Bifurcations and waves in **SO**(2) symmetry systems: The HEDGEHOG experiment as an example

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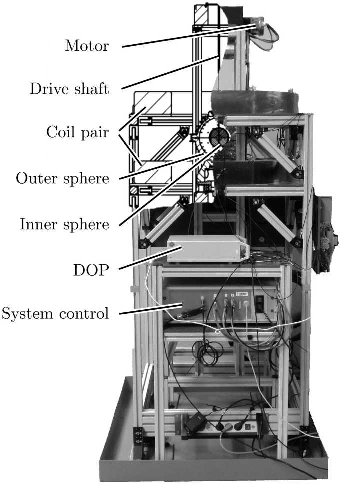
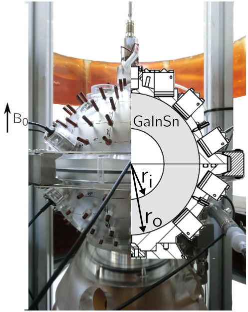
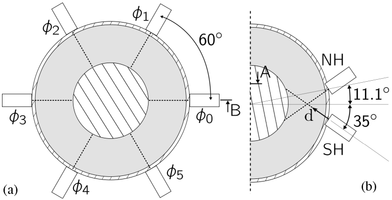
Motivation

- Dynamical systems theory provides robust mathematical tools for the study of any phenomena modeled in terms of ordinary (partial) differential equations such as the experiments at HZDR $\longrightarrow$ Kuznetsov, 1998
- Transition to turbulence usually takes place following a sequence of bifurcations: basic state, periodic orbit, quasiperiodic orbit and chaos $\longrightarrow$ Ruelle and Takens, CMP 1971. Eckmann, RMP 1981.
- Fluid flow transitions are best understood taking into account the symmetries of the system $\longrightarrow$ Crawford and Knobloch, ARMA 1991.
- HEDGEHOG experiment: Instabilities observed in differentially rotating flows in the presence of a magnetic field (magnetized spherical Couette flows (MSC)) were attributed to the magnetorotational instability (MRI), which is presently considered the most promising candidate to explain the transport mechanism of angular momentum in accretion disks around black holes and protostars $\longrightarrow$ Balbus and Hawley, AJ 1991.

HEDGEHOG Experimental Device







The Model

Navier-Stokes and induction equations

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\boldsymbol{\Omega}_o \times \mathbf{v} + \boldsymbol{\Omega}_o \times (\boldsymbol{\Omega}_o \times \mathbf{r}) = \nu\nabla^2\mathbf{v} - \frac{1}{\rho}\nabla p + \frac{1}{\rho\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}$$

$$\partial_t \mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \eta\nabla^2\mathbf{B}$$

$$\nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot\mathbf{B} = 0$$

- uniform axial magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z = B_0\cos(\theta)\hat{\mathbf{e}}_r - B_0\sin(\theta)\hat{\mathbf{e}}_\theta$
- Characteristic scales $r \to d, \quad t \to d^2/\nu, \quad \mathbf{v} \to r_i\Delta\Omega, \quad p \to \rho\nu^2/d, \quad \mathbf{B} \to B_0$

Inductionless approximation ($\Omega_o = 0$)

$$\partial_t \mathbf{v} + \mathrm{Re}\,(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla p^* + \nabla^2\mathbf{v} + \mathrm{Ha}^2(\nabla\times\mathbf{b})\times\hat{\mathbf{e}}_z, \quad \nabla\cdot\mathbf{v} = 0$$

$$0 = \nabla\times(\mathbf{v}\times\hat{\mathbf{e}}_z) + \nabla^2\mathbf{b}, \quad \nabla\cdot\mathbf{b} = 0$$

$$\mathrm{Re} = \frac{\Omega_i r_i d}{\nu} \ \text{Reynolds}, \quad \mathrm{Ha} = B_0 d\sqrt{\frac{\sigma}{\rho\nu}} \ \text{Hartmann}, \quad \chi = \frac{r_i}{r_o} \ \text{aspect ratio}$$

- $\mathbf{B} = \hat{\mathbf{e}}_z + \mathrm{Rm}\,\mathbf{b}$, terms $O(\mathrm{Rm})$ neglected, $\mathrm{Rm} = \Omega_i r_i d/\eta \ll 1$ magnetic Reynolds

Expansion in Spherical Harmonics

Toroidal/Poloidal decomposition of the velocity field (magnetic field)

$$\mathbf{v} = \nabla \times (\Psi \mathbf{r}) + \nabla \times \nabla \times (\Phi \mathbf{r})$$

