Theory of the FEL: classical and quantum aspects

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Overview

- semiclassical theory of the laser
- basic elements of FEL
- classical theory
- quantum theory and Quantum FEL
  - single-particle theory
  - many-particle theory

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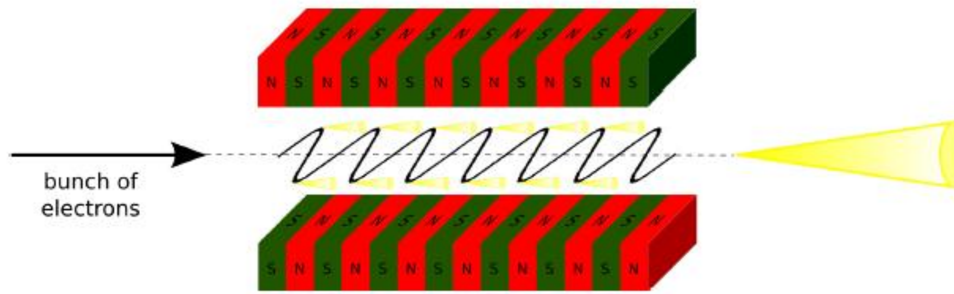
Free Electron Laser (FEL)



Classical Laser

PHYSICAL REVIEW A VOLUME 5, NUMBER 3 MARCH 1972

Classical Laser*

Matthew Borenstein† and Willis E. Lamb, Jr.
Yale University, New Haven, Connecticut 06520
(Received 26 July 1971)

In this paper a completely classical model for laser action is discussed. An active medium consisting of classical anharmonic oscillators interacts with a classical electromagnetic field in a resonant cavity. Comparison with the case of a medium consisting of harmonic oscillators shows the significance of nonlinearities for producing self-sustained oscillations in the radiation field. The results for the classical model are found to be similar to those for a semiclassical model of the ammonia-beam maser. The conclusion is that laser action is not intrinsically a quantum-mechanical effect. The classical-laser theory as given in this paper can also be applied to the case of the electron-cyclotron maser.

The following Letter should have appeared in the 1 November 1976 issue. We regret that a misunderstanding resulted in publication of reference material instead of the submitted manuscript. See Erratum, this issue, page 1368.

Strong-Signal Theory of a Free-Electron Laser*

F. A. Hopf, P. Meystre, M. O. Scully
Department of Physics and Optical Sciences Center, University of Arizona, Tucson, Arizona 85721

and

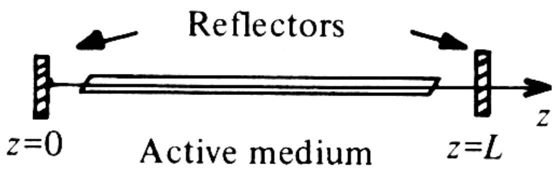
W. H. Louisell
Department of Physics, University of Southern California, Los Angeles, California 90007
(Received 13 August 1976)

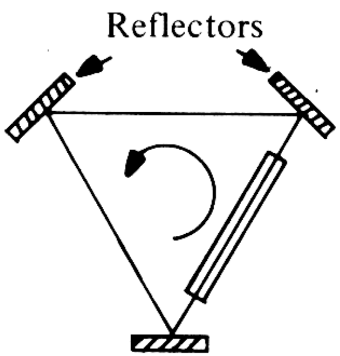
The strong-signal regime of a free-electron laser is analyzed in terms of a set of "generalized Bloch equations." We show that for current free-electron-laser configurations the saturation will be reached for a field on the order of 10^7 V/m, with an efficiency at saturation of 5×10^{-3}. However, a strong reshaping of the electron distribution may alter the efficiency of free-electron lasers in cases where the electron beam is recycled from one shot to the next.

CLASSICAL THEORY OF A FREE-ELECTRON LASER. F. A. Hopf, P. Meystre, M. O. Scully, and W. H. Louisell [Phys. Rev. Lett. 37, 1215 (1976)].

This paper was printed as a result of a misunderstanding, and duplicates material published elsewhere. Persons wishing to refer to it should cite the original publication, Opt. Commun. 18, 413 (1976), and *not* this journal.

Semiclassical laser theory (Lamb)





$$\mathbf{E}(\mathbf{r}, t) \xrightarrow[\text{mechanics}]{\text{quantum}} \langle \mathbf{p}_i \rangle \xrightarrow[\text{summation}]{\text{statistical}} \mathbf{P}(\mathbf{r}, t) \xrightarrow[\text{equations}]{\text{Maxwell's}} \mathbf{E}'(\mathbf{r}, t)$$

self-consistency

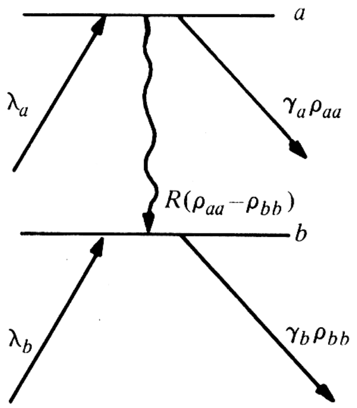
Semiclassical laser theory

Maxwell: $\Box \boldsymbol{E} = -\ddot{\boldsymbol{P}}$

$$\boldsymbol{P} = \sum_a \int_{-\infty}^{t} \mathrm{d}t_0 \, \lambda_a(z, t_0) \langle e\boldsymbol{r} \rangle$$

$\uparrow$

$\hat{\rho}$



Bloch: $\dot{\hat{\rho}} = -\frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\rho}] + \dots$

