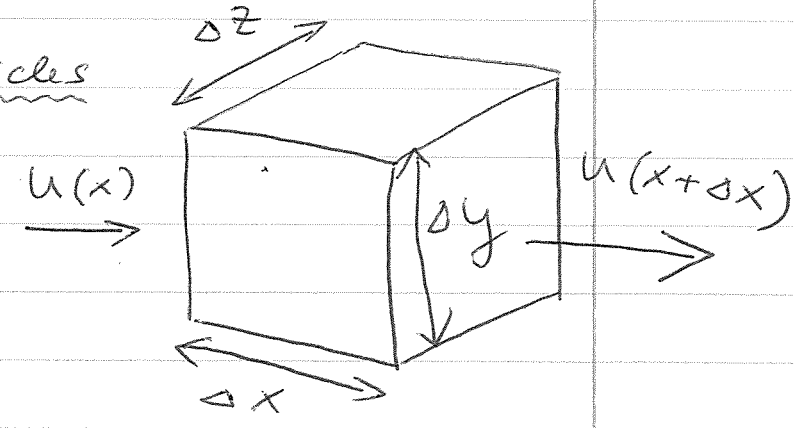
# Continuity equation

## Conservation of particles

(2 types of collisions:
- velocity averaging
- velocity dispersing)




- mass through box on LHS in $\Delta t = \rho(x)\, u(x)\, \Delta y \Delta z \cdot \Delta t$

- mass out on RHS in $\Delta t = \rho(x+\Delta x)\, u(x+\Delta x)\, \Delta y \Delta z \cdot \Delta t$

$\Rightarrow$ change in mass $= \left[-\rho(x+\Delta x)\, u(x+\Delta x) + \rho(x)\, u(x)\right] \Delta y \Delta z \cdot \Delta t$

- change in $\rho = \dfrac{\Delta m}{V} = \left[-\rho(x+\Delta x)\, u(x+\Delta x) + \rho(x)\, u(x)\right] \dfrac{\Delta t}{\Delta x}$

$= \Delta \rho$

$\Rightarrow$ as $\Delta x, \Delta t \to 0 \Rightarrow \dfrac{\partial \rho}{\partial t} = -\dfrac{\partial (\rho u)}{\partial x}$

$\Rightarrow$ in 3D: $\boxed{\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$

# The Vlasov equation

→ Collision term : $\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}$

→ Continuity equation : $\frac{\partial n_e}{\partial t} + \frac{\partial (n_e u_e)}{\partial x} = 0$

$$\Rightarrow \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e \dot{x}) = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial}{\partial \underline{r}} \cdot (f \dot{\underline{r}}) + \frac{\partial}{\partial \underline{v}} \cdot (f \dot{\underline{v}}) = 0$$

$F = ma$ is not a function of $\underline{v}$ $\Rightarrow \nabla_{\underline{v}} \cdot a = 0$

# Langmuir waves

→ as $\frac{\partial f_1}{\partial t} = -i\omega f_1(\underline{k}, \omega)$

$$\nabla \cdot \frac{\partial f_1}{\partial \underline{r}} = i \underline{k} \cdot \underline{v} f_1(\underline{k}, \omega)$$

→ $\underline{E} = \underline{E}_{k,\omega} \exp\left(i[\underline{k} \cdot \underline{r} - \omega t]\right)$

→ thermal velocity: $v_{th} = \sqrt{\frac{k_B T}{m}}$ (electrons)

→ Gaussian integrals:

$$I_0 = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}$$

$$I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}$$

$$I_2 = \int_0^\infty x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2\alpha^{3/2}}$$

$$I_4 = \int_0^\infty x^4 e^{-\alpha x^4} dx = \frac{3\sqrt{\pi}}{4\alpha^{5/2}}$$

start from: $\int e^{-\frac{mv^2}{2k_BT}} dv$ and $\int \frac{v_z e^{-\frac{mv_z^2}{2k_BT}}}{\omega - kv_z} \cdot dz$

and remember Gaussian integral: $\int e^{-x^2} dx = \sqrt{\pi}$

$$I_1 = \int_0^\infty x e^{-\alpha x^2} dx = \frac{1}{2\alpha}$$

→ Substitute: $x^2 = \frac{mv^2}{2k_BT} \Rightarrow x = \sqrt{\frac{m}{2k_BT}} \cdot v$

