MHD assumptions

→ MHD only valid if particles localized:

- by collisions: $\lambda_{mfp} \ll L$ ← plasma scale length
- by B-field: $r_{Larmor} \ll L$

⇒ MHD cannot be used without B-fields!

→ bulk motion of plasma: $V^2 = V^2_{thermal} + u^2$

particle velocity; component due to $f(v)$; bulk motion of fluid

⇒ macroscopic model

→ slow time scales (ns)

→ <u>the fluid description concept</u>: electrons / ions follow averaged EM fields

- electrons and ions are assumed to form two independent fluids that penetrate each other (2-fluid model)

→ temperatures: T_e & T_i

→ streaming velocities: $\underline{u}_e$ and $\underline{u}_i$

⇒ two shifted distribution functions for the 2 particle populations. (assume Maxwellian)

$$f_M(v_x) = n\left(\frac{m}{2\pi k_B T}\right)^{1/2} \exp\left(-\frac{m(v_x - u_x)}{2k_B T}\right)$$

mean drift velocity in x-direction

(assume ideal gas EOS: $P = \frac{2}{3}U = nk_BT$)

Maxwell's equations

→ to derive the MHD equations, must start from the proper combination of Maxwell's equations

→ Gauss law → Poisson eq.: $\boxed{\underline{E} = -\nabla \Phi}$ ← dielectric potential

→ Faraday's Law → EM induction $\overset{\text{B-flux}}{\Phi_m} = \oint_A \underline{B} \cdot d\underline{A} \Rightarrow U_{ind} = -\frac{d\Phi_m}{dt}$

voltage induced in the loop = -ve change of B-flux in the loop

→ induced voltage depends on:
- B-flux density
- change area A
- changing angle between B-field direction and the area normal ($\underline{B} \cdot d\underline{A}$)

→ Ampère's Law $\oint \underline{B} \cdot d\underline{s} = \mu_0 I$ ← integral form

→ plasmas are diamagnetic: the magnetic moment of a gyrating particle is anti-parallel to the direction of the B-field (mag. moment is a conserved quantity)

⇒ thus we do not use the mag. field strength $\underline{H}$ and dielectric displacement $\underline{D}$, as $\underline{B} = \mu_r \mu_0 \underline{H}$ typical for ferromagnetic materials is not helpful for plasmas as E field is shielded (not dielectric)

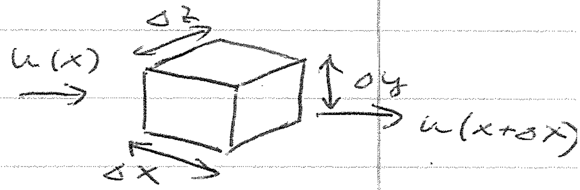
→ in plasmas polarization is a time-varying quantity, Static polarization only on plasma surface ⇒ $\underline{D}$ not suitable for static situations

Continuity equation

→ number of particles conserved:

$$N = n \cdot A \cdot \Delta x \quad \text{where area} \quad A = \Delta y \cdot \Delta z$$

→ shown in previous lecture:



→ mass flow through box on LHS in $\Delta t = \rho(x)\, u(x)\, \Delta y \Delta z \cdot \Delta t$

→ mass out on RHS in $\Delta t = \rho(x+\Delta x)\, u(x+\Delta x)\, \Delta y \Delta z \cdot \Delta t$

⇒ change in mass $= \left[-\rho(x+\Delta x)\, u(x+\Delta x) + \rho(x)\, u(x)\right] \Delta y \Delta z\, \Delta t$

⇒ change in $\rho = \frac{\Delta m}{V} = \left[-\rho(x+\Delta x)\, u(x+\Delta x) + \rho(x)\, u(x)\right] \frac{\Delta t}{\Delta x} = \Delta \rho$

$$\text{as } \Delta x, \Delta t \to 0 \Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x}$$

$$\Rightarrow \text{in 3D: } \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0}$$

→ increase / decrease of particles with changing flow:

$$-\frac{\partial N}{\partial t} = I_N(x+\Delta x) - I_N(x) \approx \frac{\partial I_N}{\partial x} \cdot \Delta x$$