$\uparrow$

$\sim \boldsymbol{E}$

Classical theory of FEL

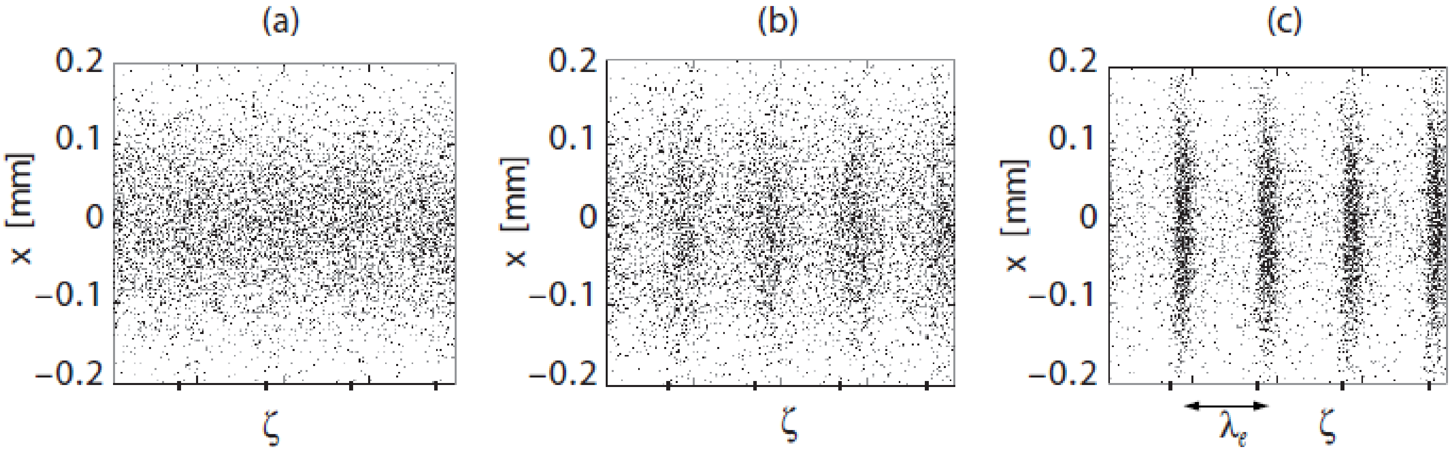
Maxwell: $\square A_L = \mu_0 j$

$$j = e\,\dot{\xi}$$

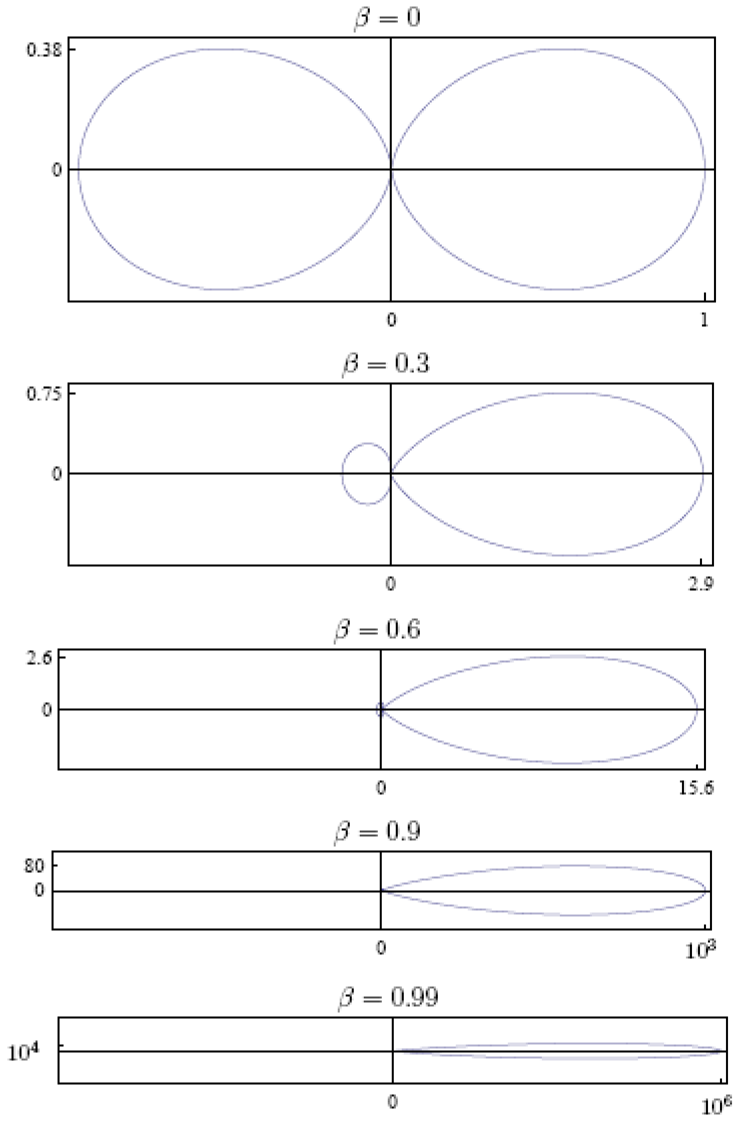
Pendulum: $\ddot{\xi} + \Omega^2 \sin(K\xi) = 0$

$$\uparrow$$

$$\sim A_L$$



Synchrotron radiation: emission angle



$$\theta_{\max} = \frac{1}{\gamma} = \sqrt{1 - \beta^2}$$

$$\beta = \frac{v}{c}$$

Emitted radiation: wavelength

$$\lambda'_W = \frac{1}{\gamma}\lambda_W$$

rest frame ↗ ↖ lab frame

$$\lambda'_L = \lambda'_W$$

$$\lambda_L = \lambda'_L \sqrt{\frac{1-\beta}{1+\beta}}$$

$$\beta = \frac{v}{c} \cong 1$$

$$\lambda_L \cong \frac{1}{2\gamma^2}\lambda_W$$

FEL: classical theory

$$H = \sqrt{(\mathbf{p} - e\mathbf{A})^2 c^2 + (m_0 c^2)^2}$$

$$\mathbf{A} = \mathbf{A}_{\mathrm{W}} + \mathbf{A}_{\mathrm{L}}$$

$$\mathbf{A}_{\mathrm{L}} = \mathcal{A}_{\mathrm{L}} \mathbf{e}\, e^{-\mathrm{i}(\omega_{\mathrm{L}} t - k_{\mathrm{L}} z)} + \mathrm{c.c.}$$

$$\mathbf{A}_{\mathrm{W}} = \mathcal{A}_{\mathrm{W}} \mathbf{e}\, e^{-\mathrm{i}(\omega_{\mathrm{W}} t + k_{\mathrm{W}} z)} + \mathrm{c.c.}$$

(Weizsäcker-Williams approximation)

612

Ausstrahlung bei Stößen sehr schneller Elektronen.

Von **C. F. v. Weizsäcker**, zur Zeit in Kopenhagen.

(Eingegangen am 28. Februar 1934.)

Die Ausstrahlung wird in dem Koordinatensystem berechnet, in dem d
Elektron anfangs ruht. Das Coulombfeld des „vorbeifliegende
Atomkerns läßt sich dann als Strahlungsfeld auffassen; die
Streuung dieser „Strahlung“ nach der Klein-Nishina-Fo
ist innerhalb der verwendeten Näherung mit dem frühere
Seine Herleitung und die Voraussetzungen für seine Gü
diskutiert.

Det Kgl. Danske Videnskabernes Selskab.
Mathematisk-fysiske Meddelelser. **XIII**, 4.

CORRELATION OF CERTAIN COLLISION PROBLEMS WITH RADIATION THEORY

BY

E. J. WILLIAMS

KØBENHAVN
LEVIN & MUNKSGAARD
EJNAR MUNKSGAARD
1935

Reduction to 1-D problem

$$\mathbf{A}_{\mathrm{W}} = \mathbf{A}_{\mathrm{W}}(z)$$

$$\Rightarrow \quad H = H(z)$$

$$\mathbf{A}_{\mathrm{L}} = \mathbf{A}_{\mathrm{L}}(z)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = 0$$

$\uparrow$

canonical momentum

$$H \cong \sqrt{\left[\mathbf{p}_z - e\left(\mathbf{A}_{\mathrm{L}} + \mathbf{A}_{\mathrm{W}}\right)\right]^2 c^2 + (m_0 c^2)^2}$$

Ponderomotive potential

$$H = \sqrt{(\mathbf{p}_z - e\mathbf{A})^2 c^2 + (m_0 c^2)^2}$$

$$\uparrow$$

$$p_z{}^2 - 2e\mathbf{p}_z \cdot \mathbf{A} + e^2 \mathbf{A}^2$$

$$\uparrow \qquad \uparrow$$

(Coulomb gauge) $\quad = 0$

$$\mathbf{A}_\mathrm{W}{}^2 + 2\mathbf{A}_\mathrm{W} \cdot \mathbf{A}_\mathrm{L} + \mathbf{A}_\mathrm{L}{}^2$$

$$H = \sqrt{p_z{}^2 c^2 + (mc^2)^2 + 2e^2 c^2 \mathbf{A}_\mathrm{W} \cdot \mathbf{A}_\mathrm{L}}$$

$$m^2 = m_0{}^2 \left(1 + \frac{e^2}{m_0{}^2 c^4} \mathbf{A}_\mathrm{W}\right)^2$$

