$$a_\theta < 0 < b_\theta \iff \|\alpha\|^2 < \lambda_1(\infty), \quad k_1(\|\alpha\|) < \|\alpha'\| < k_2(\|\alpha\|);$$

the last condition can be written equivalently as

$$\frac{\lambda_1(\infty) + \lambda_1(\theta)}{2} - \sqrt{\left(\frac{\lambda_1(\infty) - \lambda_1(\theta)}{2}\right)^2 + \lambda_1(\theta)\|\alpha\|^2} < \|\alpha'\| < \frac{\lambda_1(\infty) - \|\alpha\|^2}{\|\alpha\|}\sqrt{\lambda_1(\theta)}.$$

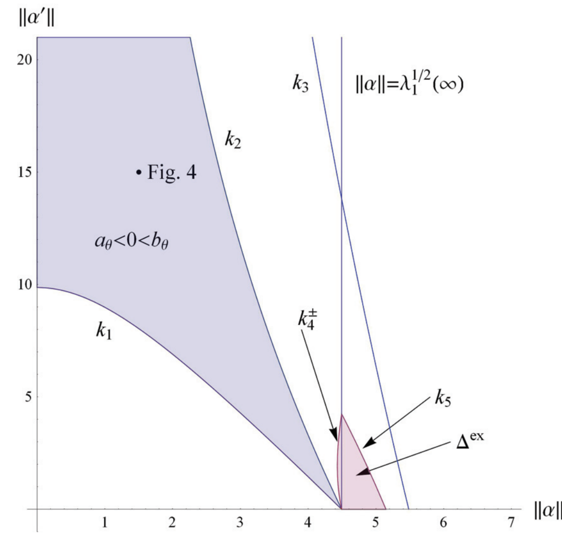

FIG. 3. *The graphs of the functions k_1, k_2, k_3, $k_4^{\pm}$, and k_5.*

Figure 3 illustrates the sets occurring in Proposition 6.4: the exceptional set Δ^{ex} where $b_\theta < a_\theta$ is the highlighted small bounded set between the graphs of k_2 and k_3; the set where $a_\theta < 0 < b_\theta$ is the highlighted unbounded set enclosed by the graphs of k_1 and k_2.

Remark 6.5. Proposition 6.4 shows, in particular, that we always have $a_\theta < b_\theta$ if one of the following holds:

(i) $\|\alpha'\| > \sqrt{\lambda_1(\infty)\lambda_1(\theta)} - \lambda_1(\theta)$,

(ii) $\|\alpha\| < -\dfrac{\sqrt{\lambda_1(\infty)}+\sqrt{\lambda_1(\theta)}}{2} + \sqrt{\Big(\dfrac{\sqrt{\lambda_1(\infty)}+\sqrt{\lambda_1(\theta)}}{2}\Big)^2 + \lambda_1(\infty)}$,

(iii) $\|\alpha\| > \dfrac{\sqrt{\lambda_1(\theta)}}{2} + \sqrt{\lambda_1(\infty) - \dfrac{3\lambda_1(\theta)}{4}}$.

This follows from Lemma 6.2 if we observe that the lower bound in (i) is the maximal value of $\|\alpha'\|$ in Δ^{ex}, the upper bound in (ii) is the lower bound for the left end-point μ of the domain of definition of the functions $k_4^{\pm}$ (see Remark 6.3), and the lower bound in (iii) is the right end-point of the domain of definition of the function k_5.

Proof of Proposition 6.4. In order to compare a_θ and b_θ, we have to distinguish the cases (i), (ii), and (iii) in Theorem 4.6 for a_θ and the two cases $\|\alpha\| \le \sqrt{\lambda_1(\infty)}$ and $\|\alpha\| > \sqrt{\lambda_1(\infty)}$ for $b_\theta = s_\theta + \|\alpha'\|$ according to the definition of s_θ in Proposition 5.8.

Elementary but lengthy and tedious calculations show that in the cases $a_\theta \le 0$ (Theorem 4.6(i)) and $a_\theta > \lambda_1(\theta)$ (Theorem 4.6(iii)), the equation $a_\theta = b_\theta$ has no solution and $a_\theta < b_\theta$. Moreover, Lemma 6.1 implies that the cases $a_\theta < 0$ and $b_\theta > 0$ appear simultaneously if and only if $\|\alpha\| < \sqrt{\lambda_1(\infty)}$ and $k_1(\|\alpha\|) < \|\alpha'\| < k_2(\|\alpha\|)$ since we have $k_1 < k_2$ by (6.1).