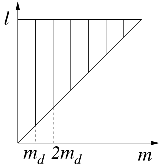
Spherical harmonics expansion up to degree (l) and order (m) $L_{\max}$

$$(\Psi, \Phi)(t, r, \theta, \varphi) = \sum_{l=0}^{L_{\max}} \sum_{\substack{m=-l \\ m=\dot{m}_d}}^{l} (\Psi_l^m, \Phi_l^m)(t, r) Y_l^m(\theta, \varphi)$$

$$Y_l^m(\theta, \varphi) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\varphi}, \; l \geq 0, \; -l \leq m \leq l$$

$\Psi_l^{-m} = \overline{\Psi_l^m}$, $\Phi_l^{-m} = \overline{\Phi_l^m}$, with $\Psi_0^0 = \Phi_0^0 = 0$.

The unknowns are $\Psi_l^m(t)$ and $\Phi_l^m(t)$ for $0 \leq l \leq L_{\max}$ and $0 \leq m = \dot{m}_d \leq l$, at a mesh of $N_r + 1$ Gauss-Lobatto points $\longrightarrow$ $n = (2L_{\max}^2 + 4L_{\max})(N_r - 1)$



Numerical Methods

- The pseudospectral method is used for the computation of the advection (nonlinear) terms. Transformation of scalars from spherical harmonics amplitudes (spectral domain) to values on a mesh in spherical coordinates (physical domain).
- The code is paralellized in the spectral as well as physical space by using OpenMP directives.
- Optimized libraries (FFTW3) for the FFTs in φ (longitude) and matrix-matrix products (dgemm MKL/OpenBlas) for the Legendre transforms in θ (colatitude) are implemented.
- High order ($k = 2, .., 5$) variable size and variable order (VSVO) Implicit-Explicit and fully implicit (DLSODPK) schemes based on backward differentiation formulae are used for time integration. Also exponential time integration methods (EXPOKIT) are implemented.

Basic flow

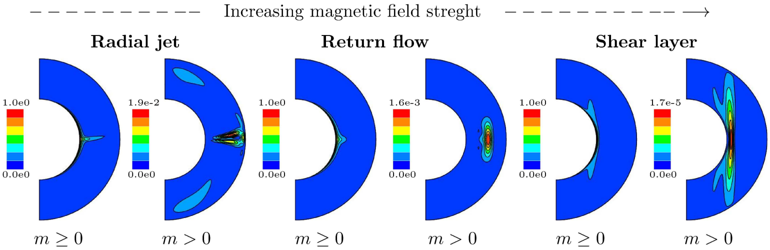
The system of equations is written as

$$\partial_t u = \mathcal{L}u + \mathcal{B}(u,u),$$

where u is a vector containing the values of the amplitudes at the mesh of collocation points in the radius, and $\mathcal{L}$ and $\mathcal{B}$ are, respectively, linear $\mathcal{L} = \mathcal{L}(\mathrm{Ha})$ and bilinear operators $\mathcal{B} = \mathcal{B}(\mathrm{Re})$.

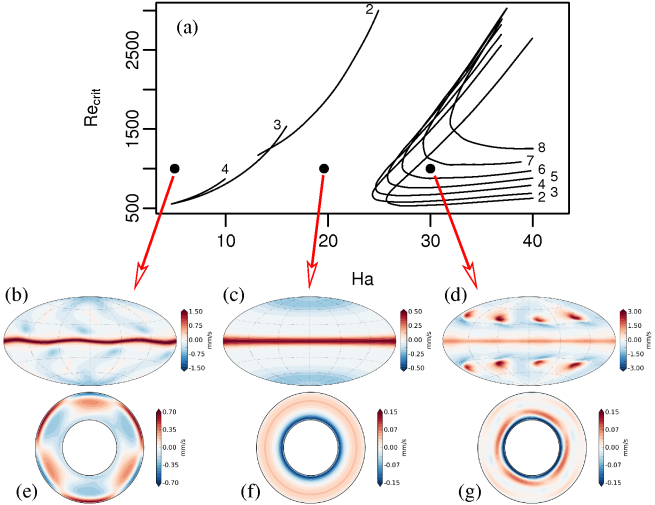
For small Re and all Ha, the solution u_0 is time independent and axisymmetric ($m = 0$). This is the **basic flow**.