Bambini-Renieri frame: $k_\mathrm{L} = k_\mathrm{W} = k$

$$\mathbf{A}_\mathrm{W} \cdot \mathbf{A}_\mathrm{L} \sim \exp\left[-\mathrm{i}(\omega_\mathrm{L} - \omega_\mathrm{W})t + \mathrm{i}(k_\mathrm{L} + k_\mathrm{W})z\right]$$

$$\mathbf{A}_\mathrm{W} \cdot \mathbf{A}_\mathrm{L} \sim \mathrm{e}^{2\mathrm{i}kz}$$

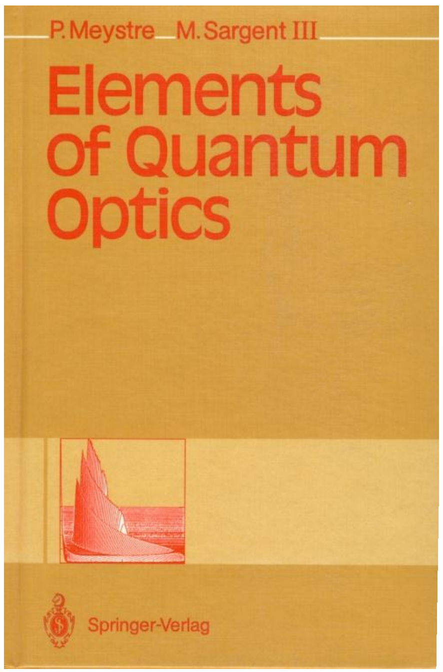
$$H = mc^2 \sqrt{1 + \left(\frac{p'_z}{mc}\right)^2 + \tilde{\mathcal{A}}\mathrm{e}^{2\mathrm{i}kz} + \mathrm{c.c.}}$$

$$H \cong \frac{{p'_z}^2}{2m} + \mathcal{A}\mathrm{e}^{2\mathrm{i}kz} + \mathrm{c.c.}$$

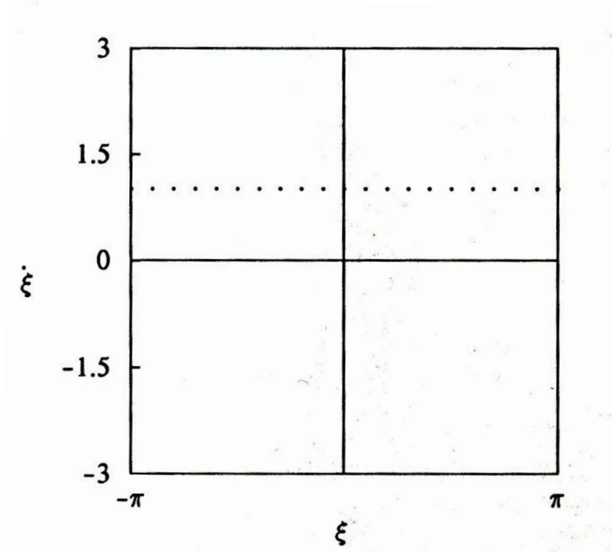
$$\uparrow$$

$$\sim \mathbf{A}_\mathrm{W} \cdot \mathbf{A}_\mathrm{L}$$

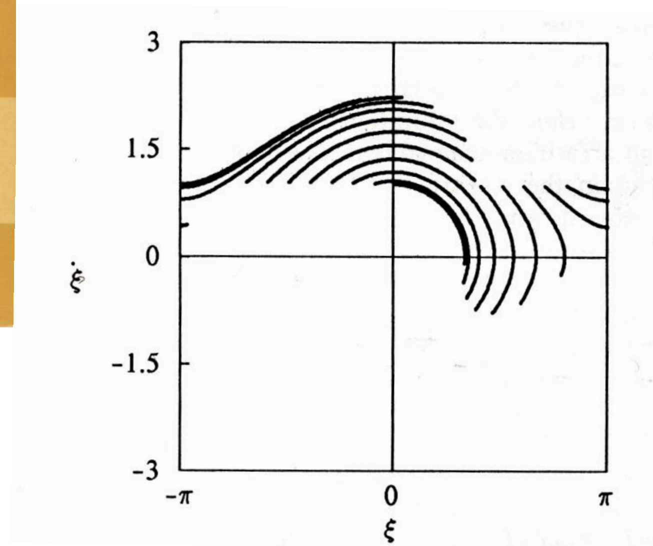
Bunching



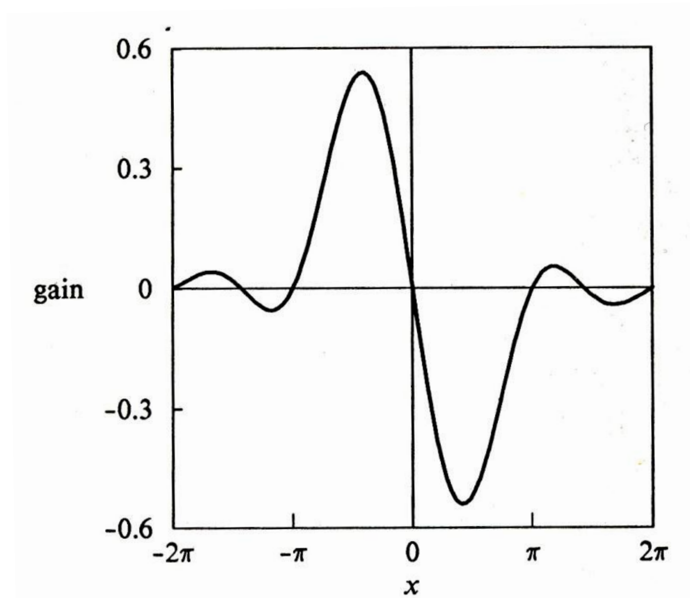
P. Meystre M. Sargent III
Elements of Quantum Optics
Springer-Verlag



3
1.5
0
-1.5
-3
$\dot{\xi}$
$-\pi$
π
ξ



3
1.5
0
-1.5
-3
$\dot{\xi}$
$-\pi$
0
π
ξ



0.6
0.3
gain
0
-0.3
-0.6
-2π
$-\pi$
0
π
2π
x

FEL: Hamiltonian and states

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hbar\,\omega\left(\hat{a}_L^\dagger \hat{a}_L + \hat{a}_W^\dagger \hat{a}_W\right)$$