In the case $0 < a_\theta \le \lambda_1(\theta)$ (Theorem 4.6(ii)), Lemma 6.2 shows that $a_\theta = b_\theta$ on the curve Γ^{ex} and $a_\theta < b_\theta$ if and only if $(\|\alpha\|, \|\alpha'\|) \notin \Delta^{\mathrm{ex}}$. ∎

In Figure 4 the boundaries of the two spectral enclosures from Theorems 4.6 and 5.10 are displayed together; the set Σ_θ from Theorem 4.6 is the set with unbounded

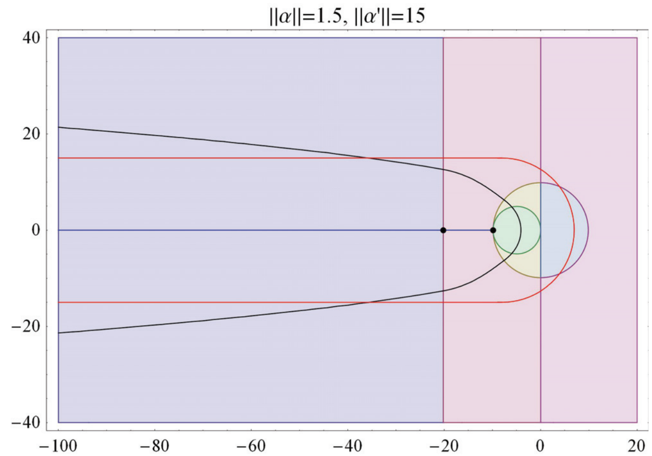

FIG. 4. *Spectral enclosures of Theorems* 4.6 *and* 5.10 *(*$a_\theta < 0 < b_\theta$*).*

imaginary part and smaller real part $\le a_\theta$, and the enclosing set from Theorem 5.10 is the set with bounded imaginary part and larger real part $\le b_\theta$. Here the values $\|\alpha\| = 1.5$ and $\|\alpha'\| = 15$ are chosen such that $a_\theta < 0 < b_\theta$; the corresponding point $(\|\alpha\|, \|\alpha'\|) = (1.5, 15)$ is marked in Figure 3 by a black dot.

Proposition 6.4 shows that, in general, the upper bound of Theorem 4.6 improves the estimate of Theorem 5.10; only for values $(\|\alpha\|, \|\alpha'\|) \in \Delta^{\mathrm{ex}}$ is the enclosure of Theorem 5.10 better. The following corollary is a direct consequence of all these results.

COROLLARY 6.6. *Let* $\theta \in [0, \infty]$, *let* a_θ, b_θ *be the bounds established in Theorems* 4.6 *and* 5.10, *respectively, and let* $\Delta^{\mathrm{ex}} \subset [0, \infty) \times [0, \infty)$ *be the bounded set defined in Lemma* 6.2. *Then every eigenvalue* λ *of the dynamo operator* $\mathcal{A}_\theta$ *satisfies*

$$|\mathrm{Im}\,\lambda| \le \|\alpha'\|, \quad \mathrm{Re}\,\lambda \le \begin{cases} a_\theta & \textit{if } \big(\|\alpha\|, \|\alpha'\|\big) \in \big([0,\infty) \times [0,\infty)\big) \setminus \Delta^{\mathrm{ex}}, \\ b_\theta & \textit{if } \big(\|\alpha\|, \|\alpha'\|\big) \in \Delta^{\mathrm{ex}}. \end{cases}$$

The case that the bound for the real part is less than 0 is of particular interest from the physical point of view since the onset of the dynamo effect requires supercritical modes, that is, eigenvalues with real part greater than 0.

Proposition 6.4 shows that if $a_\theta < 0$, then we always have $a_\theta < b_\theta$. In particular, this shows that Corollary 4.8 is stronger than Corollary 5.11; for the special case $\theta = l$ this gives the following theorem.

ANTIDYNAMO THEOREM. *The dynamo operator* $\mathcal{A}_l$ *has no spectrum in the closed right half-plane if*

$$\|\alpha\|^2 + \frac{\|\alpha\|\|\alpha'\|}{\sqrt{\lambda_1(l)}} < \lambda_1(\infty). \tag{6.3}$$

Note that since the above result has been obtained by operator theoretic estimates, a violation of the condition (6.3) provides only a necessary, not a sufficient, condition for the existence of supercritical dynamo regimes.

