Applications of Random Matrix Theory: Missing Level Corrections

G. E. Mitchell
North Carolina State University

Neutron level densities important

Assess neutron resonance data to determine best values for level densities

Essentially all evaluation methods involve random matrix theory

STATISTICAL THEORY OF THE ENERGY LEVELS OF COMPLEX SYSTEMS. V

M. L. MEHTA AND F. J. DYSON
J. MATH. PHYS. 4 (1963)

PROBLEM H. STATISTICAL EFFECTS OF MISSING AND SPURIOUS LEVELS

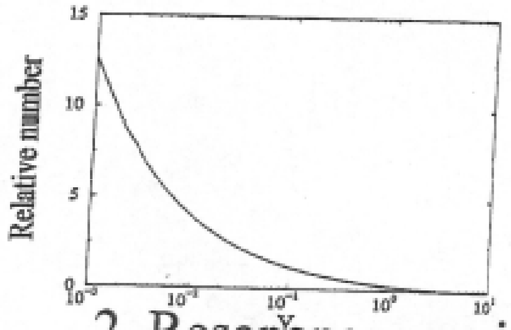
...DESIRABLE TO MAKE THE RESULTS ... MORE PRECISE BY CALCULATING QUANTITATIVELY THE EFFECTS OF MISSING AND SPURIOUS LEVELS. TO CARRY THROUGH SUCH CALCULATIONS WOULD NOT BE DIFFICULT, ONLY RATHER LABORIOUS.

Random Matrix Theory (RMT)

$\Rightarrow$ Assume RMT

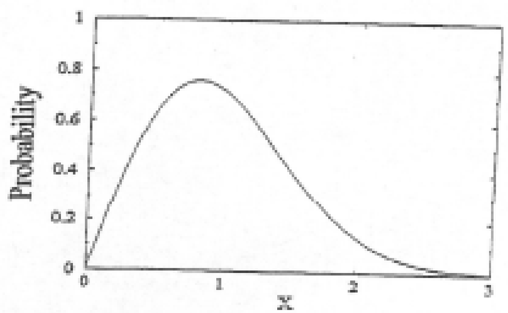
1. Reduced widths obey **Porter-Thomas distribution**

$$P(y)dy = \frac{1}{\sqrt{2\pi y}}\exp(-\frac{y}{2})dy$$



$$y = \frac{\gamma^2}{\langle \gamma^2 \rangle}$$

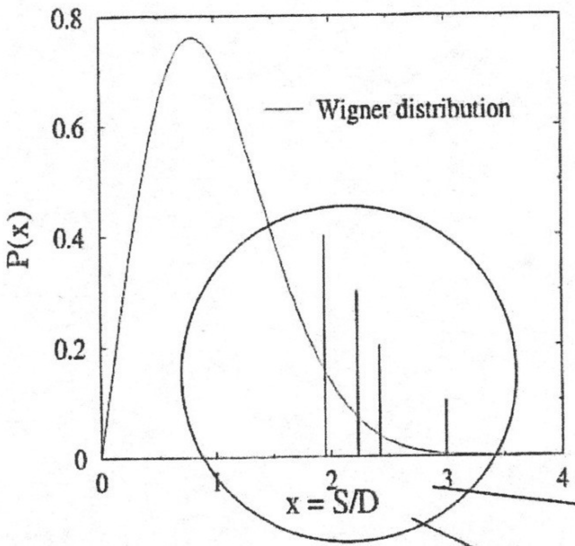
2. Resonance spacings are distributed according to **Wigner distribution**

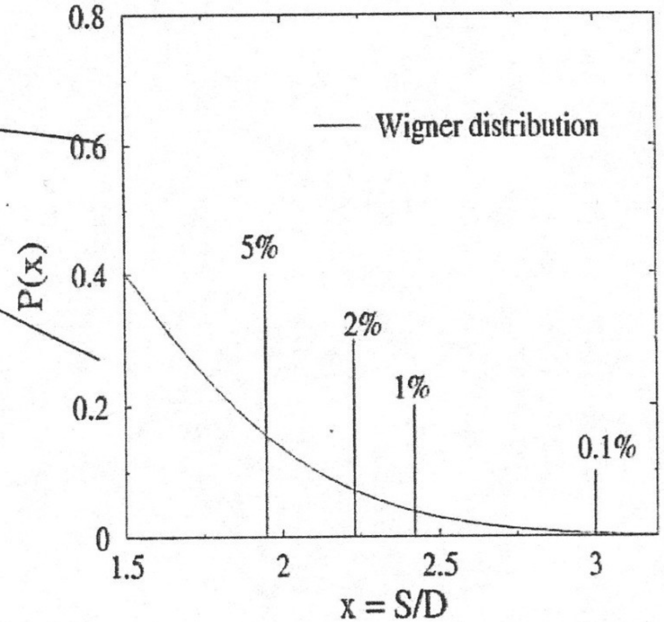


$$P(x)dx = \frac{\pi x}{2}\exp(-\frac{\pi x^2}{4})dx$$

$$x = \frac{D}{\langle D \rangle}$$

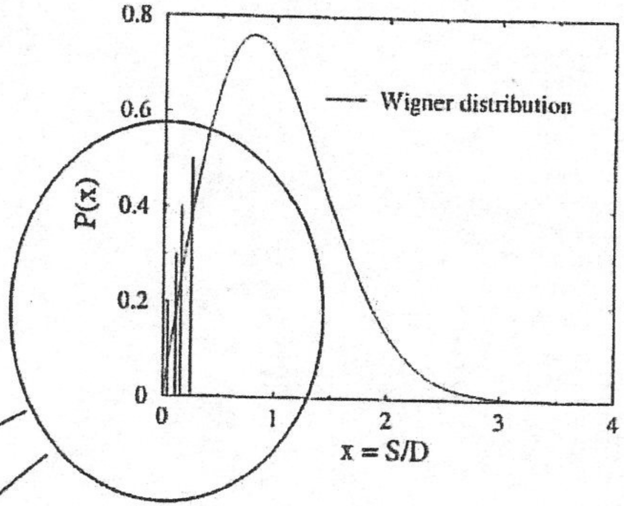
Percentage of Large Spacings

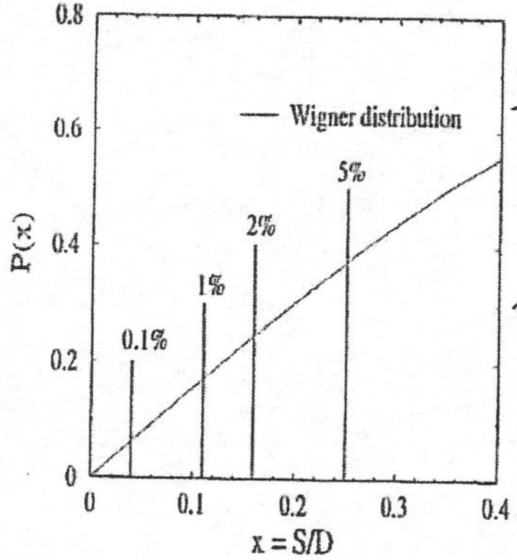




Percentage	x
5%	1.95
2%	2.23
1%	2.42
0.1 %	3.0

Percentage of Small Spacings





Percentage	x	1/x
5%	0.25	4.0
2%	0.16	6.25
1%	0.11	8.84
0.1%	0.04	25.0

Levels with $y < y_0 = \dfrac{\gamma_0^2}{\langle \gamma^2 \rangle}$ are undetectable