Taylor expand and retain only the differential change in flux

→ divide by $\Delta V = A \cdot \Delta x$ and take the limit $\Delta V \to 0$

$$\Rightarrow \frac{\partial n}{\partial t} + \frac{\partial (n u_x)}{\partial x} = 0 \quad \text{as} \quad I_n = n \cdot \Delta y \Delta z \cdot u_x$$

$$\Rightarrow \text{in 3D: } \boxed{\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{u}) = 0}$$

Momentum equation

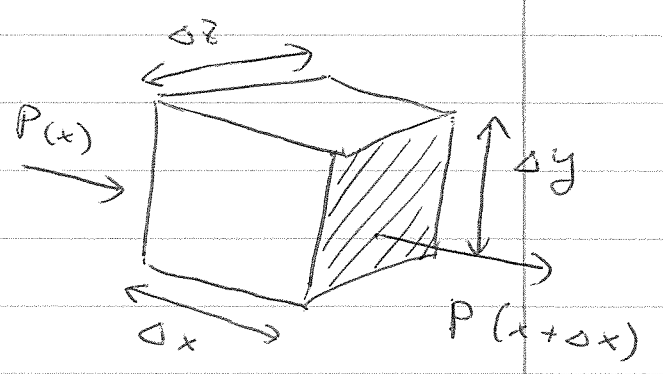
→ net force: sum of all forces acting within the cell

$$F = m \cdot a = m\frac{dv}{dt} = q(E + \underline{v} \times \underline{B})$$

d/dt calculated at the position of point-like particle

→ accelerating fluid:

pressure: $P = \dfrac{F}{A}$



$$\Rightarrow \Delta F = \Delta P \cdot A \quad \text{, where } A = \Delta y \Delta z$$

→ in frame moving with the fluid:

$$\Delta F \to P(x)\Delta y \Delta z - P(x - \Delta x)\Delta y \Delta z = \underbrace{\rho \Delta x \Delta y \Delta z}_{m = \rho \cdot V} \frac{du}{dt}$$

→ then $\Delta x, \Delta y, \Delta z, \Delta t \longrightarrow 0$

⇒ force per volume: $-\dfrac{\partial P}{\partial x} = \rho \dfrac{du}{dt}$

⇒ in 3D: $\boxed{\rho \dfrac{du}{dt} = -\nabla P}$

↳ where $\dfrac{d\underline{u}}{dt} = \dfrac{\partial u}{\partial t} + (\underline{u} \cdot \nabla)\, \underline{u}$

convective derivative

→ Substitute: $\rho\left[\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right] = -\nabla P$

fixed point in space

fluid with different velocity arriving at and leaving at this point ⇒ viscosity

→ number density $\Delta n(v_x)$ of the particle group is related to the distribution function: (of velocity between v_x and Δv_x)

$$\Delta n(v_x) = \Delta v_x \iint f(v_x, v_y, v_z)\, dv_y\, dv_z$$

Note: flow in x-direction only

⇒ momentum flux through a boundary <u>per unit time</u>

→ $|v_x|$ is the rate at which the particles pass through the boundary

→ net gain / loss: since $I_p = (mv_x)\Delta n(v_x)\,|v_x|\,\Delta y\,\Delta z$

v_x^2

$$\Rightarrow \frac{\partial p_x}{\partial t} = -m\frac{\partial}{\partial x}\left(n\langle v_x^2\rangle \Delta x\,\Delta y\,\Delta z\right)$$

and $p_x - p_{x+\Delta x} = \Delta y\,\Delta z\cdot m\left(n\langle v_x^2\rangle_{x_0-\Delta x} - n\langle v_x^2\rangle_{x_0}\right)$

$$= \Delta y\,\Delta z\cdot m\,(-\Delta x)\,\frac{\partial}{\partial x}\left(n\langle v_x^2\rangle\right)$$