By increasing Re it becomes unstable against nonaxisymmetric ($m > 0$) perturbations and the flow pattern depends strongly on Ha. These are the magnetic instabilities:



Basic flow marginal stability curves: $\chi = 0.5$

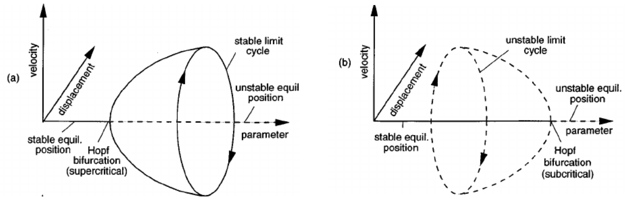
From Kasprzyk et al., MJ 2017, and Hollerbach, PRSA 2009.



Rotating Waves

The system is $\mathbf{SO}(2)\times\mathbf{Z}_2$-equivariant, $\mathbf{SO}(2)$ generated by azimuthal rotations, and $\mathbf{Z}_2$ by reflections with respect to the equatorial plane.

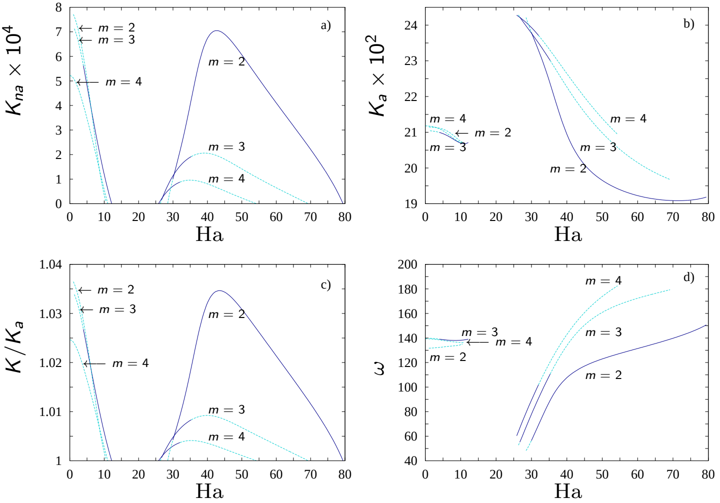
The basic axisymmetric ($m = 0$) flow is unstable to nonaxisymmetric perturbations via Hopf bifurcation and thus the emerging solution is a rotating wave $\longrightarrow$ Ecke et al., EL 1992.



RWs are solutions in which a fixed flow pattern with m_1-fold azimuthal symmetry is rotating at a frequency ω in the azimuthal direction $u(t, r, \theta, \varphi) = \tilde{u}(r, \theta, \tilde{\varphi})$, $\quad \tilde{\varphi} = \varphi - \omega t \longrightarrow$ Rand, ARMA 1982

RW bifurcation diagrams: $\mathrm{Re} = 10^3$, $\chi = 0.5$

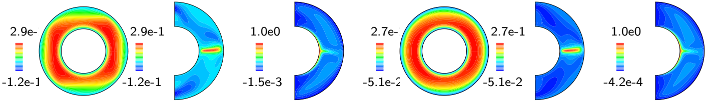
Garcia and Stefani, PRSA 2018.



Rotating wave patterns

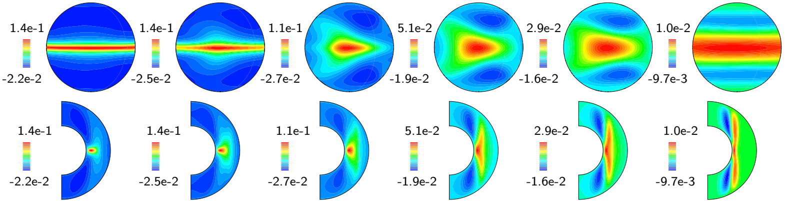
- Radial Jet Instability

$\mathrm{Ha} = 4.32$ $\mathrm{Ha} = 9.29$

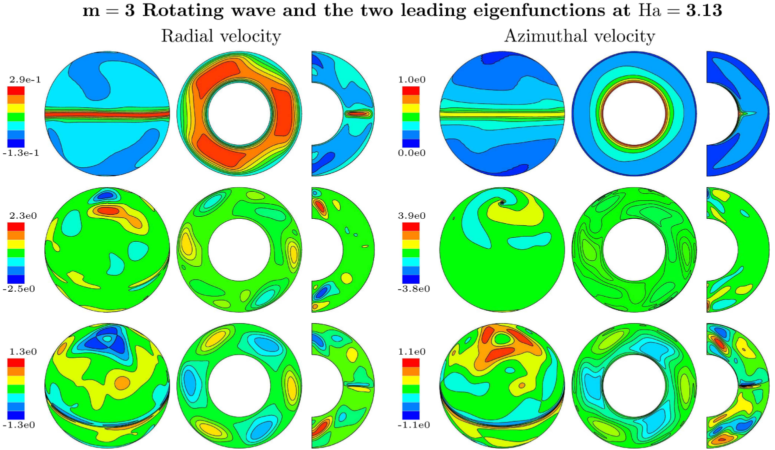


- Return Flow Instability $\longrightarrow$ Shear Layer Instability

$\mathrm{Ha} = 29$ $\mathrm{Ha} = 32.9$ $\mathrm{Ha} = 44.1$ $\mathrm{Ha} = 60.2$ $\mathrm{Ha} = 72$ $\mathrm{Ha} = 79.4$