---

Standard method:

$$\{\gamma_i^2\}, i = 1, N_0 \quad \Rightarrow \quad \langle \gamma^2 \rangle = \frac{\Sigma \gamma_i^2}{N_0}$$

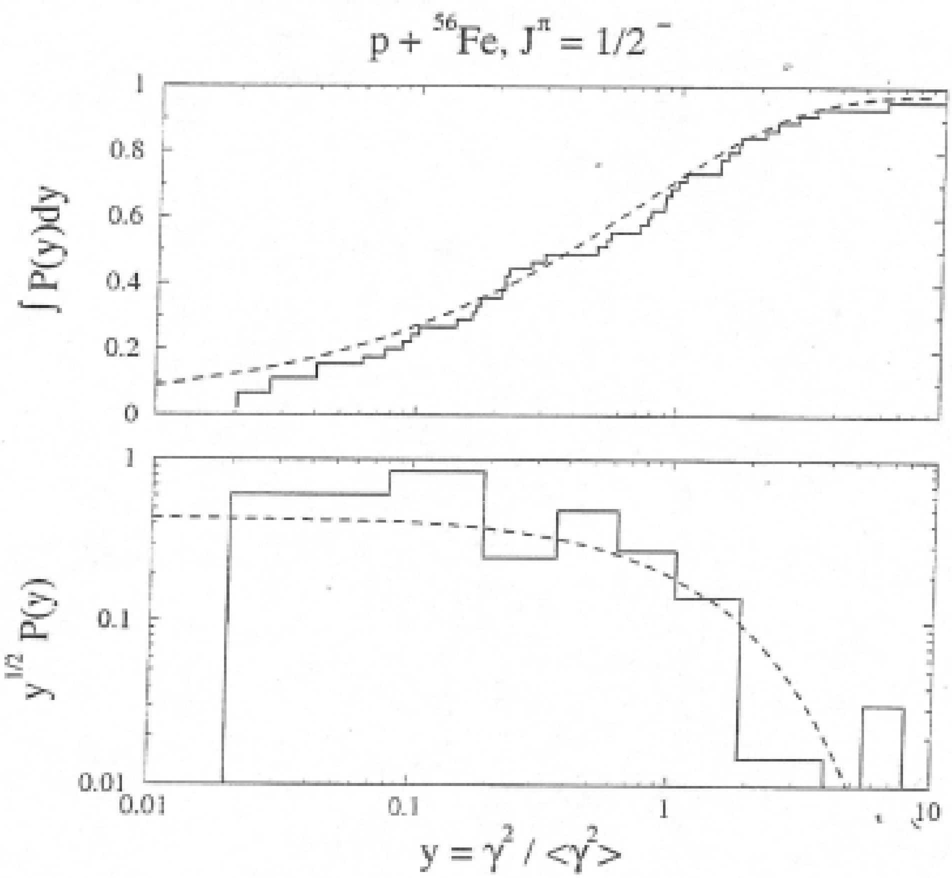
$$- - - - - \longrightarrow \qquad y_{\min} = \frac{\gamma_{\min}^2}{\langle \gamma^2 \rangle}$$

$$\textbf{MF} = \text{Missing fraction} = \int_0^{y_{\min}} P(y)dy$$

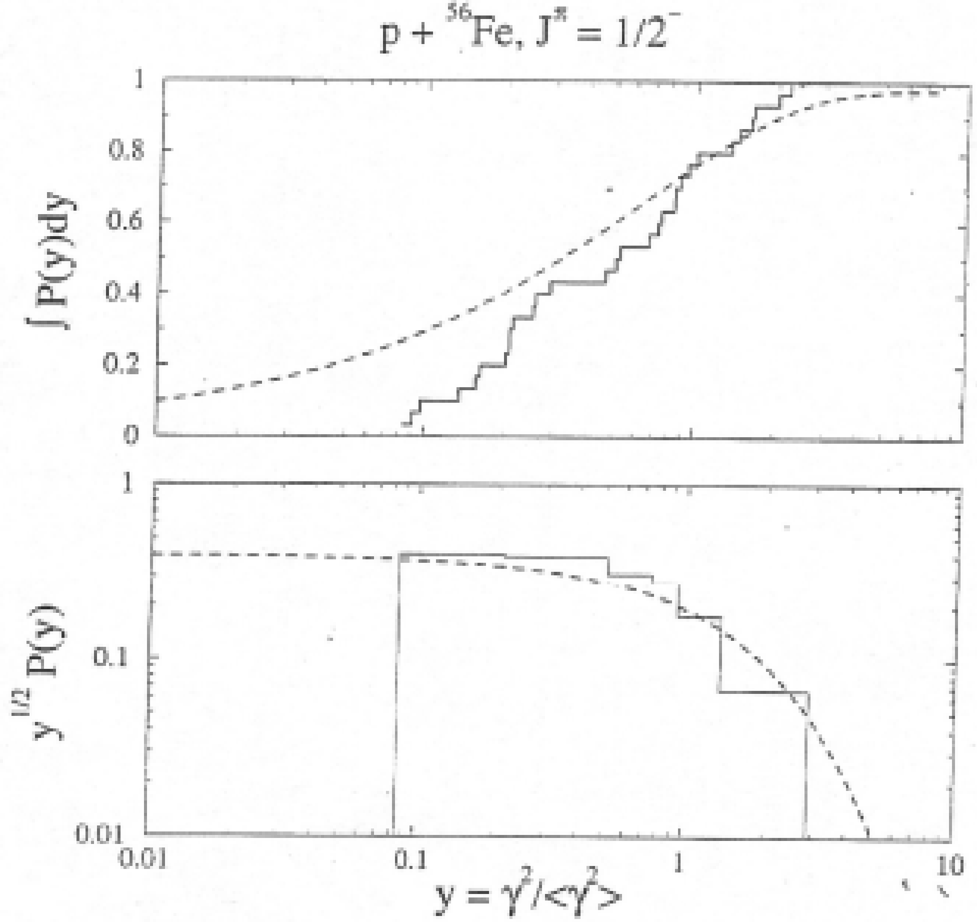
$$\textbf{MS} = \text{Missing strength} = \int_0^{y_{\min}} yP(y)dy$$

$$\longleftarrow - - - - - \qquad \langle \gamma^2 \rangle = \frac{\Sigma \gamma^2 + \text{MS}}{N_0 / (1 - \text{MF})}$$

until converges



p + 56Fe, Jπ = 1/2 −
∫ P(y)dy
y1/2 P(y)
y = γ2 / <γ2>
0.01
0.1
1
10
0
0.2
0.4
0.6
0.8



p + 56Fe, Jπ = 1/2−
∫P(y)dy
y1/2 P(y)
y = γ2/<γ2>

New Method for Missing Level Correction

U. Agvaanluvsan
M.P. Pato
J.F. Shriner, Jr.
G.E. Mitchell

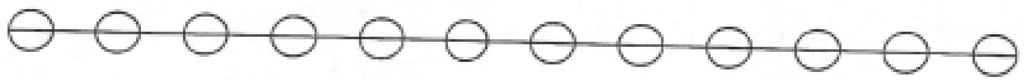
$\langle \Gamma D \rangle = 0 \implies$ use spacing analysis as independent test

Spacing: minimal effect due to non-statistical phenomena. However, levels missed at **random**, therefore analysis is harder to formulate

Question:

Given the spacing distribution and missed levels at random, how can one determine a missing fraction of levels?