$$\Rightarrow dv = \sqrt{\frac{2k_BT}{m}} \cdot dx \quad \text{(change of variable)}$$

$$\Rightarrow \int e^{-\frac{mv^2}{2k_BT}} \cdot dv = \sqrt{\frac{2k_BT}{m}} \int e^{-x} \cdot dx = \sqrt{\frac{2\pi k_BT}{m}} \;//$$

and:

$$\int v \cdot e^{-\frac{mv^2}{2k_BT}} dv = \left(\frac{2k_BT}{m}\right) \int x e^{-x^2} dx = \frac{k_BT}{m} \;//$$

⇒ then use $\omega_p = \sqrt{\frac{n_0 e^2}{\varepsilon_0 m}}$ ↘ any odd function integrates to zero for $-\infty$ to $\infty$

$$\Rightarrow 1 - \frac{2}{\pi^{1/2}} \cdot \frac{\omega_p^2}{\omega^2 \cancel{k}} \cdot \left(\frac{m}{2k_BT}\right)^{3/2} \int \left(v^{\cancel{2}} + \frac{k^{\cancel{2}}}{\omega^2} \cdot v^4 + \dots\right) e^{-\frac{m}{2k_BT} \cdot v^2} dv = 0$$

k cancell out ↑ k « less

Langmuir waves

higher orders: $\int x^2 e^{-\alpha x^2} dx = \frac{\pi^{1/2}}{2\alpha^{3/2}}$

$$\int x^4 e^{-\alpha x^2} dx = \frac{3\pi^{1/2}}{4\alpha^{5/2}}$$

$\Rightarrow$ integrate higher terms in expansion: $v_z \left(1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + \dots\right)$

$$\Rightarrow \int v^2 \cdot e^{-\frac{m}{2k_B T} \cdot v^2} \cdot dv = \frac{\sqrt{\pi}}{2 \cdot \left(\frac{m}{2k_B T}\right)^{3/2}}$$

$$\Rightarrow \frac{k^2}{\omega^2} \int v^4 \cdot e^{-\frac{m}{2k_B T} \cdot v^2} dv = \frac{k^2}{\omega^2} \cdot \frac{3}{4} \cdot \frac{\sqrt{\pi}}{\left(\frac{m}{2k_B T}\right)^{5/2}}$$

$$\Rightarrow 1 - \frac{2}{\sqrt{\pi}} \cdot \frac{\omega_p^2}{\omega^2} \left(\frac{m}{2k_B T}\right)^{3/2} \cdot \frac{\sqrt{\pi}}{2 \cdot \left(\frac{m}{2k_B T}\right)^{3/2}}$$

$$- \frac{2}{\sqrt{\pi}} \frac{\omega_p^2}{\omega^2} \cdot \left(\frac{m}{2k_B T}\right)^{3/2} \cdot \frac{k^2}{\omega^2} \cdot \frac{3}{4} \cdot \frac{\sqrt{\pi}}{2} \left(\frac{m}{2k_B T}\right)^{5/2} = 0$$

$$\Rightarrow 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\omega^2} \cdot \frac{k^2}{\omega^2} \cdot \frac{3}{2} \cdot \frac{2k_B T}{m} = 0$$

$\frac{2k_B T}{m} \leftarrow v_{th}^2$

$$\Rightarrow 1 - \frac{\omega_p^2}{\omega^2} \left(1 + 3 \frac{k^2}{\omega^2} \cdot v_{th}^2 + \dots\right) = 0$$

