Collision time

de Broglie wavelength: $\Lambda_e = \dfrac{h}{2\pi m_e k_B T_e} \approx \dfrac{h}{mv}$

notice: $\Delta v = \dfrac{Ze^2}{2\pi \varepsilon_0 m b v} \Rightarrow \Delta v \propto \dfrac{1}{b}$

i.e head on collision

$\Rightarrow$ infinite velocity $\rightarrow$ impossible

$\Rightarrow$ smallest meaningful impact parameter gives $\Delta v \approx v$

$$\Rightarrow b_{min} \simeq \frac{Ze^2}{4\pi \varepsilon_0 m v^2}$$

$\rightarrow$ usually larger than Λ_e

$\Rightarrow$ de Broglie approx. justified

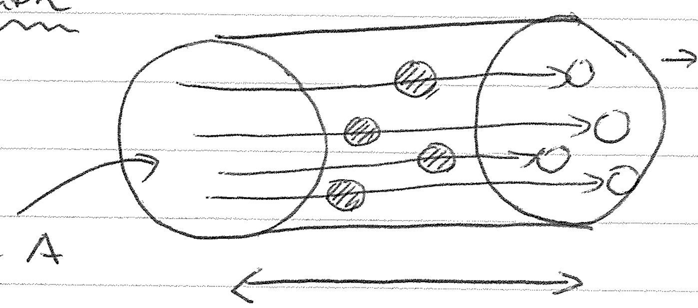
$\rightarrow$ Coulomb logarithm:

$$\int_B^A \frac{1}{x}\,dx = \left[\ln x\right]_B^A \Rightarrow \ln B - \ln A \Rightarrow \ln\left(\frac{B}{A}\right)$$

$b_{max} = \lambda_D$

$b_{min} = \Lambda_e$

Mean free path



$\rightarrow$ differential probability for a collision: $\Delta W = \dfrac{N_A \sigma}{A} = n_a \sigma \Delta Z$

blocked area; total area; density of target; cross section

$n_a = N_a (A \Delta Z)^{-1}$

⇒ Collision frequency defined as: $\boxed{\nu_{coll} = \sigma v n_a}$

$$\tau_{coll} = \frac{1}{\nu_{coll}}$$

→ multiply probabilities:

$$W(z) = \lim_{\Delta z \to 0} \prod_{i=1}^{z/\Delta z} (1 - n_a \sigma \Delta z) = \lim_{\Delta z \to 0} (1 - n_a \sigma \Delta z)^{z/\Delta z}$$

$$= \exp(-n_a \sigma z) = \exp(-z/\lambda_{mfp})$$

→ where $\lambda_{mfp} = \frac{1}{n_a \sigma}$ ⇒ inversely proportional to density

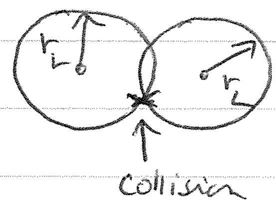
Effects of collisions

→ energy confinement time: $\tau_E \sim \tau_e \left(\frac{a}{\delta}\right)^2$

plasma radius (a)

step length (δ)

→ classical diffusion: $\delta = r_L$

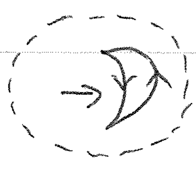


tokamak: $\delta \sim r_L \sim 2\,mm$

$a \cong 1.2\,m$

⇒ $\tau \sim$ few minutes

→ neoclassical diffusion – vertical drift motion

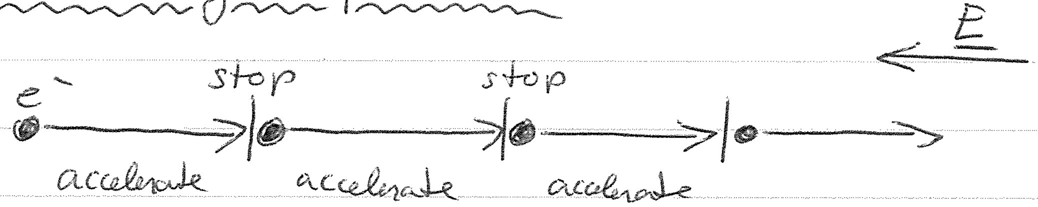


→ each collision moves particle by 1 banana orbit

$\rightarrow$ anomalous diffusion - due to turbulence
- macroscopic fluctuations
$\rightarrow$ increase frequency & scale
$\Rightarrow$ increase τ_E by increasing a (plasma radius)

$\Rightarrow$ BIGGER TOKAMAK !