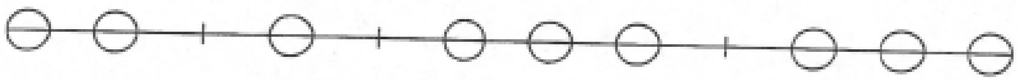
A. Perfect sequence = all levels are observed.



Number of spacings type 0 = 11

(0+1)11 = 11

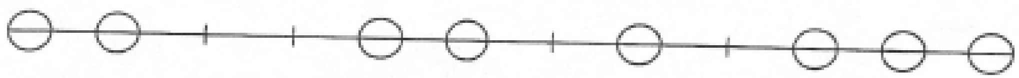
B. Imperfect sequence = levels are missing.



Number of spacings type 0 = 5
type 1 = 3

(0+1)5+(1+1)3 = 11

C. Imperfect sequence = levels are missing.



Number of spacings type 0 = 4
type 1 = 2
type 2 = 1

(0+1)4+(1+1)2+(2+1)1 = 11

Expand the spacing distribution in terms of higher order distributions:

$$P(z)dz = \sum_k a_k \lambda \, p(k, \lambda z) dz$$

Normalizations

$$\int_0^\infty P(z)dz = 1 \qquad \int_0^\infty z\,P(z)dz = 1$$

$$\int_0^\infty p(k,z)dz = 1 \qquad \int_0^\infty z\,p(k,z)dz = k+1$$

lead to conditions:

$$\textbf{(a).} \quad \sum_{k=0}^{\infty} a_k = 1$$

$$\textbf{(b).} \quad \sum_{k=0}^{\infty} a_k (1+k) = \lambda$$

Introduce entropy:

$$S\{a_k\} = -\sum_{k=0}^{\infty} a_k \ln a_k$$

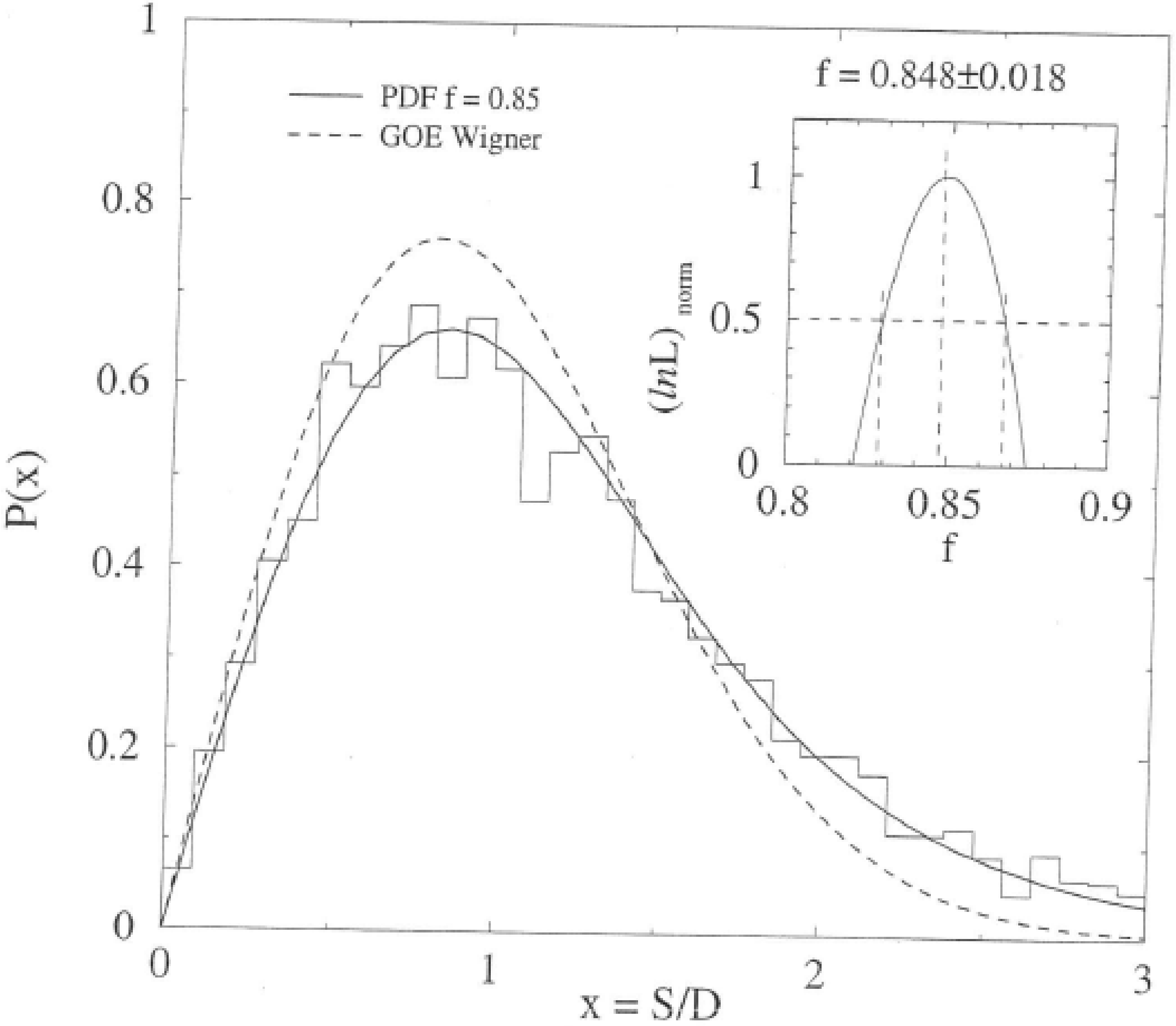
and introduce two Lagrange multipliers α and β (because there are 2 constraint equations)

$$\delta\{S - \alpha\sum a_k - \beta\sum a_k(1+k)/\lambda\} = 0$$

$$\boxed{\begin{aligned} a_k &= (1-f)^k f \\ \lambda &= 1/f \end{aligned}}$$

Using these results and denoting $\lambda z = x$ one obtains:

$$\boxed{\mathrm{P}(x)dx = \sum_k (1-f)^k f\,\mathrm{p}(k,x)dx}$$


PDF f = 0.85
GOE Wigner
f = 0.848±0.018
$(ln\mathrm{L})_{\mathrm{norm}}$
f
P(x)
x = S/D
0
0.2
0.4
0.6
0.8
1
0
1
2
3
0.5
0.8
0.85
0.9