RW and dominant eigenfunctions (Floquet modes)



Modulated Rotating Waves

Secondary Hopf bifurcations of rotating waves RW (periodic flows) $\longrightarrow$ branches of modulated rotating waves MRW (quasiperiodic flows)

MRW are τ-periodic solutions of the MSC system in a reference frame rotating with frequency ω:

$$\partial_t u = \mathcal{L}u + \mathcal{B}(u,u) + \omega \partial_\varphi u.$$

There exist a basic (minimal) time $\tau_{\rm min} > 0$ and an integer $0 \le n < m_1/s$:

$$u(t,r,\theta,\varphi) = u(t+\tau_{\rm min}, r, \theta, \varphi + 2\pi n/m_1) \quad \forall t,\ \forall \varphi.$$

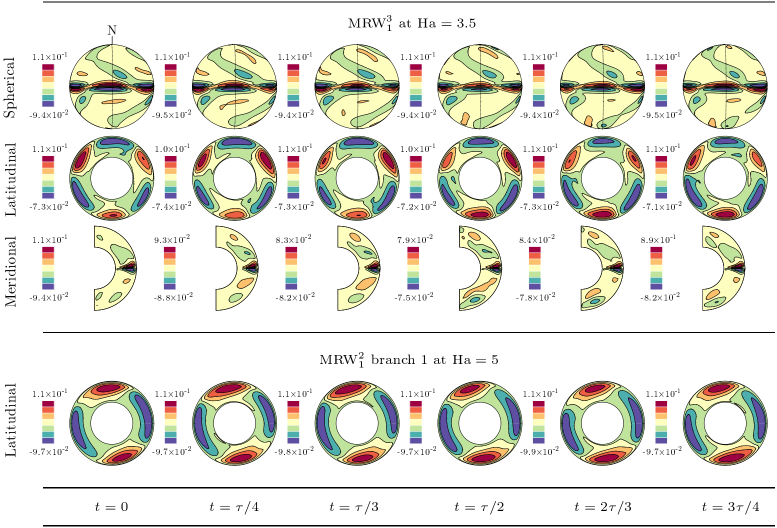
The spatio-temporal symmetry of MRW described by $(m_1, n, s) \in \mathbb{Z}^3$:

- m_1-fold azimuthal symmetry of the parent RW
- n related with $\tau_{\rm min}$ (angle of azimuthal rotation)
- s-fold azimuthal symmetry of the MRW

Alternative (rough) classification: RW/MRW$_s^{m_{\rm max}}$, s the azimuthal symmetry of the waves and $m_{\rm max} \neq 0$ their most energetic azimuthal wave number.

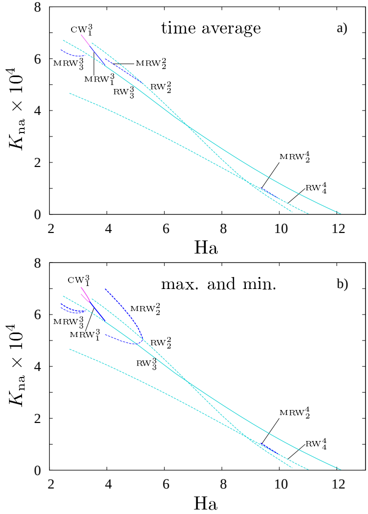
The dominant Floquet multiplier (of a RW with m_1) has m_2 azimuthal symmetry $\longrightarrow$ the azimuthal symmetry of the bifurcated MRW is $s = \mathrm{GCF}(m_1, m_2)$.