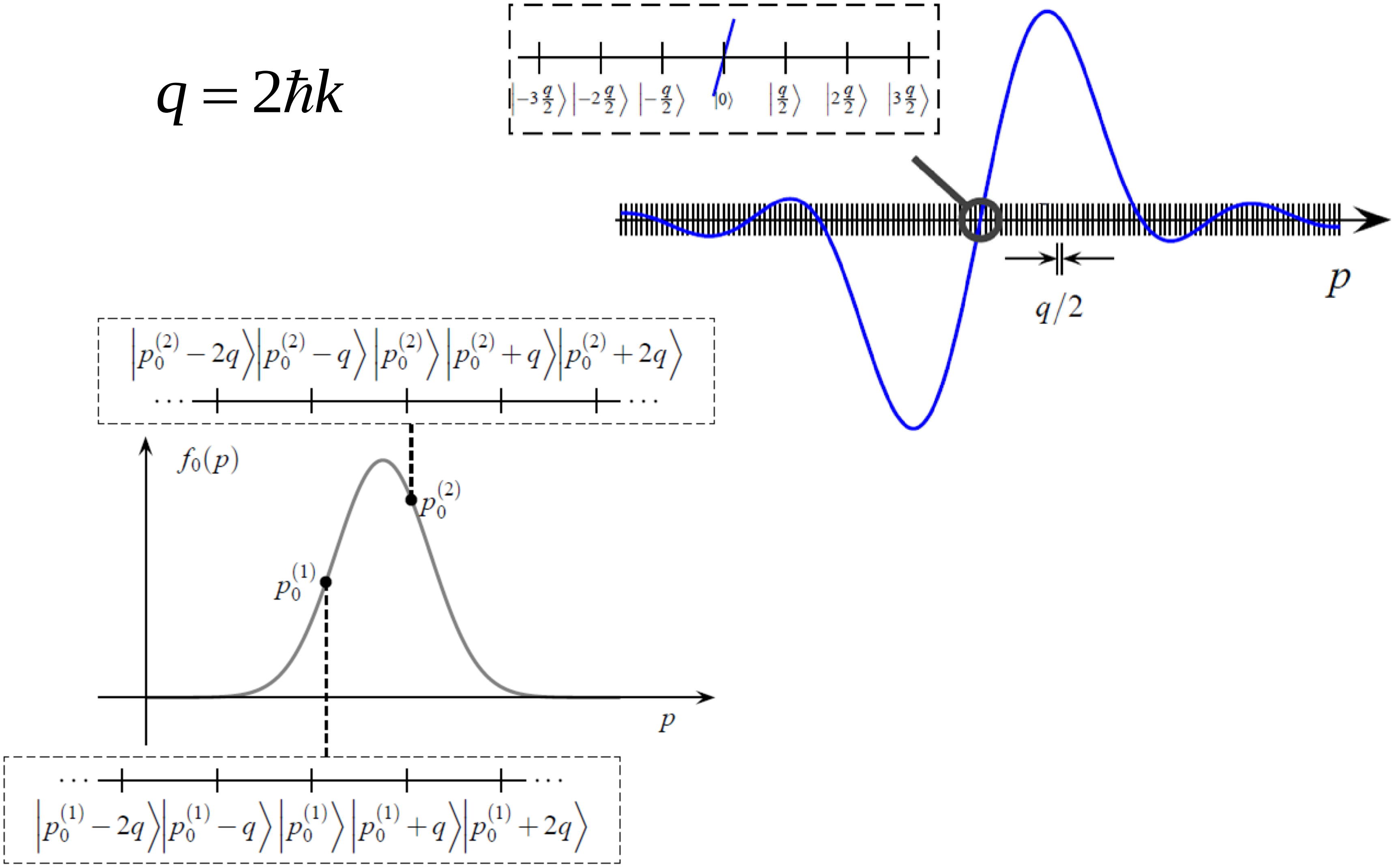
$$+\,\hbar g\left(\hat{a}_L^\dagger \hat{a}_W\,\mathrm{e}^{-2\mathrm{i}k\hat{z}} + \hat{a}_L \hat{a}_W^\dagger\,\mathrm{e}^{2\mathrm{i}k\hat{z}}\right)$$

$$|n\rangle \equiv |p^{(0)} - n2\hbar k\rangle |n_L^{(0)} + n\rangle |n_W^{(0)} - n\rangle$$

↑ electron ↑ laser ↑ wiggler

Momentum states: Bambini-Renieri Frame

$$q = 2\hbar k$$



Dynamics

$$|\Psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

↑

$$|p^{(0)} - n2\hbar k\rangle |n_L^{(0)} + n\rangle |n_W^{(0)} - n\rangle$$

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

↑

$$\sim \hat{a}_L^\dagger \hat{a}_W \, \mathrm{e}^{-2\mathrm{i}k\hat{z}}$$

$$\hat{a}_L \hat{a}_W^\dagger \, \mathrm{e}^{2\mathrm{i}k\hat{z}}$$

$$\mathrm{i}\dot{c}_n = (\ldots)c_n + g\Big[(\ldots)c_{n-1} + (\ldots)c_{n+1}\Big]$$

Dynamics: Raman-Nath equations

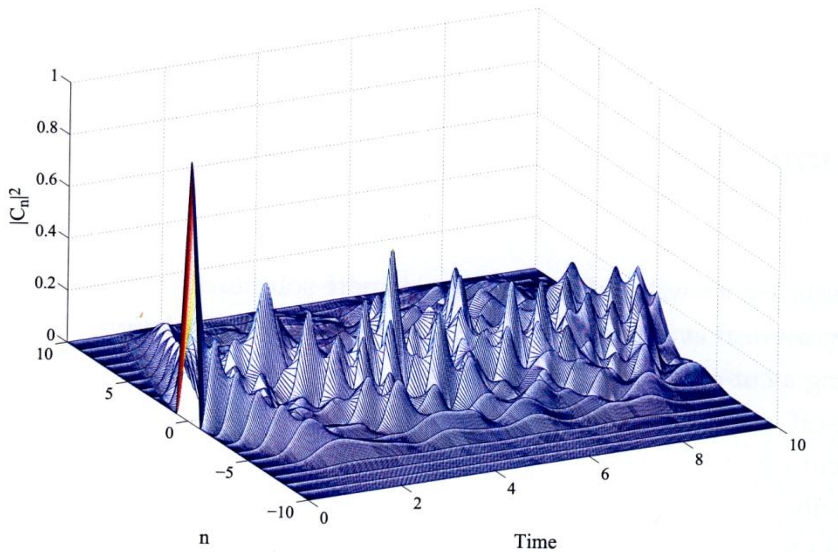
$$\mathrm{i}\dot{c}_n = (-\delta n + \epsilon n^2)c_n + g\left(\sqrt{(n_W^{(0)} - n)(n_L^{(0)} + n + 1)}\, c_{n+1} + \sqrt{(n_W^{(0)} - n + a)(n_L^{(0)} + n)}\, c_{n-1}\right)$$

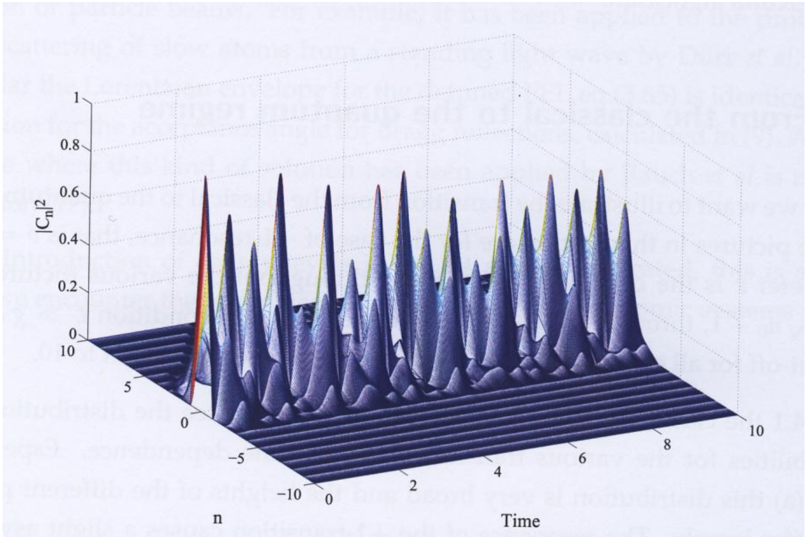
$$\delta \equiv 2\,\frac{kp_0}{m} \qquad \varepsilon \equiv \frac{2\hbar k^2}{m}$$