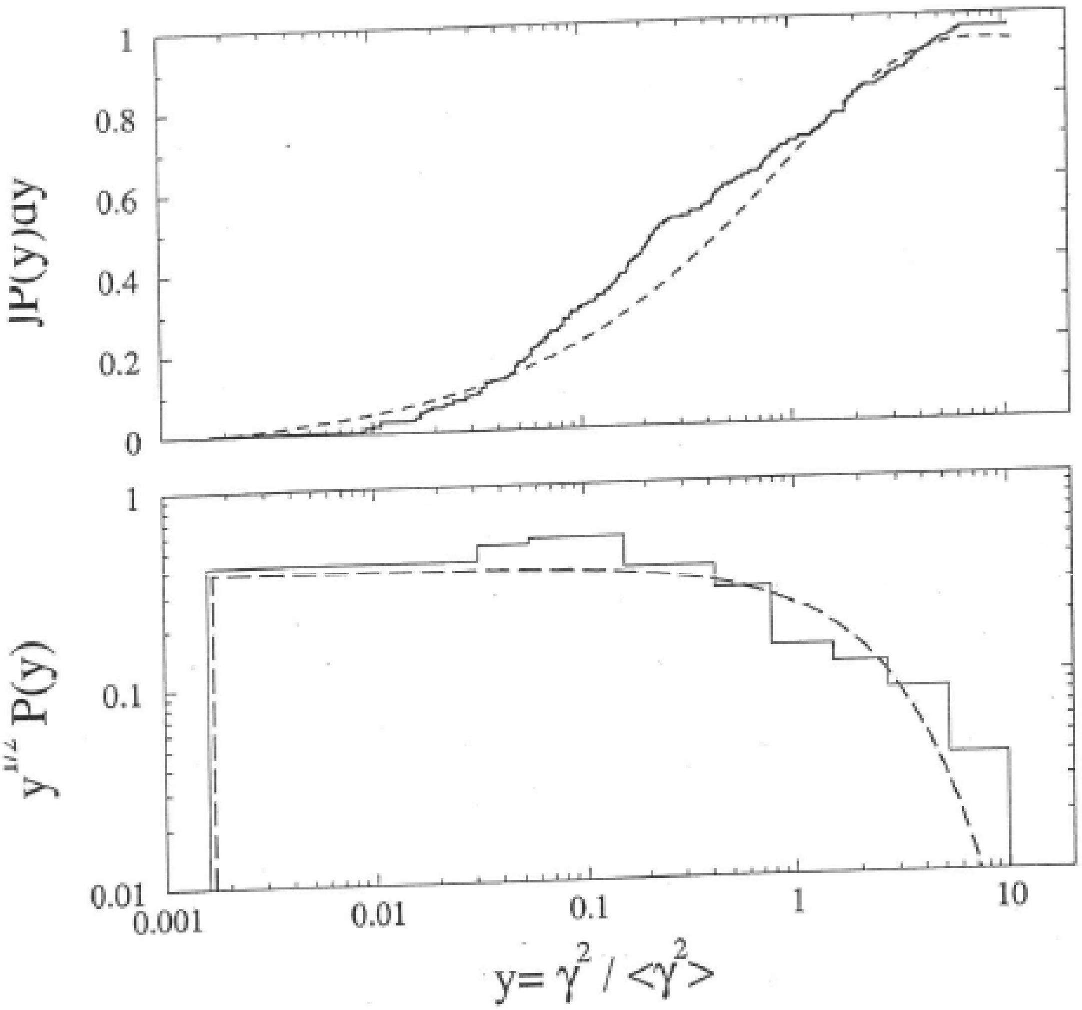
^{48}Ti s-wave resonances

Width analysis $f = 0.87^{+0.13}_{-0.11}$

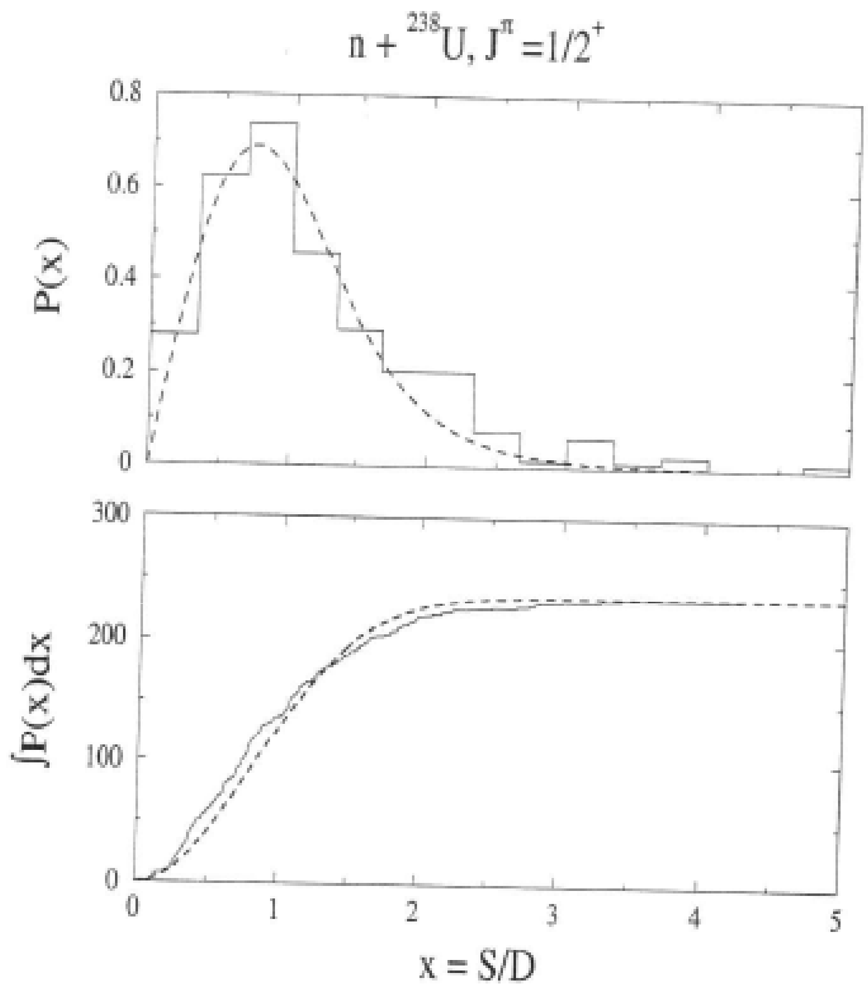
Spacing analysis $f = 0.89 \pm 0.07$

so $\bar{f} = 0.88 \pm 0.06$

$n + {}^{238}U, J^{\pi} = 1/2^{+}$



$f = 0.97^{+0.03}_{-0.08}$



Method 2: Spacing analysis $f = 0.89 \pm 0.06$
less than $f = 0.97^{+0.03}_{-0.08}$ from method 1.

Can one identify a single missing level?

What is the information content of a level in a spectrum?

Depends on nature of spectrum

For a picket fence
remove one level – 100% obvious

For Poisson
remove one level – 0% chance to tell

Normal correlated spectra
somewhere in between

Need information theory for spectra
Entropy, etc.

ANSWER

Dyson did this in 1963!