Resistivity in plasmas



τ_e = time between collisions
η = resistivity = $\frac{1}{\sigma}$ $\nwarrow$ conductivity

$\rightarrow$ previously derived collision time: $\tau = \frac{2\pi \varepsilon_0^2 m^2 v^3}{n_i Z^2 e^4 \ln(\Lambda)}$

$\rightarrow$ assume uniform v (increased, but the difference is small)

where $v = v_{th} = \sqrt{\frac{3k_B T_e}{m_e}}$

$$\Rightarrow \eta = \frac{m_e}{n_e e^2 \tau} = \frac{m_e}{n_e e^2} \cdot \frac{n_i Z^2 e^4 \ln\Lambda}{2\pi \varepsilon_0^2 m^2 v^3} \quad \text{and } n_i = \frac{n_e}{Z}$$

$$= \frac{m_e^{2/2}}{\cancel{n_e} \cancel{e^2}} \cdot \frac{n_e}{\cancel{Z}} \cdot \frac{Z^{\cancel{2}} \cancel{e^4}^{\,e^2} \ln\Lambda}{2\pi \varepsilon_0^2 m_e^{4/2} (3k_B T_e)^{3/2}} \cdot m_e^{3/2}$$

$$\approx \frac{m_e^{1/2} e^2}{\varepsilon_0^2 (k_B T)^{3/2}} \cdot Z \ln\Lambda \; //$$

Bremsstrahlung

$$\Delta W = \Delta t \frac{dW}{dt} = \frac{4}{3} \cdot \frac{Z^2 e^6}{(4\pi\varepsilon_0)^3 m_e^2 c^3 b^3 v}$$

as $\Delta t = \frac{2b}{v}$

$\Rightarrow$ no. of scattering events for a single ion:

electron flux $\times$ area of ring

$= n_e v \times 2\pi b \, db$

energy loss per volume per second

$$\Rightarrow \text{multiple events} = dP_{br} = \underbrace{\frac{8\pi}{3} \cdot \frac{Z^2 e^6 n_e n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3}}_{B_{br}} \cdot \frac{db}{b^2}$$

$\rightarrow$ singularity @ $b = 0$, but the smallest possible scale is the de Broglie wavelength: $\lambda_B = \frac{h}{m_e v} \approx \frac{h}{\sqrt{3 k_B T m_e}}$

$$\left(\frac{1}{2} m_e v^2 = \frac{3}{2} k_B T_e\right)$$

$\rightarrow$ integrating: $P_{Br} = \int_{\lambda_B}^{\infty} B_{br} \cdot \frac{db}{b^2}$

$$= \frac{8\pi}{\sqrt{3}} \cdot \frac{(k_B T_e)^{1/2}}{(4\pi\varepsilon_0)^3 m_e^{3/2} c^2 h} \cdot n_e n_i Z^2$$

Bremsstrahlung in plasma

→ nonrelativistic expression for the radiated energy per unit of time at an acceleration a:

$$\frac{dW}{dt} = \frac{e^2}{6\pi\varepsilon_0 c^3}\cdot a^2 = \frac{e^2}{6\pi\varepsilon_0 c^3}\cdot\left(\frac{dv}{dt}\right)^2$$

→ acceleration due to attraction of e^- to an ion:

$$a \approx \frac{Ze^2}{4\pi\varepsilon_0 b^2 m_e} \quad \text{as} \quad m_e\frac{dv}{dt} = \frac{Ze^2}{4\pi\varepsilon_0 \underset{\substack{\uparrow \\ \text{impact parameter}}}{b^2}}$$

and $\Delta t = \frac{2b}{v}$ (interaction time)

estimate radiated energy:

$$\Rightarrow \Delta W = \Delta t\,\frac{dW}{dt} = \frac{\cancel{2b}}{v}\cdot\frac{e^2}{\underset{3}{\cancel{6}}\pi\varepsilon_0 c^3}\cdot\frac{Z^2e^4}{(4\pi)^2\varepsilon_0^2 b^{\underset{3}{\cancel{4}}} m_e^2}\times\frac{4\pi}{4\pi}$$

$$= \frac{4}{3}\cdot\frac{Z^2e^6}{(4\pi\varepsilon_0)^3 m_e^2 c^3 b^3 v} \quad \text{(radiated energy for a single event)}$$

→ no. of scattering events per second for a single ion given by: $\underset{\substack{\uparrow \\ \text{current density}}}{\overset{\text{(electron flux)}}{n_e v}}$ × area of ring $2\pi b\,db$

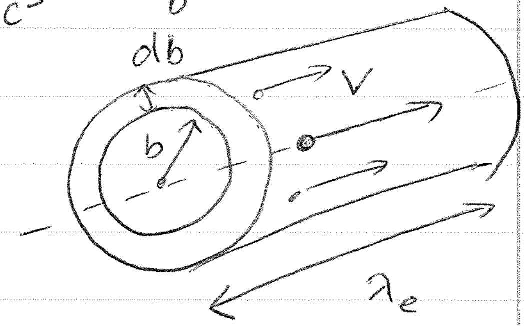
→ event rate per volume (many ions) ⇒ multiply by $\times\, n_i$