→ remember plasma frequency: $\omega_{pe}^2 = \frac{n_0 e^2}{\varepsilon_0 m}$

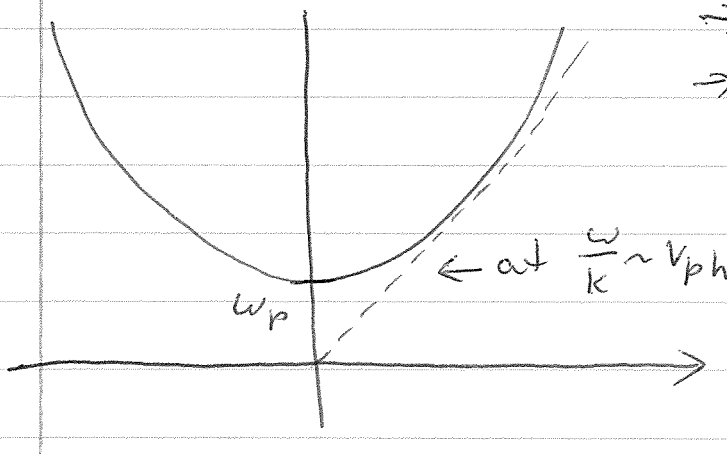
→ assumption: $v_z \gg v_{th}$ (cold plasma)

⇒ thus approx. solution $\omega \approx \omega_{pe}$

$$1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2} \cdot v_e^2\right) = 0$$

$$1 = \frac{\omega_p^2}{\omega^2} + 3\frac{k^2}{\omega^2} v_e^2 \cdot \frac{\omega_p^2}{\omega^2} \quad \text{and } \omega \approx \omega_p$$

$$\Rightarrow \omega^2 = \omega_p^2 + 3k^2 v_e^2 \;//$$




→ no solution for $\omega < \omega_p$

→ can drive longitudinal waves for $\omega > \omega_p$ (otherwise $e^-$ too fast ⇒ quasi neutrality)

$$\Rightarrow \frac{d\omega}{dk} \cong \sqrt{\frac{k_B T}{m}}$$

(our old assumption)

⇒ information travels @ sound speed, here thermal speed

⇒ similar to sound waves!

# Landau damping

→ resonance at $v_z = \frac{\omega}{k}$ (the pole)

$$\Delta v \approx \left(\frac{eE}{mk}\right)^{1/2} \sim \left(\frac{2e\phi_0}{m}\right)^{1/2}$$

estimate $\phi_0$:

$$\phi_0 = \frac{Q}{4\pi\varepsilon_0 r} \quad \text{and} \quad \underset{\text{(energy)}}{V_E} = q\phi \rightarrow F_E = \frac{qQ}{4\pi\varepsilon_0 r^2}$$

$$\rightarrow \phi_0 = \frac{F_E \cdot r}{q} = r \cdot E \Rightarrow \phi_0 \sim \lambda \cdot E_{ampl}$$

and set $E_{Ampl} \sim \frac{E}{2}$

→ estimate energy density by capatitor approtimation:

- E stored in capacites: $U = \frac{1}{2} \cdot \frac{Q^2}{C}$
- E stored by field generated by charge separatia between plates: $U = \frac{1}{2} C V^2$

- for capacitor with area of plates A and separation d:

$$V = E \cdot d \quad \text{and} \quad C = \varepsilon_0 \frac{A}{d}$$

$\Rightarrow$ stored energy: $U = \frac{1}{2}\varepsilon_0 E^2 (A \cdot d) \Rightarrow$ energy density $= \frac{1}{2}\varepsilon E^2$ //

Power: $P = \frac{dU}{dt} \times$ no. of particles

$$\Delta E = m v \Delta v$$

→ energy loss $\Delta E$ (in the same frame of the oscillating $e^-$)

$$\Delta E = F \cdot s = m \frac{\Delta v}{\Delta t} \cdot s = m \Delta v \cdot v$$

as $\frac{dE}{dt} = \frac{P}{\varepsilon_0 E}$

and $P \approx \left(\frac{\partial f}{\partial v}\right) v \cdot \frac{e^2 E^2}{mk} \Rightarrow \frac{P}{E^2} \approx \left(\frac{\partial f}{\partial v}\right) \frac{e^2 v}{mk}$

and $\omega_{pe}^2 = \frac{n_e e^2}{\varepsilon_0 m}$ (plasma frequency)

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_0 e^2}}$$ (Debye Length)

$$\Rightarrow \gamma = \left(\frac{\partial f}{\partial v}\right) \frac{\omega_{pe}^3}{k^2 n_e}$$

→ for small rate / disturbance : $f \sim \frac{\partial f}{\partial v}$