Thermodynamics for circular ensembles

Coulomb gas – point particles on a circle

Parameter beta (inverse temperature)

Beta = 1,2,4 corresponds to GOE, GUE, GSE

Calculates energy, specific heat, free energy, entropy

Values are given in Mehta's monograph

To our knowledge, nothing has ever been done experimentally.

So we have started a program to examine random matrix thermodynamics

2D Coulomb Gas – GOE

Potential Energy W

$$\mathbf{W} = -\sum_{j>i} \ln \left| e^{i\theta_i} - e^{i\theta_j} \right|$$

Minimum value W_o

Internal energy per particle U

$NU = \langle W - W_o \rangle$

Large N limit – U = 0.365

LINEAR SPECTRUM – NEED CONFINING POTENTIAL

$$W = -\sum_{j>i} \ln|E_i - E_j| + \sum_i V(E_i)$$

N LEVELS – ENERGY INTERVAL $2\mathcal{L}$ – CONFINING POTENTIAL AVERAGE TO ZERO

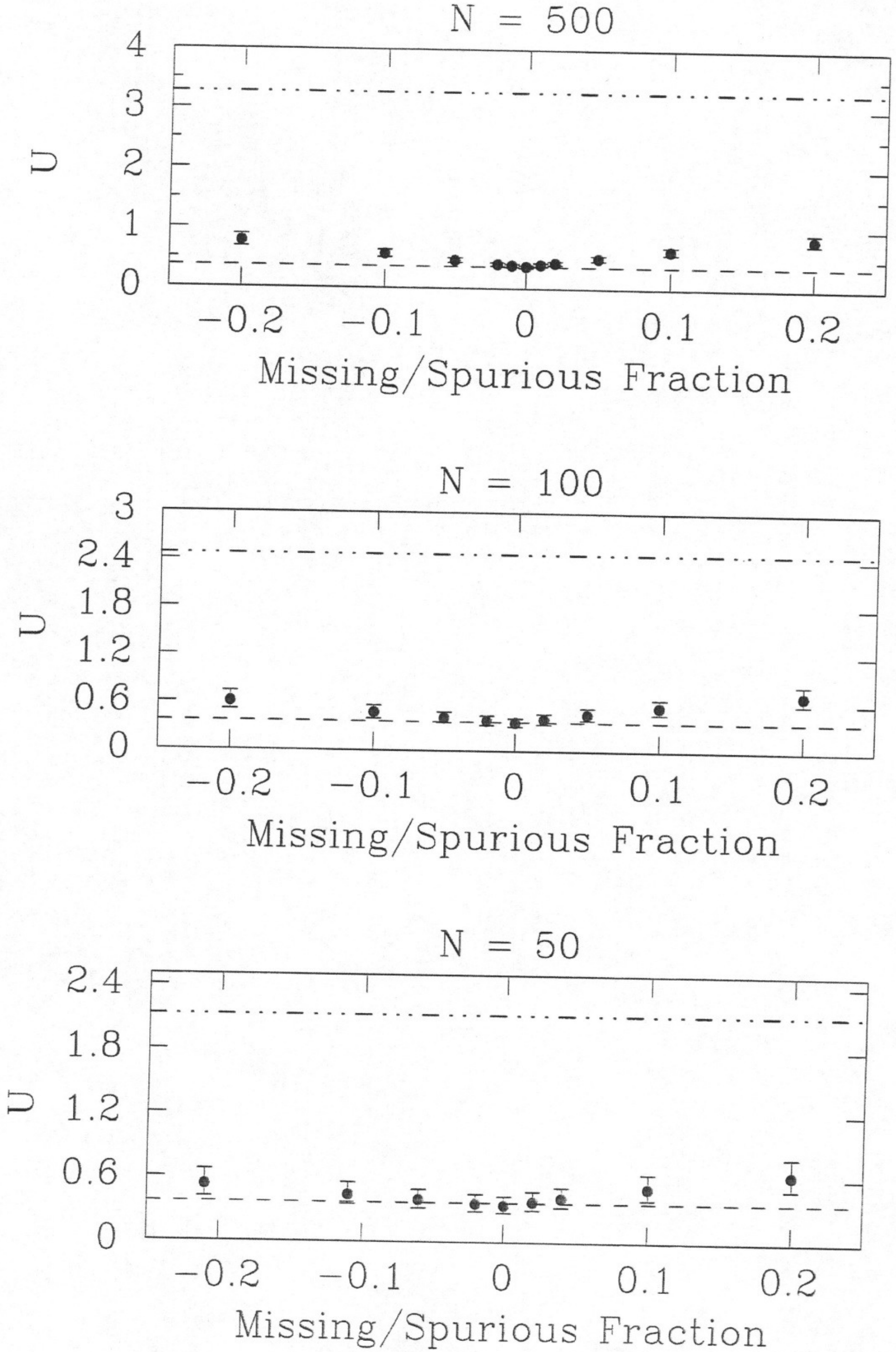
$$V(E) = (\mathcal{L} - E) * \left[\frac{1}{2} + \ln\left(\frac{\mathcal{L}-E}{2\mathcal{L}}\right)\right]$$
$$+ (\mathcal{L} + E) * \left[\frac{1}{2} + \ln\left(\frac{\mathcal{L}+E}{2\mathcal{L}}\right)\right]$$

MINIMUM W_0 FOR PICKET FENCE SPECTRUM

$$\xi_i = -\mathcal{L} + i - \tfrac{1}{2}$$

COMBINING YIELDS FINAL RESULT

$$NU = -\sum_{j>i} \ln \left| \frac{E_j - E_i}{j-i} \right| + \sum_i \left[V(E_i) - V(\xi_i) \right].$$


N = 500
U
4
3
2
1
0
−0.2
−0.1
0
0.1
0.2
Missing/Spurious Fraction
N = 100
U
3
2.4
1.8
1.2
0.6
0
−0.2
−0.1
0
0.1
0.2
Missing/Spurious Fraction
N = 50
U
2.4
1.8
1.2
0.6
0
−0.2
−0.1
0
0.1
0.2
Missing/Spurious Fraction

Dyson-Mehta Δ_3 Statistic

$$\Delta_3 = min_{A,B} \frac{1}{E_{max} - E_{min}} \int_{E_{min}}^{E_{max}} [N(E) - AE - B^2] dE$$

Center levels on E=0 in interval [-L,L]