Remark 6.7. By Lemma 3.3, we know that $\lambda_1(l)$ and $\lambda_1(\infty)$ are the first nonzero zeros of the Bessel functions $J_{l-1/2}(\sqrt{\lambda})$ and $J_{l+1/2}(\sqrt{\lambda})$, respectively. If we indicate this dependence on the parameter $l \in \mathbb{N}$ for a moment and write $\lambda_1(l) = \lambda_1(l;l)$ and $\lambda_1(\infty) = \lambda_1(\infty, l)$, we see that $\lambda_1(l_1; l_1) < \lambda_1(l_2; l_2)$ and $\lambda_1(\infty; l_1) < \lambda_1(\infty; l_2)$ for $l_1, l_2 \in \mathbb{N}$ with $l_1 < l_2$ (see [1, equation 9.5.2]). Therefore, if condition (6.3) is satisfied for $l_1 \in \mathbb{N}$, then it is also satisfied for $l_2 \in \mathbb{N}$ with $l_2 > l_1$.

This provides an explanation of the numerical observation that, for increasing $\|\alpha\|$ and $\|\alpha'\|$, criticality usually starts from dipole modes ($l = 1$), and quadrupole and higher-degree modes ($l > 1$) become supercritical only for larger values of $\|\alpha\|$ and $\|\alpha'\|$ (see [22]).

7. Examples. In this section we illustrate our results by applying them to the physical dynamo problem (1.1), (1.2) and to the idealized dynamo problem (1.1), (1.3). Here we have to set $\theta = l$ and $\theta = \infty$, respectively, in the previous statements.

The simplest case of constant α, where all eigenvalues are real, already shows that the physical problem is far more complex than the idealized problem: explicit formulas for the eigenvalues are known for the latter, whereas the eigenvalues of the former are given only implicitly as the zeros of an equation involving 4 different Bessel functions.

Example 7.1. Let $\alpha \equiv \alpha_0$ be constant and consider the idealized dynamo problem (1.1), (1.3) for which $\theta = \infty$. The eigenvalues of this problem, or equivalently of the operator $\mathcal{A}_\infty$, can be calculated explicitly (see, e.g., [10]). In fact, since α is constant and hence $\alpha' \equiv 0$, the operator $\mathcal{A}_\infty$ is given by

$$\mathcal{A}_\infty = \begin{pmatrix} -A_\infty & \alpha_0 \\ \alpha_0 A_\infty & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_\infty) = \mathcal{D}(A_\infty) \oplus \mathcal{D}(A_\infty).$$

A point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_\infty$ if and only if there is a $y = (y_1, y_2)^{\mathrm{t}} \in \mathcal{D}(\mathcal{A}_\infty)$, $y \neq 0$, such that

$$\begin{aligned} (-A_\infty - \lambda)y_1 + \alpha_0 y_2 &= 0, \\ \alpha_0 A_\infty y_1 + (-A_\infty - \lambda)y_2 &= 0. \end{aligned}$$

Since $y_2 \in \mathcal{D}(A_\infty)$, the first equation yields $y_1 \in \mathcal{D}(A_\infty^2)$, and thus the eigenvalue equations are equivalent to the relations

$$\big((A_\infty + \lambda)^2 - \alpha_0^2 A_\infty\big)y_1 = 0, \quad \alpha_0 y_2 = (A_\infty + \lambda)y_1.$$

Hence $\lambda \in \sigma_{\mathrm{p}}(\mathcal{A}_\infty)$ if and only if $0 \in \sigma_{\mathrm{p}}\big((A_\infty + \lambda)^2 - \alpha_0^2 A_\infty\big)$. Now the spectral mapping theorem shows that

$$\sigma(\mathcal{A}_\infty) = \big\{-\lambda_n(\infty) \pm \alpha_0 \sqrt{\lambda_n(\infty)} : n \in \mathbb{N}\big\},$$

where $\lambda_n(\infty)$, $n \in \mathbb{N}$, are the eigenvalues of the Bessel operator A_∞ with Dirichlet boundary conditions (that is, $\lambda_n(\infty)$ is the nth zero of $\lambda \mapsto J_{l+1/2}(\sqrt{\lambda})$; see Definition 2.1 and Lemma 3.3(i)). Thus we have

$$\max \sigma(\mathcal{A}_\infty) = -\lambda_1(\infty) + |\alpha_0|\sqrt{\lambda_1(\infty)} \tag{7.1}$$

and the estimate

$$\max \sigma(\mathcal{A}_\infty) = -\left(\sqrt{\lambda_1(\infty)} - \frac{|\alpha_0|}{2}\right)^2 + \frac{\alpha_0^2}{4} \leq \frac{\alpha_0^2}{4}. \tag{7.2}$$