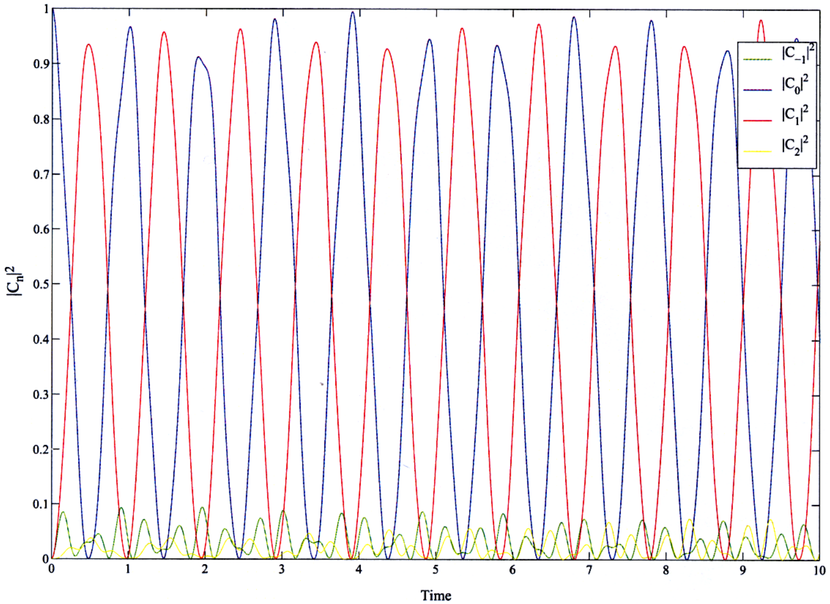
classical wiggler $1 \ll n_W^{(0)} \equiv \mathrm{const.}$

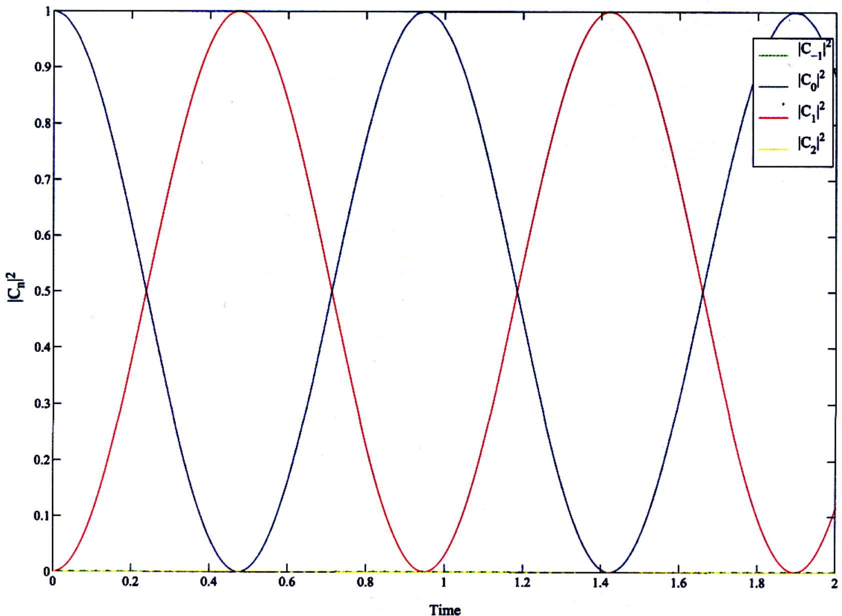
$$\mathrm{i}\dot{c}_n = (-\delta n + \epsilon n^2)c_n + \tilde{g}\left(\sqrt{n_L^{(0)} + n + 1}\, c_{n+1} + \sqrt{(n_L^{(0)} + n)}\, c_{n-1}\right)$$

Transition to Quantum FEL









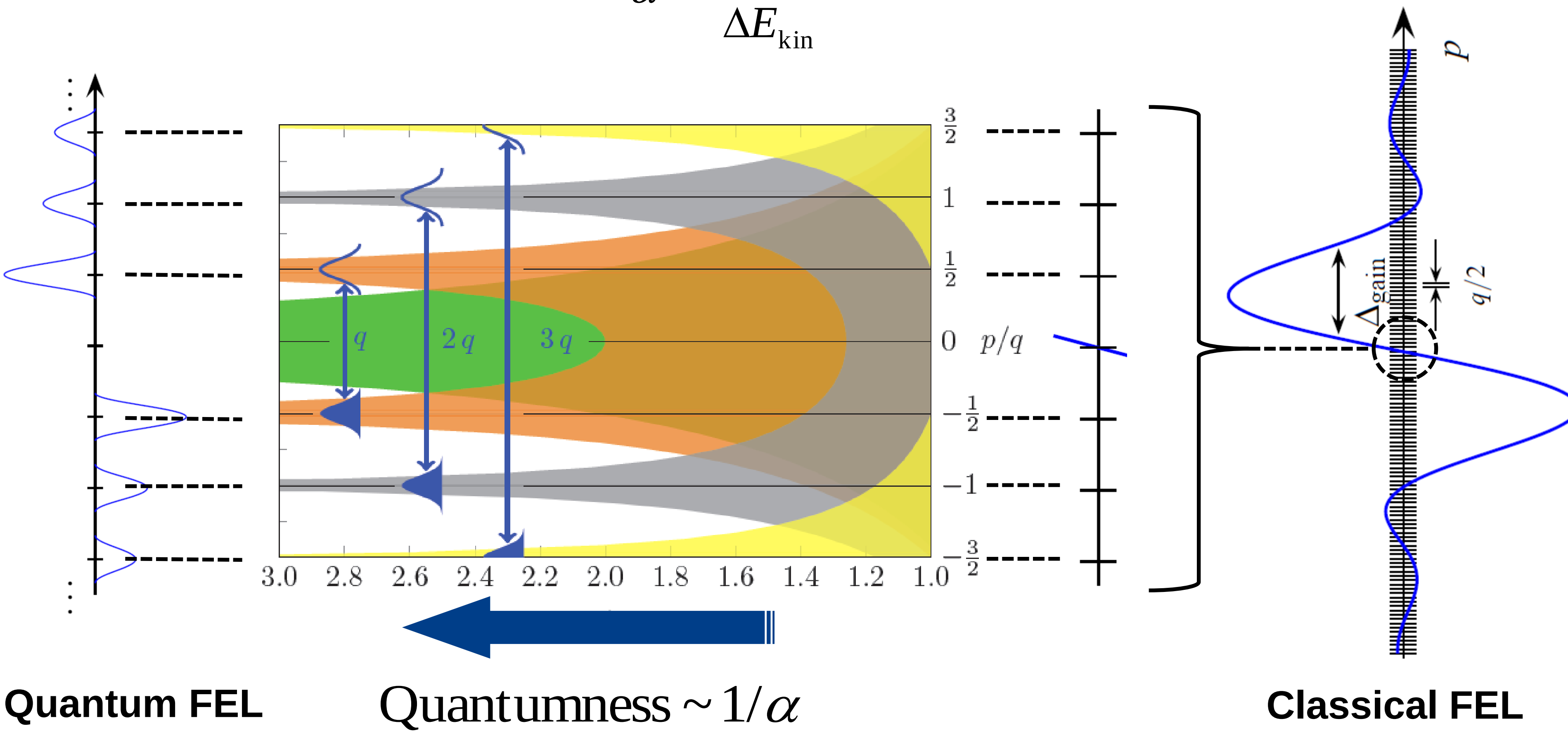
Quantum regime: Effective two-level behavior