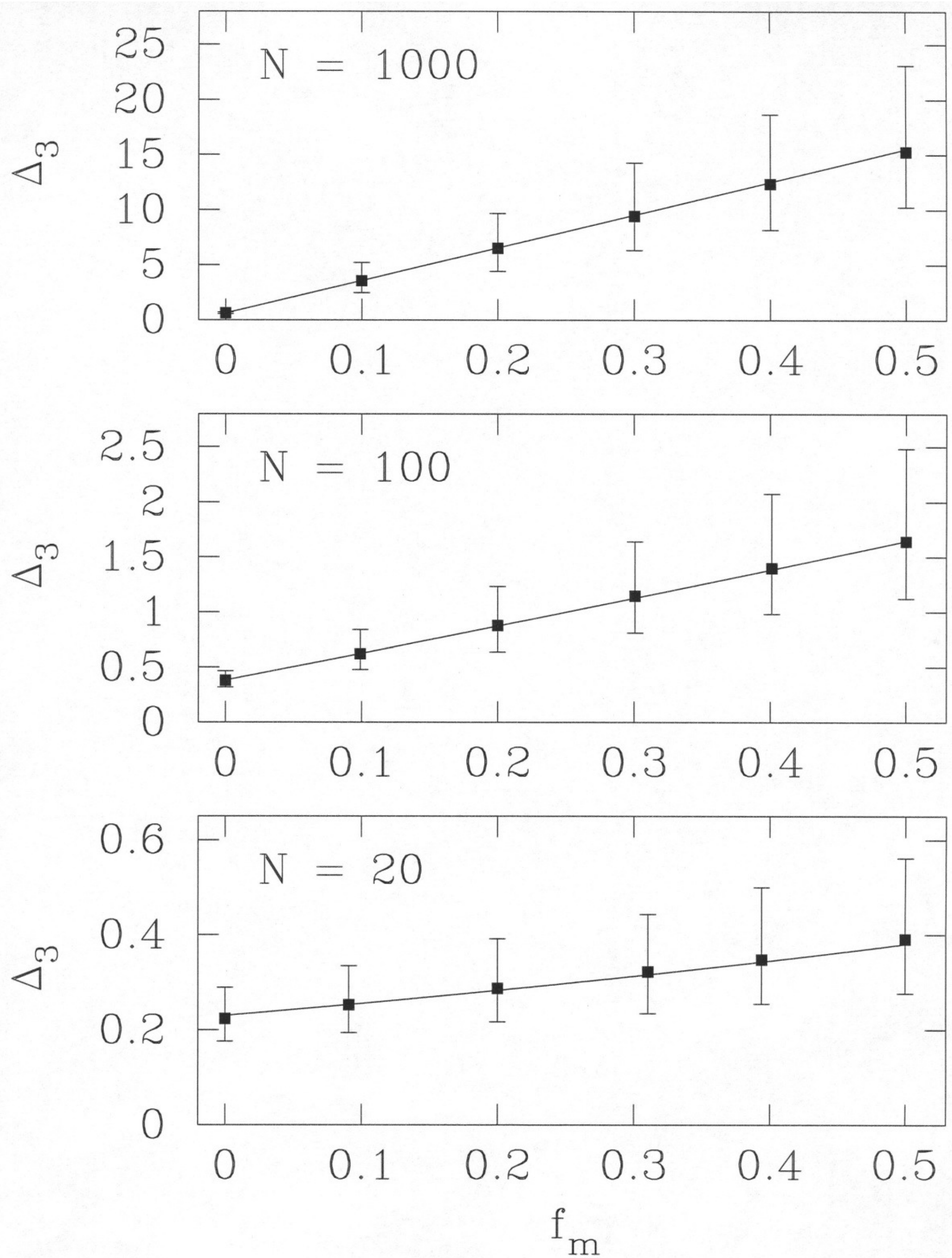
$$\epsilon_i \equiv E_i - (E_{min} + E_{max})/2$$
$$L \equiv (E_{min} - E_{max})/2$$

$$\Delta_3 = \frac{N^2}{16} - \frac{1}{4L^2}\left[\sum_{i=1}^{N} \epsilon_i\right]^2 + \frac{3N}{8L^2}\left[\sum_{i=1}^{N} \epsilon_i^2\right] + \frac{3}{16L^4}\left[\sum_{i=1}^{N} \epsilon_i^2\right]^2 + \frac{1}{2L}\left[\sum_{i=1}^{N} (N - 2i + 1)\epsilon_i\right]$$

for GOE $\quad \Delta_3(N) \approx \frac{1}{\pi^2}[lnN - 0.0687]$

effect of missing levels

$$\Delta_3(f_m, N) = f_m \frac{N}{15} + (1 - f_m)^2 \Delta_{3,GOE}\left(\frac{N}{1 - f_m}\right)$$



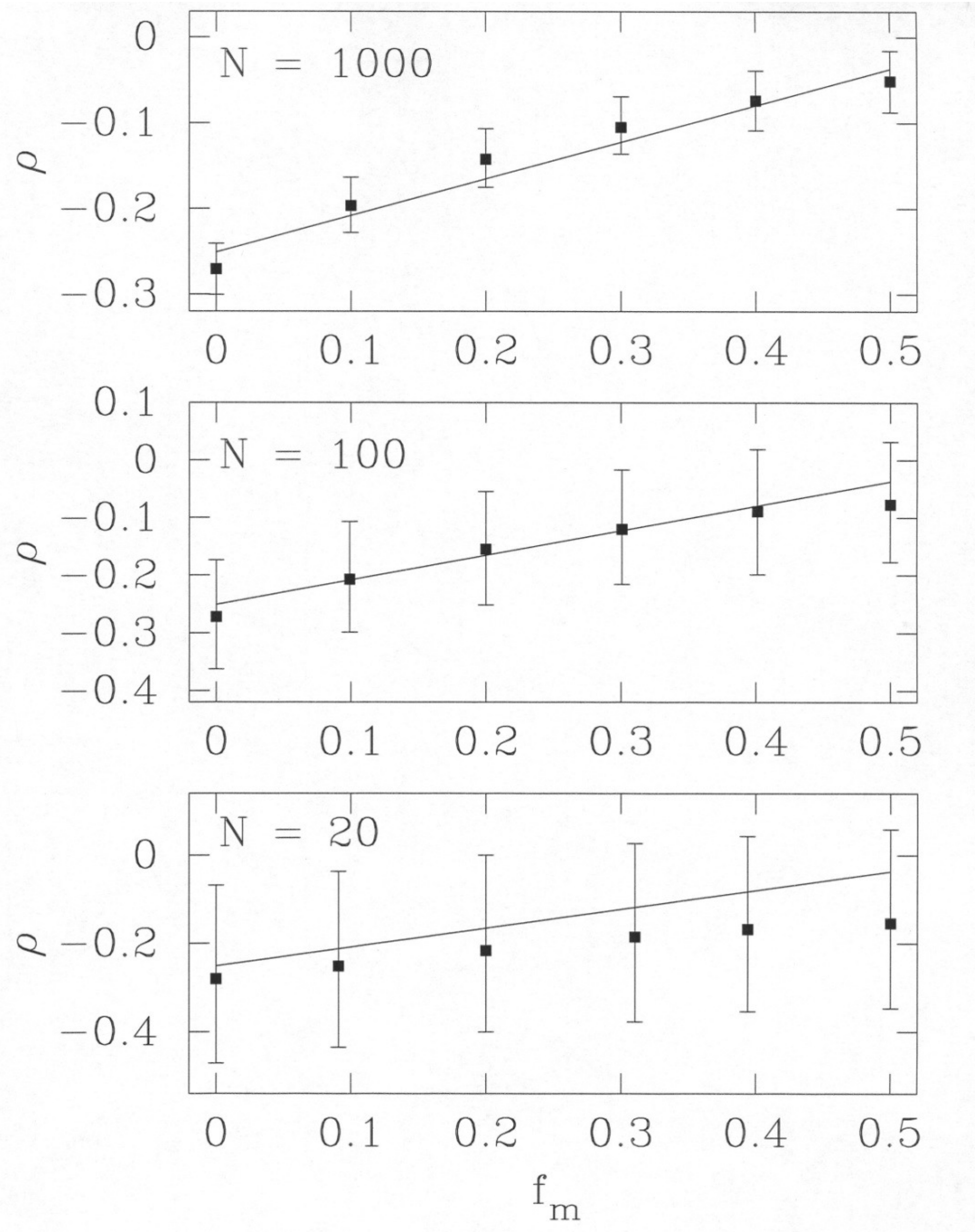
N = 1000
N = 100
N = 20
Δ_3
f_m
0 0.1 0.2 0.3 0.4 0.5
0 5 10 15 20 25
0 0.5 1 1.5 2 2.5
0 0.2 0.4 0.6