⇒ energy loss rate per volume from collisions with $\overbrace{b \text{ to } b+db}^{\text{ring}}$:

$$dP_B = \frac{8\pi}{3}\cdot\frac{Z^2e^6 n_e \cancel{v}\times n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3 b^{\underset{2}{\cancel{3}}}\cancel{v}}\cdot\cancel{b}\cdot db$$

→ simplify:

$$dP_B = \frac{8\pi}{3} \cdot \frac{Z^2 e^6 n_e n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3} \cdot \frac{db}{b^2}$$



⇒ integrate to get total power

→ ignore singularity @ $b=0$
as smallest possible radius is the de Broglie wavelength

$$\lambda_B = \frac{h}{m_e v} \approx \frac{h}{(3k_B T_e m_e)^{1/2}}$$

where we assume: $\frac{1}{2} m_e v^2 = \frac{3}{2} k_B T_e$

⇒ integrate from λ_B to infinity over all possible b (impact parameter)

$$\int_{\lambda_B}^{\infty} \frac{8\pi}{3} \cdot \frac{Z^2 e^6 n_e n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3} \cdot \frac{db}{b^2}$$

$\frac{-1}{\infty} - \frac{-1}{\lambda_B}$
$\uparrow$ 0 " $\frac{1}{\lambda_B}$

$$= \frac{8\pi}{3} \cdot \frac{Z^2 e^6 n_e n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3} \cdot \left[\frac{-1}{b} \right]_{\lambda_B}^{\infty}$$

$$= \frac{8\pi}{3} \cdot \frac{Z^2 e^6 n_e n_i}{(4\pi\varepsilon_0)^3 m_e^2 c^3} \cdot \frac{\sqrt{3}\,(k_B T_e m_e)^{1/2}}{h}$$

$$\Rightarrow P_B = \frac{8\pi}{\sqrt{3}} \cdot \frac{(k_B T_e)^{1/2} \cdot e^6}{(4\pi\varepsilon_0)^3 m_e^{3/2} c^3 h} \cdot n_e n_i Z^2 \quad //$$

Other radiation losses

photon E = E of free e^- + ionisation potential

cut-off energy

- $h\nu < E_{ion}$ ⇒ no radiation
- $h\nu > E_{ion}$ ⇒ spectrum like Bremsstrahlung

→ total radiation power output $\cong P_{Bremsstrahlung} \times \frac{E_{ion}}{\tau}$
(for each ionisation stage)

→ Complications: opacity
scattering
synchrotron radiation

Maxwell's equations:

$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0}$ (Gauss Law)

$\nabla \cdot \underline{B} = 0$ or $\nabla \cdot H = 0$ (Gauss law for magnetism)

$\nabla \times \underline{E} = -\mu \frac{\partial \underline{H}}{\partial t}$ or $\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$ (Faraday's law)

$\nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$ or $\nabla \times H = \underline{J} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$ (Ampere's Law)

$\nabla \times H = \underline{J} + \frac{\partial D}{\partial t}$

J – conduction current, charges moving

$\frac{\partial D}{\partial t}$ – displacement current, $\propto$ rate of change in E-field $\propto \frac{\partial E}{\partial t}$

since $\frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t}$ $\Rightarrow$ and for plasmon $E = \frac{nex}{\varepsilon_0}$

$\Rightarrow \frac{\partial D}{\partial t} = ne \frac{\partial x}{\partial t} = -J_{\parallel}$

Electromagnetic waves in plasma

$\Rightarrow$ for electromagnetic waves: $k \perp E \Rightarrow \nabla \cdot E = 0$

$k \cdot E = 0$

i.e no charge separation for transverse waves (EM waves)

$\Rightarrow$ Gauss law: $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 0$ //

$\Rightarrow$ Ampère's Law: $\nabla \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t} = \underbrace{\sigma(\omega)\underline{E}}_{\underline{J}} + \frac{\partial D}{\partial t}$

Ohm's Law

$\Rightarrow$ vector identity: $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$

here Ampère's law: $\nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$

$\nabla \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t}$ where $D = \varepsilon_r \varepsilon_0 E$

and $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ and $c' = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} = \frac{c}{\sqrt{\varepsilon_r}}$

refractive index: $n = \frac{c}{c'} = \sqrt{\varepsilon_r}$

$\Rightarrow k^2 = -\mu_0(i\omega\sigma - \varepsilon_0\omega^2) = \mu_0(\varepsilon_0\omega^2 - i\omega\sigma)$

$= \omega^2 \cdot \underbrace{\mu_0 \varepsilon_0}_{1/c^2} \left(1 - \frac{i\sigma}{\varepsilon_0 \omega}\right)$