The abstract eigenvalue estimates in Theorems 4.6 and 5.10, combined with Remark 5.9(iii), show that, for $\theta = \infty$ and $\|\alpha'\| = 0$,

$$\max \sigma(\mathcal{A}_\infty) \le a_\infty = b_\infty = -\lambda_1(\infty) + |\alpha_0| \max\left\{|\alpha_0|, \sqrt{\lambda_1(\infty)}\right\}; \tag{7.3}$$

note that $\mathcal{B}_\infty = \mathcal{S}_\infty$ and thus $b_\infty = s_\infty$ because α is constant (see Proposition 5.1 and Theorem 5.3).

Comparing (7.1) and (7.3), we see that the abstract upper bounds a_∞ and b_∞ for $\sigma(\mathcal{A}_\infty)$ in Theorems 4.6 and 5.10 are sharp for the case $\|\alpha\| \le \sqrt{\lambda_1(\infty)}$. Moreover, (7.2) coincides with the abstract estimate proved in Proposition 5.7 for constant α.

For physical boundary conditions, even in the case of constant α, only implicit formulas are known for the eigenvalues $\lambda_n(\mathcal{A}_l)$ of $\mathcal{A}_l$ (cf. [15, section 14], in particular, [15, (14.39), (14.41), (14.42)]).

Example 7.2. Let $\alpha \equiv \alpha_0$ be constant and consider the physical dynamo problem (1.1), (1.2) for which $\theta = l$. There are no explicit formulas known for the eigenvalues of this problem, or equivalently of the operator $\mathcal{A}_l$ given by

$$\mathcal{A}_l = \begin{pmatrix} -A_l & \alpha_0 \\ \alpha_0 A_l & -A_\infty \end{pmatrix}, \quad \mathcal{D}(\mathcal{A}_l) = \mathcal{D}(A_l) \oplus \mathcal{D}(A_\infty).$$

Note that here the operators A_l and A_∞ are given by the same differential expression $\tau = -\partial_r^2 + l(l+1)/r^2$, but, unlike the idealized case, they are equipped with different boundary conditions (see Definition 3.1).

A point $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{A}_l$ if and only if there exists a $y = (y_1, y_2)^{\mathrm{t}} \in L_2(0,1) \oplus L_2(0,1)$, $y \ne 0$, with $y_i, y_i' \in \mathrm{AC}_{\mathrm{loc}}((0,1])$, $\tau y_i \in L_2(0,1)$ for $i = 1, 2$, such that

$$(-\tau - \lambda) y_1 + \alpha_0 y_2 = 0, \tag{7.4}$$
$$\alpha_0 \tau y_1 + (-\tau - \lambda) y_2 = 0, \tag{7.5}$$

and

$$y_1'(1) + l y_1(1) = 0, \quad y_2(1) = 0. \tag{7.6}$$

Clearly, (7.4), (7.5) are equivalent to the relations

$$\big((\tau + \lambda)^2 - \alpha_0^2 \tau\big) y_1 = 0, \tag{7.7}$$
$$\alpha_0 y_2 = (\tau + \lambda) y_1. \tag{7.8}$$

By Proposition 5.7, we already know that for constant α all eigenvalues are contained in the interval $(-\infty, \alpha_0^2/4)$. For $\lambda \le \alpha_0^2/4$, we denote by

$$v_\pm(\lambda) := -\lambda + \frac{\alpha_0^2}{2} \pm \alpha_0 \sqrt{\frac{\alpha_0^2}{4} - \lambda} = \left(\frac{\alpha_0}{2} \pm \sqrt{\frac{\alpha_0^2}{4} - \lambda}\right)^2 \ge 0$$

the two solutions of the quadratic equation $(v + \lambda)^2 - \alpha_0^2 v = 0$. Then we can factorize the differential expression in (7.7) as

$$(\tau + \lambda)^2 - \alpha_0^2 \tau = \big(\tau - v_+(\lambda)\big)\big(\tau - v_-(\lambda)\big). \tag{7.9}$$