$$|\Psi(t)\rangle = \sum_n c_n(t)|n\rangle$$

$$|p^{(0)} - n2\hbar k\rangle |n_L^{(0)} + n\rangle |n_W^{(0)} - n\rangle$$

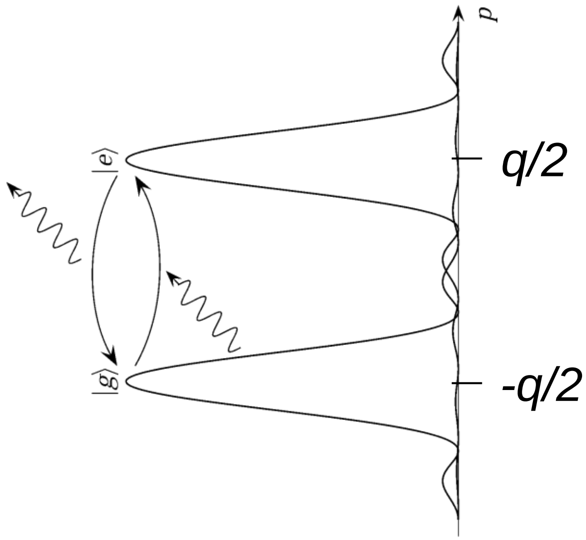
$$i\hbar \dot{c}_n = \Delta E_{\text{kin}} c_n + E_{\text{int}}\left(\sqrt{n+1}\,c_{n+1} + \sqrt{n}\,c_{n-1}\right)$$