⇒ note: the sum over Δn results in the average $\langle v_x^2 \rangle$ over the velocity distribution

$$\text{since } n\langle v_x^2\rangle = \int f(v_x)\, v_x^2\, dv_x$$

$$\text{and } \Delta n = \Delta v_x \iint f(v_x, v_y, v_z)\, dv_y\, dv_z$$

⇒ by substituting: $v_x = \underset{\text{flow}}{u_x} + \underset{\text{thermal motion}}{\tilde{v}_x}$

$$\frac{\partial}{\partial t}(n m u_x) = -m\frac{\partial}{\partial x}\Big[n\big(u_x^2 + 2u_x\underbrace{\langle \tilde{v}_x\rangle}_{=0} + \underbrace{\langle \hat{v}_x^2\rangle}_{\text{thermal energy}}\big)\Big]$$

$$\tfrac{1}{2} m \langle \hat{v}_x^2\rangle = \tfrac{1}{2} k_B T$$

$\langle \hat{v}_x\rangle = 0$; random motion

$$\text{and } \frac{\partial(n u^2)}{\partial x} = u_x \frac{\partial(n u_x)}{\partial x} + n u_x \frac{\partial u_x}{\partial x}$$

$$\Rightarrow mn\frac{\partial u_x}{\partial t} + m u_x \frac{\partial n}{\partial t} = -m u_x \frac{\partial(n u_x)}{\partial x} - m n u_x \frac{\partial u_x}{\partial x} - \frac{\partial(n k_B T)}{\partial x}$$

but continuity equation: $\dfrac{\partial n}{\partial t} + \dfrac{\partial(n u_x)}{\partial x} = 0$

$$\Rightarrow \frac{\partial(n u_x)}{\partial x} = -\frac{\partial n}{\partial t}$$

$$\Rightarrow mn\frac{\partial u_x}{\partial t} + \cancel{m u_x \frac{\partial n}{\partial t}} = +\cancel{m u_x\frac{\partial n}{\partial t}} - m n u_x \frac{\partial u_x}{\partial x} - \frac{\partial(n k_B T)}{\partial x}$$

$$\Rightarrow nm\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x} \;//$$

→ Since we have shown that:

$$\boxed{\frac{d\underline{u}}{dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}}$$

MHD = HD + Lorentz force

$$\Rightarrow F_L = \underbrace{-e\,(\underline{v}_e \times \underline{B})}_{\text{electrons}} + \underbrace{Z e\,(\underline{v}_i \times \underline{B})}_{\text{ions}}$$

and we have current density: $\boxed{j = nqv}$

$$F = q\,\underline{v} \times B \rightarrow F.n = \underbrace{nq\underline{v}}_{} \times \underline{B}$$

$$\underline{j} \times \underline{B}$$

$$\Rightarrow \frac{F}{V} = F.n = \underline{j} \times \underline{B} = \nabla P$$

• <u>two fluid model:</u>

$$n_e m_e \left[\frac{\partial \underline{u}_e}{\partial t} + (\underline{u}_e \cdot \nabla)\,\underline{u}_e\right] = -n_e e\,(\underline{E} + \underline{u}_e \times \underline{B}) - \nabla p_e$$

$$n_i m_i \left[\frac{\partial \underline{u}_i}{\partial t} + (\underline{u}_i \cdot \nabla)\,\underline{u}_i\right] = +n_i e\,(\underline{E} + \underline{u}_i \times \underline{B}) - \nabla P_i$$

→ space charge: $\rho = n_i e - n_e e$

→ current density: $\underline{j} = n_i e\,\underline{u}_i - n_e e\,\underline{u}_e$

→ and since $\frac{F_L}{V} = \underline{j} \times \underline{B} = \nabla P$ (force per volume)

⇒ momentum equation for MHD:

$$\rho \left[\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right] = \underline{j} \times \underline{B} - \nabla P$$

→ can add other forces

Moments of the distribution function

→ moment = weighted average (averaged over the distribution function)

$n(\underline{r}) = \int f(\underline{r}, \underline{v})\, d^3 v$ ⇒ number density