Since the differential expression τ is in the limit point case at 0 (see the proof of Proposition 3.1), every fundamental system of a differential equation $(\tau - \mu)x = 0$ has exactly one solution in $L_2(0,1)$. This and relation (7.9) imply that every fundamental system of the differential equation (7.7) has exactly two solutions in $L_2(0,1)$, which may be chosen as the solutions $y_{1,l}^{\pm} \in L_2(0,1)$ of $\big(\tau - v_{\pm}(\lambda)\big)x = 0$ given by (see (3.4))

$$y_{1,l}^{\pm}(r,\lambda) = \sqrt{\frac{\pi}{2}}\sqrt{r\,k_{\pm}(\lambda)}\,J_{l+1/2}(r\,k_{\pm}(\lambda)), \quad r \in [0,1],$$

with

$$k_{\pm}(\lambda) := \sqrt{v_{\pm}(\lambda)} = \frac{\alpha_0}{2} \pm \sqrt{\frac{\alpha_0^2}{4} - \lambda}. \tag{7.10}$$

Note that $k_+(\lambda) + k_-(\lambda) = \alpha_0$ and $k_+(\lambda)k_-(\lambda) = \lambda$; that is, $k_{\pm}(\lambda)$ are the solutions of the quadratic equation $k^2 - \alpha_0 k + \lambda = 0$. By definition of $v_{\pm}(\lambda)$ and $k_{\pm}(\lambda)$, we have $(v_{\pm}(\lambda) + \lambda)^2 = \alpha_0^2 v_{\pm}(\lambda) = \alpha_0^2 k_{\pm}(\lambda)^2$. Thus (7.8) implies that

$$y_{2,l}^{\pm}(r,\lambda) = \frac{1}{\alpha_0}(\tau + \lambda)y_{1,l}^{\pm}(r,\lambda) = \frac{1}{\alpha_0}\big(v_{\pm}(\lambda) + \lambda\big)y_{1,l}^{\pm}(r,\lambda) = k_{\pm}(\lambda)y_{1,l}^{\pm}(r,\lambda).$$

Therefore a fundamental system $\{y_l^+, y_l^-\}$ in $L_2(0,1) \oplus L_2(0,1)$ of the system (7.4), (7.5) is given by

$$y_l^+(r,\lambda) = \begin{pmatrix} 1 \\ k_+(\lambda) \end{pmatrix} y_{1,l}^+(r,\lambda), \quad y_l^-(r,\lambda) = \begin{pmatrix} 1 \\ k_-(\lambda) \end{pmatrix} y_{1,l}^-(r,\lambda).$$

If we take into account the boundary conditions and use relation (3.10), we see that there exists a nonzero solution of the boundary eigenvalue problem (7.4), (7.5), (7.6) if and only if

$$\begin{vmatrix} k_+(\lambda)y_{1,l-1}^+(1) & k_-(\lambda)y_{1,l-1}^-(1) \\ k_+(\lambda)y_{1,l}^+(1) & k_-(\lambda)y_{1,l}^-(1) \end{vmatrix} = 0$$

or, equivalently,

$$J_{l-1/2}\big(k_+(\lambda)\big)J_{l+1/2}\big(k_-(\lambda)\big) - J_{l+1/2}\big(k_+(\lambda)\big)J_{l-1/2}\big(k_-(\lambda)\big) = 0 \tag{7.11}$$

with $k_{\pm}(\lambda)$ given by (7.10).

This shows that the eigenvalues of the physical dynamo problem (1.1), (1.2) are only given implicitly as the solutions of (7.11). As a consequence, even in the simplest case of constant α, it is difficult to obtain any analytic information about the eigenvalues of $\mathcal{A}_l$.