$$\alpha \sim \frac{E_{\text{int}}\sqrt{n}}{\Delta E_{\text{kin}}}$$



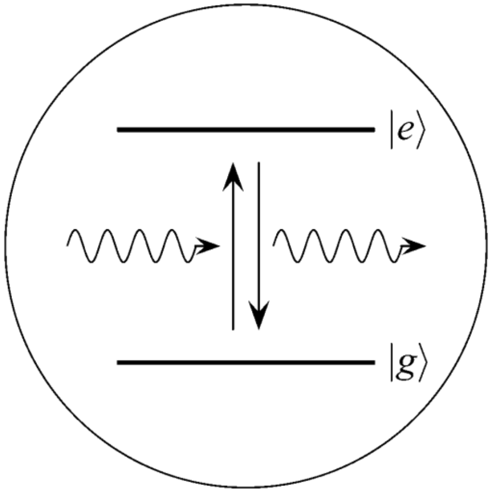
QFEL: Two-level dynamics

QFEL



center-of-mass motion of electron

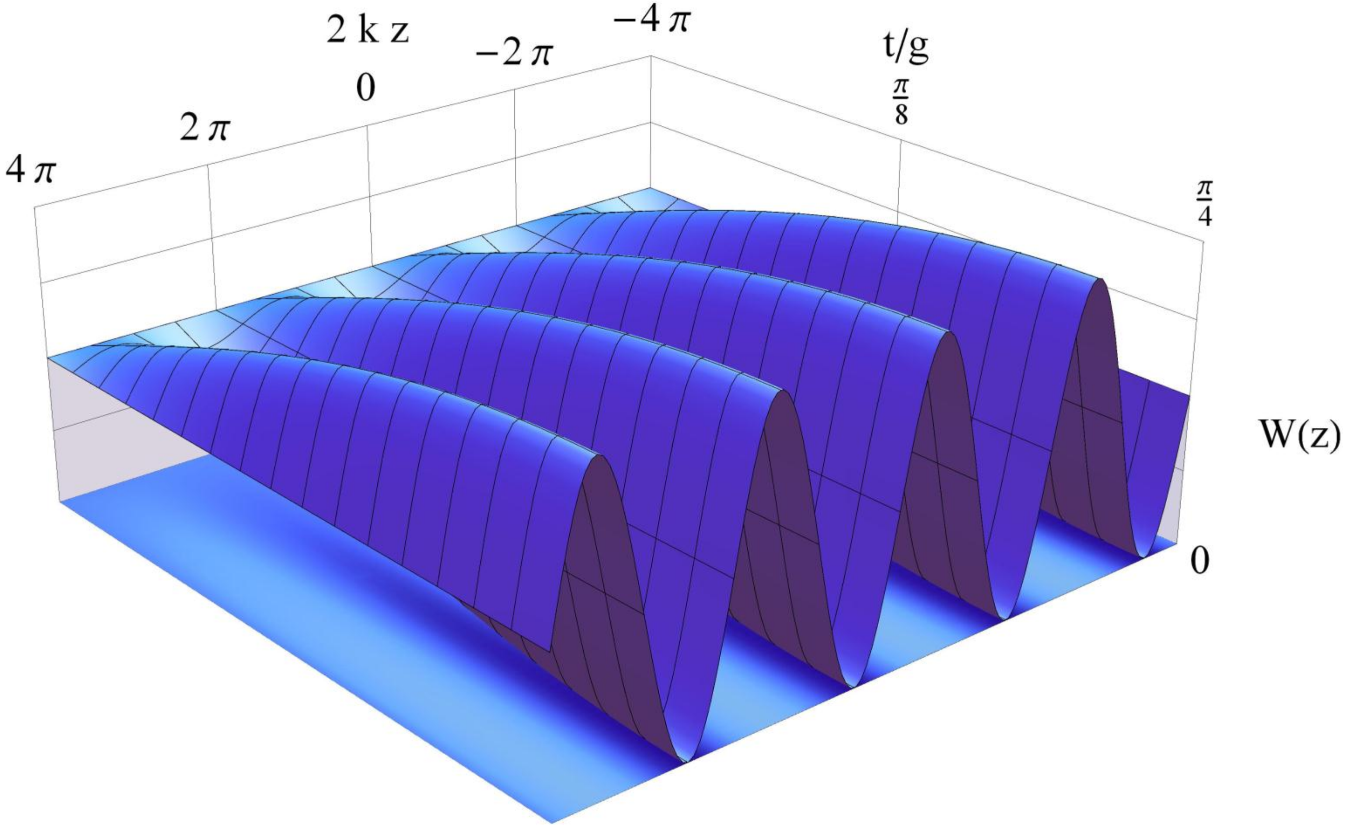
two-level system



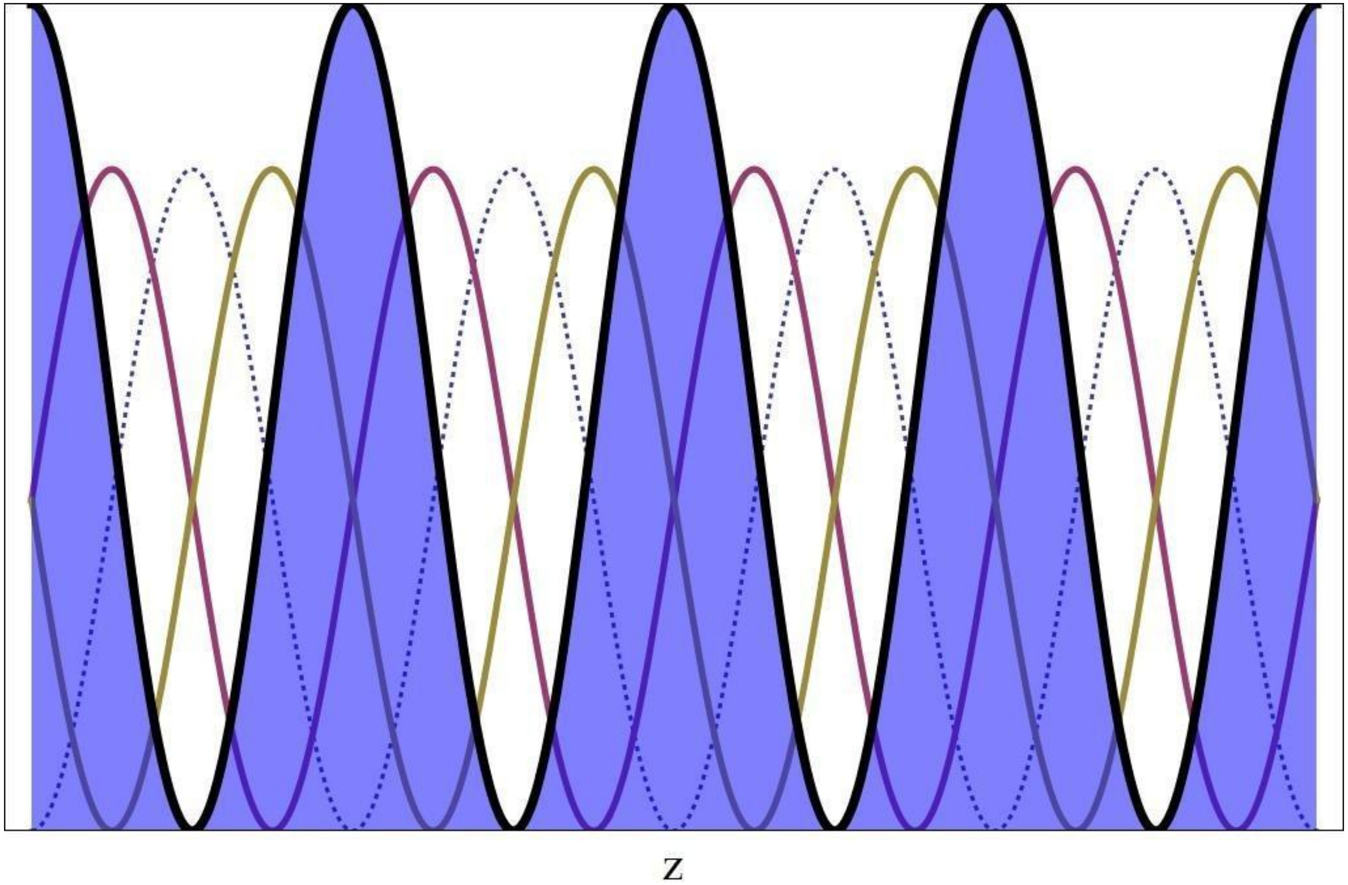
internal degrees of freedom

QFEL: Probability distribution for the electrons

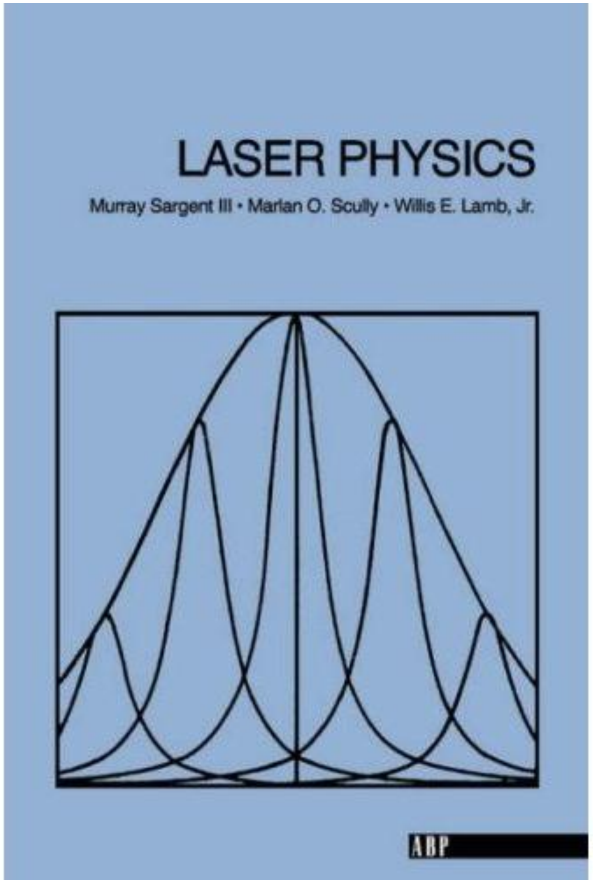
$$W(z,t) \sim [1 - \sin(2\Omega t)\sin(2kz)]$$

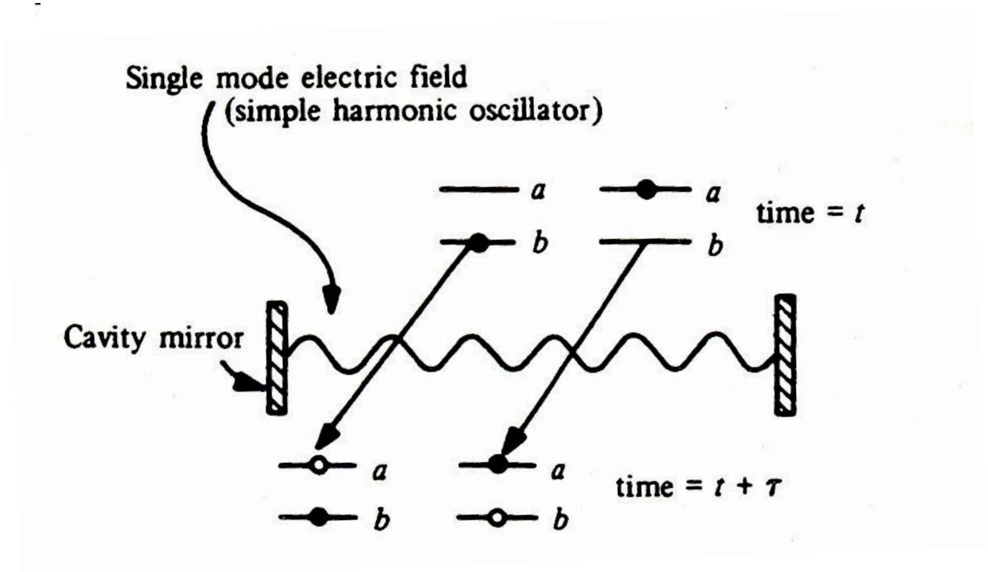


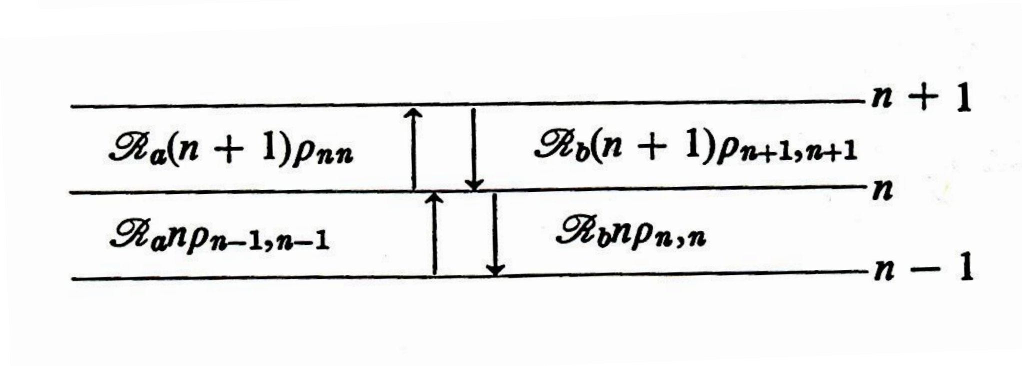
Phase-shift compared to classical potential



Atomic laser: Scully – Lamb (1967)







$$\frac{\mathrm{d}}{\mathrm{d}t} < n > = (\mathcal{R}_a - \mathcal{R}_b) < n > + \mathcal{R}_a$$