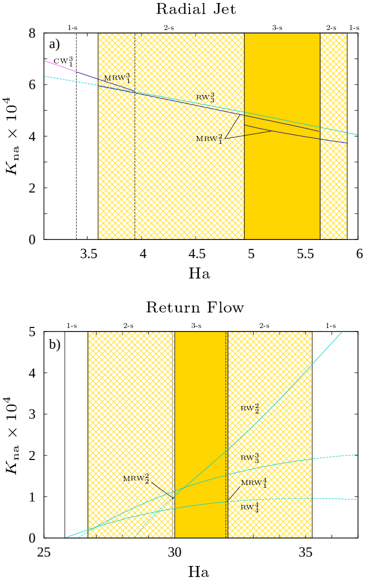
Radial velocity patterns of MRW



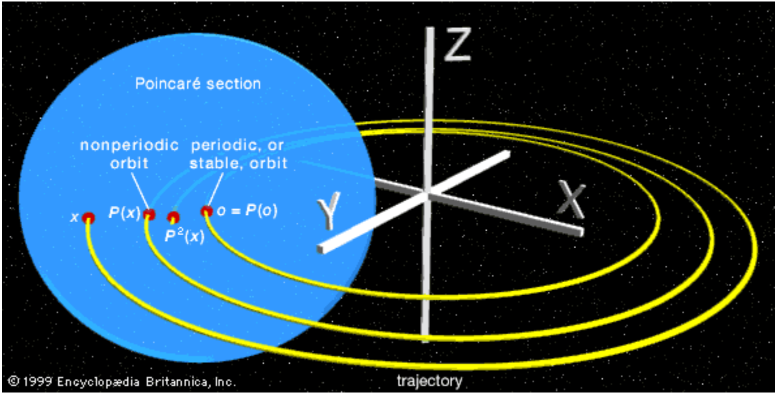
MRW bifurcation diagrams: $\mathrm{Re} = 10^3$, $\chi = 0.5$

Garcia et al., JNS 2019 submitted.

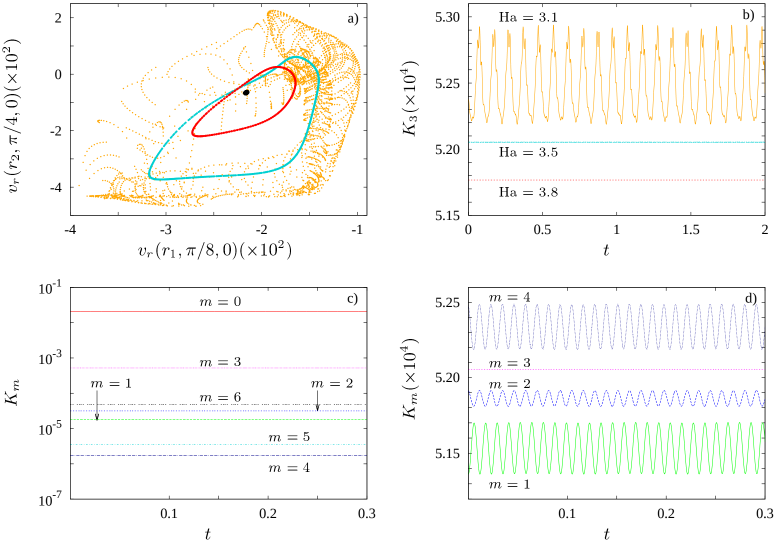




Poincaré Section Concept



Time series and Poincaré sections



Ongoing work

- Bifurcation diagrams for chaotic solutions at small magnetic forcing (absent at large forcing). We have found several Ruelle-Takens and period doubling scenarios for obtaining chaotic flows.
- By decreasing $\mathrm{Ha} \to 0$ several quasiperiodic and chaotic purely hydrodynamic attractors coexist at $\mathrm{Re} = 1000$.
- The azimuthal drifting behaviour is strong and persist for all the types of flows found.
- Simulation of the HEGDEGOG UDV measurements $\longrightarrow$ Preliminary comparisons of flow velocities are in good agreement, both in amplitudes and time dependence.
- Time scales of the azimuthal drift range from roughly 0.007 to 0.01 Hz, (around 2 min) depending on the main azimuthal symmetry of the flow.
- For the radial jet instability MRW the time scales of the modulation can be around $10^{-3} - 10^{-4}$ Hz, i. e. $\sim 15 - 60$ min. Could be experimentally detected?
- Computation of electric potential differences for estimating voltages in the HEGDGEHOG experiment $\longrightarrow$ Preliminary results point to $10^{-7} - 10^{-6}$ V in the latitudinal direction and $10^{-8} - 10^{-9}$ V in the azimuthal direction.
- Analyse experiments with different magnetic forcing $\mathrm{Ha} \in [25, 80]$ (the return flow/shear layer regimes) to see if the behaviour of the rotating frequency seen on the bifurcation diagrams is reproduced.

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