The estimates in Theorems 4.6 and 5.10, however, provide the global bound

$$\max \sigma(\mathcal{A}_l) \le \min\{a_l, b_l\} = b_l,$$

where

$$b_l = \begin{cases} -\dfrac{\lambda_1(\infty)+\lambda_1(l)}{2} + \sqrt{\left(\dfrac{\lambda_1(\infty)-\lambda_1(l)}{2}\right)^2 + \lambda_1(l)|\alpha_0|^2} & \text{if } |\alpha_0|^2 \le \lambda_1(\infty), \\ -\lambda_1(\infty) + |\alpha_0|^2 & \text{if } |\alpha_0|^2 > \lambda_1(\infty). \end{cases}$$

In addition, the estimate in Proposition 5.7 for constant α yields the bound

$$\max \sigma(\mathcal{A}_l) \le \frac{|\alpha_0|^2}{4},$$

which was already used to establish the eigenvalue relation (7.11).

Moreover, since $\mathcal{B}_l = \mathcal{S}_l$ for constant α, Proposition 5.13 yields that the number k_0 of positive eigenvalues of $\mathcal{A}_l$ coincides with the number of positive eigenvalues of the operator $-A_\infty + \alpha_0^2$ or, equivalently,

$$\#\big(\sigma(\mathcal{A}_l)\cap(0,\infty)\big) = \#\big(\sigma(A_\infty)\cap(0,\alpha_0^2)\big);$$

if we enumerate the eigenvalues of $\mathcal{A}_l$ and of A_∞ as $0<\lambda_{k_0}(\mathcal{A}_l)\le\cdots\le\lambda_1(\mathcal{A}_l)$ and $\lambda_1(\infty)\le\cdots\le\lambda_{k_0}(\infty)<|\alpha_0|^2$, respectively, we have the estimate

$$0<\lambda_j(\mathcal{A}_l)\le -\lambda_j(\infty)+|\alpha_0|^2, \quad j=1,2,\dots,k_0.$$

Finally, we consider the physical dynamo problem with nonconstant helical turbulence function α. As shown for the first time in [22], there exist special α-profiles which provide dipole-dominated criticality for oscillatory dynamo regimes (that is, nonreal eigenvalues passing from the left to the right half-plane first for dipole modes ($l=1$) and later for quadrupole and higher-degree modes ($l>1$)). Such regimes are of high physical interest due to their close relation to polarity reversal processes of the magnetic field (see [23], [24]).

Example 7.3. Consider the physical dynamo problem (1.1), (1.2) for $l=1$ with α given by

$$\alpha(r) = C(-21.46 + 426.41\,r^2 - 806.73\,r^3 + 392.28r^4), \quad r\in[0,1], \tag{7.12}$$

where $C\ge 0$ is a constant. For $C=0$ the eigenvalues are real and coincide with the interlacing sequences of eigenvalues of the Bessel operators $-A_\infty$ and $-A_l$. According to the numerical computations in [22], if C increases, the largest two eigenvalues merge at $C=0.818$ and form a complex conjugate pair for $0.818<C<1.097$. In between, at $C=1$ this pair crosses the imaginary axis.

According to the abstract result of Corollary 4.11, the largest two eigenvalues can meet only if condition (4.20) is violated. Indeed, for $l=1$, we have

$$\lambda_1(l) = \pi^2 \approx 9.87, \quad \lambda_1(\infty)\approx 20.19,$$

and, for α given by (7.12) with $C=0.818$,

$$\|\alpha\| = \max_{r\in[0,1]}|\alpha(r)| \approx 17.55, \quad \|\alpha'\| \approx \max_{r\in[0,1]}|\alpha'(r)|\approx 71.36,$$

and hence condition (4.20) does not hold since

$$\|\alpha\|^2 + \frac{\|\alpha\|\,\|\alpha'\|}{\sqrt{\lambda_1(\infty)}} \approx 586.72 > 1.32 \approx \frac{\big(\lambda_1(\infty)-\lambda_1(\theta)\big)^2}{4\lambda_1(\infty)}.$$

Acknowledgments. We would like to thank Frank Stefani and Gunter Gerbeth (Research Center Dresden-Rossendorf) for useful discussions on the physics of dynamos and for sharing their numerical data with us. Special thanks also go to Markus Wagenhofer (now Psylock, Regensburg), who performed various numerical calculations while this manuscript was being prepared.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1964.

[2] N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space*, Dover, New York, 1993.

[3] S. Albeverio, R. Hryniv, and Y. Mykytyuk, *Inverse spectral problems for Bessel operators*, J. Differential Equations, 241 (2007), pp. 130–159.