LCC for adjacent spacings ρ

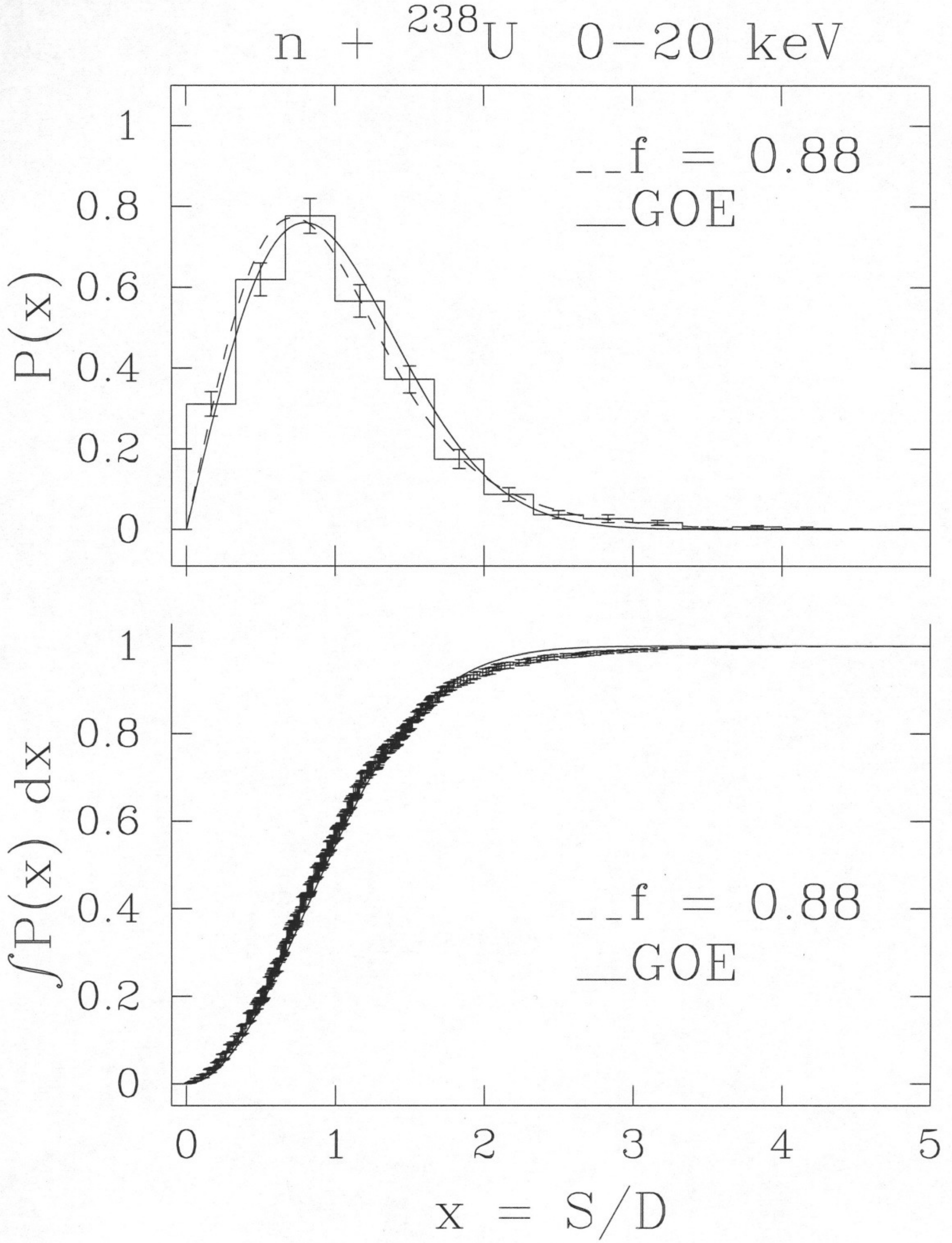
$$S_i = E_{i+1} - E_i$$

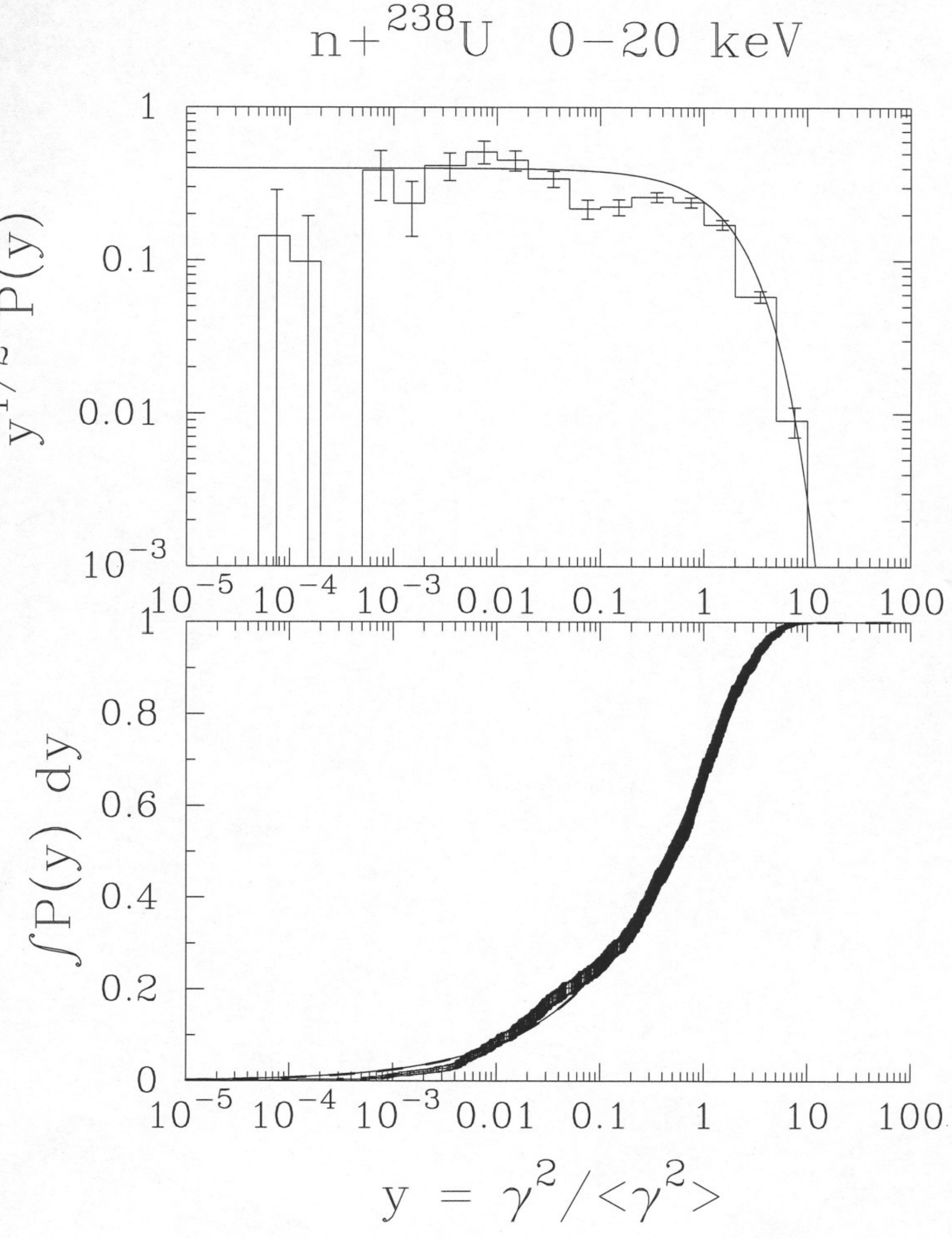
$$\rho(S_i, S_{i+1}) = \frac{\Sigma_i(S_i - < S_i >)(S_{i+1} - < S_{i+1} >)}{[\Sigma_i(S_i - < S_i >^2)\Sigma_i(S_{i+1} - < S_{i+1} >)^2]^{1/2}}$$