$= \frac{\omega^2}{c^2}\left(1 - \frac{i\sigma}{\varepsilon_0\omega}\right)$, remember $\frac{\omega}{k} = c' = \frac{c}{\sqrt{1 - \frac{i\sigma}{\varepsilon_0 \omega}}}$

speed of light in medium

dielectric constant ε_r

⇒ electric field drives the electrons ⇒ derive relationship between the magnitude of E and resultant electron velocity:

$$m \frac{dv}{dt} = i\omega m v(\omega)\, exp(\cancel{i\omega t}) = -e\, E(\omega)\, exp(\cancel{i\omega t})$$

$$\Rightarrow v(\omega) = -\frac{e}{im\omega}\, E(\omega) \quad //$$

Conductivity of plasma

⇒ remember the plasma frequency $\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}}$

$$\Rightarrow J(\omega) = \sigma(\omega)\, E(\omega)$$

$$\text{while } v(\omega) = -\frac{e}{im\omega}\, \overset{\times i}{\underset{\times i}{E(\omega)}} = \frac{ie}{m\omega}\, E(\omega)$$

$$\Rightarrow J(\omega) = -nev(\omega) = -i \underbrace{\frac{ne^2}{m\omega}}_{=\sigma(\omega)} E(\omega)$$

and remember: $\varepsilon(\omega) = 1 - \frac{i\sigma}{\varepsilon_0 \omega} = 1 - \frac{\overbrace{i\cdot(-i)}^{=1}\, ne^2}{\varepsilon_0 m\, \omega \cdot \omega}$

$$= 1 - \frac{ne^2}{\varepsilon_0 m \omega^2}$$

⇒ as $\omega \nearrow$ ⇒ $\varepsilon(\omega) = 1$

as $\omega \searrow$ ⇒ $\varepsilon(\omega = \omega_p) = 0$! and at $\omega \ll \omega_p$, $\varepsilon(\omega) < 0$

⇒ light at $\omega < \omega_p$ cannot propagate into plasma

⇒ reflection ⇒ reflecitivity: $R = \left| \frac{1-\sqrt{\varepsilon}}{1+\sqrt{\varepsilon}} \right|^2$ ⇒ light reflected

$$\sigma(\omega) = -\frac{i n e^2}{m\omega}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

⇒ back to EM waves in plasma

⇒ again start from: $\nabla(\nabla\cdot\underline{E}) - \nabla^2\underline{E} = -\frac{\partial(\nabla\times\underline{B})}{\partial t}$

⇒ derive in r for: $E = E_0 \exp[i(\omega t - \underline{k}\cdot\underline{r})]$

$$\Rightarrow -\underline{k}(\underline{k}\cdot\underline{E}) + k^2\underline{E} = -\mu_0 \frac{\partial}{\partial t}\left(J + \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \underline{k}\cdot(\underline{k}\cdot\underline{E}) - k^2\underline{E} = \underbrace{\mu_0\varepsilon_0}_{1/c^2} \cdot \frac{\partial}{\partial t}\left(-i\underbrace{\left(\frac{ne^2}{\varepsilon_0 m}\right)}_{=\omega_p^2}\underline{E} + \frac{\partial E}{\partial t}\right)$$

$$\frac{\partial E}{\partial t} = i\omega\underline{E} \qquad = \frac{1}{c^2}\cdot\left[\frac{\overbrace{(-i)\cdot i}^{=1}\cdot\cancel{\omega}}{\cancel{\omega}}\cdot\omega_p^2\underline{E} - \omega^2\underline{E}\right]$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega\underline{E}$$

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \Rightarrow \varepsilon(\omega)\omega^2 = \omega^2 - \omega_p^2$$
$$\Rightarrow \omega_p^2 = \omega^2 - \omega^2\varepsilon(\omega)$$

→ general expression: $\underbrace{(\omega^2\varepsilon(\omega)}_{=\omega^2-\omega_p^2} - c^2k^2)\underline{E} + c^2\underline{k}^2(\underline{k}\cdot\underline{E}) = 0$ (as $k \perp E$)

⇒ critical density for plasma: $\omega^2 = \omega_p^2$

$$\Rightarrow \omega^2 = \frac{n_{crit}e^2}{\varepsilon_0 m} \Rightarrow n_{crit} = \frac{\varepsilon_0 m \omega^2}{e^2}$$

as $n_e\nearrow$ $\omega_p\nearrow$ ⇒ group velocity $V_g = \frac{\partial\omega}{\partial k}\searrow$, phase velocity $v_{ph} = \frac{\omega}{k}\nearrow$

→ refractive index $n = \frac{c}{v_{ph}} = \sqrt{\varepsilon(\omega)} \to 1$ as $v_{ph} \to c$

when $\omega = \omega_p$ ⇒ $n \to 0$, $\frac{\partial\omega}{\partial k} \to 0$, $\frac{\omega}{k} \to \infty$ → no propagation