Density operator: Dynamics

von Neumann:
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{\mathrm{i}}{\hbar}[\hat{H},\hat{\rho}]$$

propagation over time τ:

$$\hat{\rho}(t+\tau) - \hat{\rho}(t) = \sum_{n=1}^{\infty}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \int_t^{t+\tau}\mathrm{d}t_n \int_t^{t_n}\mathrm{d}t_{n-1}\ldots\int_t^{t_2}\mathrm{d}t_1$$

$$\left[\hat{H}(t_n),[\hat{H}(t_{n-1}),\ldots[\hat{H}(t_1),\hat{\rho}(t)]\ldots]\right]$$

↑ initial time

Quantum theory of amplification

reservoir

$$\hat{\rho} = \hat{\rho}_{s+r}$$

system

density operator of system

$$\hat{\rho}_s \equiv \mathrm{Tr}_r \, \hat{\rho}_{s+r}$$

coarse-grained derivative: dynamics of system

$$\frac{\mathrm{d}}{\mathrm{d}t} \hat{\rho}_s(t) \equiv \frac{1}{\tau} \left[\hat{\rho}_s(t+\tau) - \hat{\rho}_s(t) \right]$$

Coarse-grained dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_s = \frac{1}{\tau}\sum_{n=1}^{\infty}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \int_t^{t+\tau}\mathrm{d}t_n \int_t^{t_n}\mathrm{d}t_{n-1}\ldots\int_t^{t_2}\mathrm{d}t_1$$

$$\mathrm{Tr}_r\left\{\left[\hat{H}(t_n),[\hat{H}(t_{n-1}),\ldots[\hat{H}(t_1),\hat{\rho}_{s+r}(t)]\ldots]\right]\right\}$$

$$\hat{\rho}_{s+r} = \hat{\rho}_s \otimes \hat{\rho}_r$$

H ≡ interaction between system and reservoir

FEL: quantum theory

$$\hat{H} = \frac{\hat{p}_z^2}{2m} + \mathcal{A}\left(\hat{a}\ \mathrm{e}^{2\mathrm{i}kz} + \hat{a}^\dagger\ \mathrm{e}^{-2\mathrm{i}kz}\right)$$

$$[\hat{p}_z\,, \hat{z}] = \frac{\hbar}{\mathrm{i}}$$

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

Comparison: laser versus FEL

$$\dot{W}_n = -\left[\mathcal{R}_a\ (n+1) + \mathcal{R}_b\ n\right] W_n + \mathcal{R}_a\ n\ W_{n-1} + \mathcal{R}_b\ (n+1)\ W_{n+1}$$

$$\left.\begin{matrix}\mathcal{R}_a \\ \mathcal{R}_b\end{matrix}\right\} \text{ injection rate of atoms in } \left\{\begin{matrix}\text{excited} \\ \text{ground}\end{matrix}\right. \text{ state}$$

$$\dot{W}_n = -\left[\mathcal{R}_-\ (n+1) + \mathcal{R}_+\ n\right] W_n + \mathcal{R}_-\ n\ W_{n-1} + \mathcal{R}_+\ (n+1)\ W_{n+1}$$

$$\mathcal{R}_\pm \sim \frac{\sin^2 \varphi_\pm}{\varphi_\pm^2} \qquad\qquad \varphi_\pm \sim p \pm \hbar k$$

FEL: gain (small signal)

$$\frac{\mathrm{d}}{\mathrm{d}t} < n >= (\mathcal{R}_- - \mathcal{R}_+) < n > + \mathcal{R}_+$$

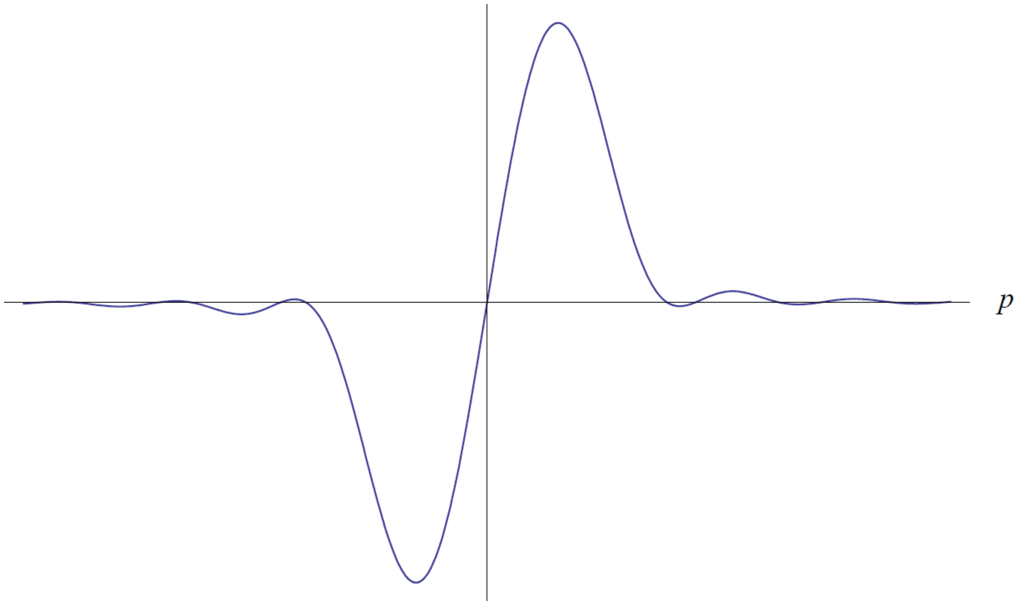
$\uparrow$

gain

$$\mathcal{R}_- - \mathcal{R}_+ \sim \frac{\partial \mathcal{R}}{\partial p}$$

$\uparrow$

$\hbar k \ll p$



QFEL: Experimental parameters

- Quantum FEL requires $\rho < 0.8\,\frac{\gamma\lambda_C}{\lambda_W(1+a_0{}^2)}$ and $\Delta E < \frac{\gamma^2 hc}{\lambda_W(1+a_0{}^2)} \equiv \Delta E_{\max}$

α	**0.2***
γ	70
λ_W	0.4 µm
a_0	0.3
λ_L	44.5 pm
ρ	$1.06\cdot 10^{-4}$
$\left(\frac{I}{\pi\sigma^2}\right)_{e^-}$	$4.86\cdot 10^6\,\frac{\mathrm{kA}}{\mathrm{cm}^2}$
I_0	$1.95\cdot 10^{17}$ W/cm²
$\Delta E_{\max}$	13.9 keV
L_b	600 µm
$\frac{\Delta\omega}{\omega}$	$7.4\cdot 10^{-8}$

For helical laser wiggler with wavelength $\lambda_W = \frac{\lambda_0}{2}$ we find

$$\rho = \frac{1}{2\gamma}\left(\frac{I}{I_A}\right)^{1/3}\left(\frac{\lambda_W a_0}{2\pi\sigma}\right)^{2/3} \quad [1]$$

with the Alfven current $I_A = 17.045\,\mathrm{kA}$, electron current I and electron rms beam size σ and

$$a_0 = 0.85\cdot 10^{-9}\lambda_0[\mu\mathrm{m}]\sqrt{I_0\left[\frac{\mathrm{W}}{\mathrm{cm}^2}\right]} \quad [2]$$

with the laser intensity I_0

[1] M. Xie, Proceedings of the Particle Accelerator Conference, Dallas 1995, p. 183
[2] A.D. Debus et al., Appl. Phys. B (2010) 100, p.61

* i.e. $\bar{\rho} = 0.14$

Summary

- semiclassical theory of the laser
- basic elements of FEL
- classical theory
- quantum theory and Quantum FEL
  - single-particle theory
  - many-particle theory