[4] V. I. Arnold and B. A. Khesin, *Topological Methods in Hydrodynamics*, Appl. Math. Sci. 125, Springer-Verlag, New York, 1998.

[5] M. S. Birman and M. Z. Solomjak, *Spectral Theory of Selfadjoint Operators in Hilbert Space*, Math. Appl. (Soviet Ser.), D. Reidel, Dordrecht, The Netherlands, 1987.

[6] C. Chicone, Y. Latushkin, and S. Montgomery-Smith, *The spectrum of the kinematic dynamo operator for an ideally conducting fluid*, Comm. Math. Phys., 173 (1995), pp. 379–400.

[7] D. Eschwé and M. Langer, *Variational principles for eigenvalues of self-adjoint operator functions*, Integral Equations Operator Theory, 49 (2004), pp. 287–321.

[8] F. Gesztesy and M. Mitrea, *Self-adjoint Extensions of the Laplacian and Kreĭn-type Resolvent Formulas in Nonsmooth Domains*, preprint, 2009; available online from http://arxiv.org/PS_cache/arxiv/pdf/0907/0907.1750v1.pdf.

[9] S. K. Godunov, *Modern Aspects of Linear Algebra*, Transl. Math. Monogr. 175, AMS, Providence, RI, 1998.

[10] U. Günther and O. N. Kirillov, *A Krein space related perturbation theory for MHD α^2-dynamos and resonant unfolding of diabolical points*, J. Phys. A, 39 (2006), pp. 10057–10076.

[11] U. Günther and F. Stefani, *Isospectrality of spherical MHD dynamo operators: Pseudo-Hermiticity and a no-go theorem*, J. Math. Phys., 44 (2003), pp. 3097–3111.

[12] D. Hinrichsen and A. J. Pritchard, *Mathematical Systems Theory.* I, *Modelling, State Space Analysis, Stability and Robustness*, Texts Appl. Math. 48, Springer-Verlag, Berlin, 2005.

[13] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed., Grundlehren Math. Wiss. 132, Springer-Verlag, Berlin, 1976.

[14] O. N. Kirillov, U. Günther, and F. Stefani, *Determining role of Kreĭn signature for three-dimensional Arnold tongues of oscillatory dynamos*, Phys. Rev. E, 79 (2009), article 016205.

[15] F. Krause and K.-H. Rädler, *Mean-field Magnetohydrodynamics and Dynamo Theory*, Akademie-Verlag, Berlin, Pergamon Press, Oxford, 1980.

[16] M. G. Kreĭn, *Concerning the resolvents of an Hermitian operator with the deficiency-index* (m, m), C. R. (Doklady) Acad. Sci. URSS (N.S.), 52 (1946), pp. 651–654.

[17] M. G. Kreĭn, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications.* II, Mat. Sbornik N.S., 21 (63) (1947), pp. 365–404.

[18] A. S. Markus, *Introduction to the Spectral Theory of Polynomial Operator Pencils*, Transl. Math. Monogr. 71, AMS, Providence, RI, 1988.

[19] M. A. Naĭmark, *Linear Differential Operators. Part* II: *Linear Differential Operators in Hilbert Space*, Frederick Ungar, New York, 1968.

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* IV. *Analysis of Operators*, Academic Press, New York, 1978.

[21] M. Reed and B. Simon, *Methods of Modern Mathematical Physics.* I. *Functional Analysis*, 2nd ed., Academic Press, New York, 1980.

[22] F. Stefani and G. Gerbeth, *Oscillatory mean-field dynamos with spherically symmetric, isotropic alpha*, Phys. Rev. E, 67 (2003), article 027302.

[23] F. Stefani and G. Gerbeth, *Asymmetric polarity reversals, bimodal field distribution, and coherence resonance in a spherically symmetric mean-field dynamo model*, Phys. Rev. Lett., 94 (2005), article 184506.

[24] F. Stefani, G. Gerbeth, U. Günther, and M. Xu, *Why dynamos are prone to reversals*, Earth and Planetary Sci. Lett., 243 (2006), pp. 828–840.

[25] E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations. Part* I, 2nd ed., Clarendon Press, Oxford, 1962.

[26] C. Tretter, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008.

[27] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil* II: *Anwendungen*, Math. Leitfäden, B. G. Teubner, Stuttgart, 2003.

[28] A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues*, Math. Sci. Engrg. 89, Academic Press, New York, 1972.