for GOE $\rho = -0.27$

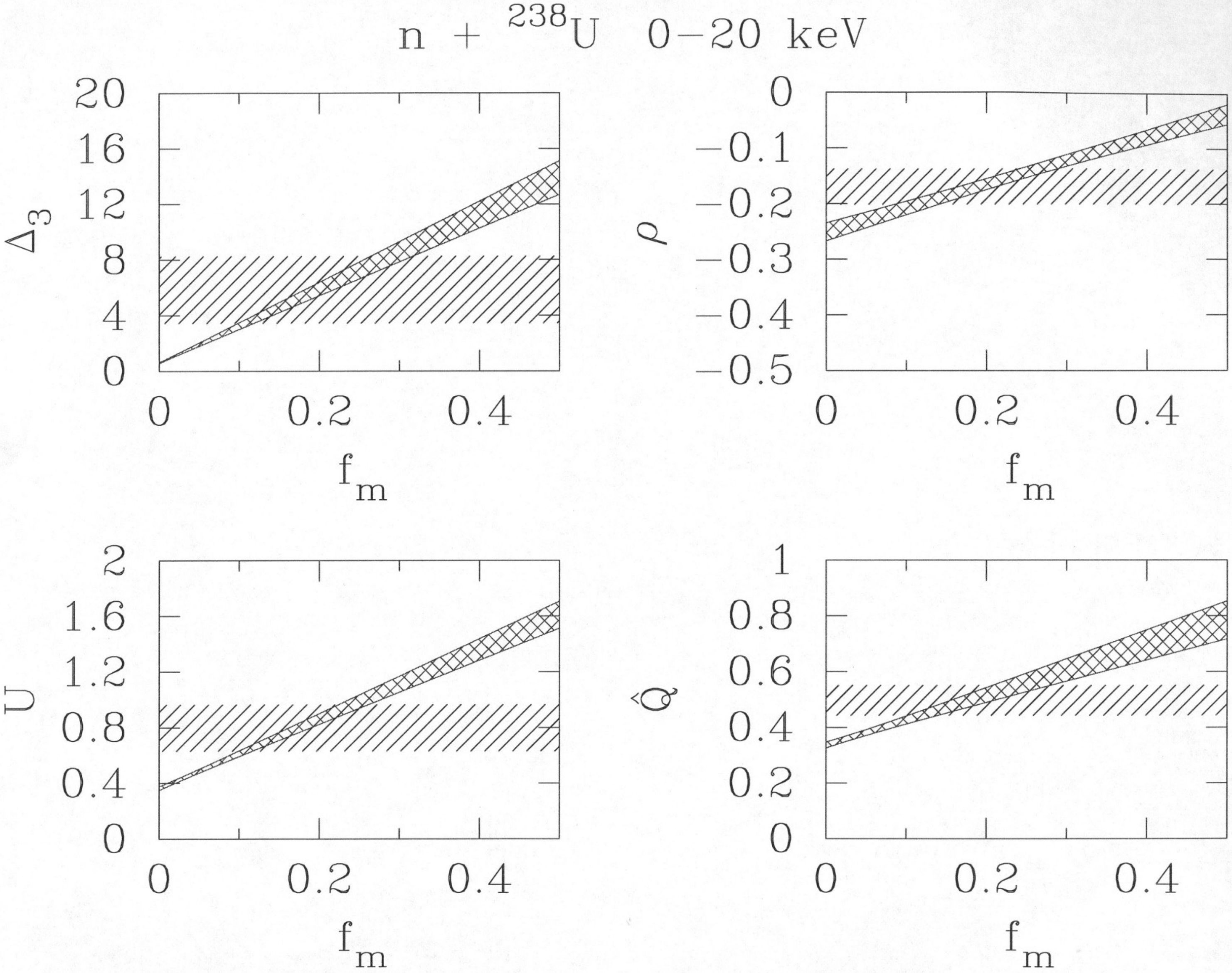


N = 1000
N = 100
N = 20
ρ
0
−0.1
−0.2
−0.3
0.1
−0.4
0.1
0.2
0.3
0.4
0.5
f_m


n + 238U 0–20 keV
__f = 0.88
__GOE
P(x)
1
0.8
0.6
0.4
0.2
0
__f = 0.88
__GOE
∫P(x) dx
0
1
2
3
4
5
x = S/D



n+238U 0–20 keV
y^1/2 P(y)
1
0.1
0.01
10^-3
10^-5
10^-4
10^-3
0.01
0.1
1
10
100
∫P(y) dy
1
0.8
0.6
0.4
0.2
0
y = γ²/<γ²>



n + ^{238}U 0–20 keV
Δ_3
ρ
U
$\hat{Q}$
f_m
f_m
f_m
f_m
0
0.2
0.4
20
16
12
8
4
0
−0.1
−0.2
−0.3
−0.4
−0.5
2
1.6
1.2
0.8
0.4
1
0.8
0.6
0.4
0.2

IAEA
International Atomic Energy Agency

INDC(NDS)-0561
Distr. G+NM

INDC International Nuclear Data Committee

Missing Level Corrections using Neutron Spacings

G.E. Mitchell
North Carolina State University and
Triangle Universities Nuclear Laboratory

and

J.F. Shriner, Jr.
Tennessee Technological University

November 2009

IAEA Nuclear Data Section, Vienna International Centre, 1400 Vienna, Austria

Random Matrices and Chaos in Nuclear Physics:

Nuclear Structure

H.A. Weidenmueller and G.E. Mitchell

Reviews of Modern Physics 81, 539 (2009)