→ bulk motion velocity $u(\underline{r})$ integral of $f(v)$ over all v's

⇒ weighted average of $\underline{v} = \int \underline{v}\, f(\underline{r}, \underline{v})\, d^3\underline{v} = u(\underline{r})\, n(\underline{r})$

$$\therefore \quad u(\underline{r}) = \frac{\int \underline{v}\, f(\underline{r}, \underline{v})\, d^3 v}{\int f(\underline{r}, \underline{v})\, d^3 v}$$

→ 0^{th} order moment: $n(\underline{v}) \propto \int \underline{v}^0 f\, d^3\underline{v}$

→ 1^{st} order moment: $u(\underline{v}) \propto \int \underline{v}^1 f\, d^3\underline{v}$

→ 2^{nd} order moment: $\underset{\text{thermal energy}}{U(v)} \propto \int \underline{v} \cdot \underline{v}\, f\, d^3\underline{v}$

obtain P and T through the equation of state

Generalized Ohm's Law

Start from momentum equations for electrons and ions:

ions: $$n m_i \frac{\partial \underline{u}_i}{\partial t} = ne\left(\underline{E} + \underline{u}_i \times \underline{B}\right) - \nabla p_i + n m_i g + \underbrace{n \nu_{ei} m_e (\underline{u}_e - \underline{u}_i)}_{\sim \rho_{ei}} \quad (1)$$

electrons: $$n m_e \frac{\partial \underline{u}_e}{\partial t} = -ne\left(\underline{E} + \underline{u}_e \times \underline{B}\right) - \nabla p_e + \underbrace{n m_e g}_{\text{gravity}} + \underbrace{n \nu_{ei} m_e (\underline{u}_i - \underline{u}_e)}_{\text{collisions}} \quad (2)$$

neglecting the $\underline{u} \cdot \nabla \underline{u}$ term for slowly evolving plasma

with momentum exchange between electrons and ions being described by the collision frequency ν_{ei}

→ the mean mass motion:

total current density → Lorentz force

$$\rho_m \frac{\partial \underline{V}_m}{\partial t} = \underline{j} \times \underline{B} - \nabla p + \rho_m g$$

(static situation when $\frac{\partial V_m}{\partial t} = 0$)

where: $\rho_m = n(m_i + m_e)$

→ and total pressure: $p = p_e + p_i$

→ mean mass velocity: $$\underline{V}_m = \left(\frac{m_i \underline{u}_i + m_e \underline{u}_e}{m_e + m_i}\right)$$

reduced mass

→ in the Lorentz force $\underline{j} \times \underline{B}$ acts on the total current density. There is no effect of friction between electrons and ions (fluids) as it does not change the total momentum (only influences redistribution).

$\Rightarrow$ multiply eq. (1) by m_e and (2) by m_i & subtract:

$$\Rightarrow n m_i m_e \frac{\partial}{\partial t}(\underline{u}_i - \underline{u}_e) =$$

$$ne(m_e + m_i)\underline{E} + ne(m_e \underline{u}_i + m_i \underline{u}_e) \times \underline{B}$$
$$- m_e \nabla P_i + m_i \nabla P_e + n(m_e + m_i)\nu_{ei} m_e (\underline{u}_e - \underline{u}_i)$$

$\Rightarrow$ neglect m_e in the sum of masses $\rightarrow$ very small

$\Rightarrow$ the mixed term:

$$m_e \underline{u}_i + m_i \underline{u}_e = m_i \underline{u}_i + m_e \underline{u}_e$$
$$+ m_i(\underline{u}_e - \underline{u}_i) + m_e(\underline{u}_i - \underline{u}_e)$$
$$= \frac{1}{n}\rho_m \underline{v}_m - (m_i - \cancel{m_e})\frac{1}{ne}\underline{j}$$

where we define $\underline{j} = (\underline{u}_i - \underline{u}_e)ne$ and $\rho_{e,i} = n \cdot m_{e,i}$

$$\Rightarrow \frac{m_i m_e}{e} \cdot \frac{\partial \underline{j}}{\partial t} = e\rho_m\left(\underline{E} + \underline{v}_m \times \underline{B} - \frac{\nu_{ei} m_e}{ne^2}\underline{j}\right)$$

$$- m_i \underline{j} \times \underline{B} - \underbrace{\cancel{m_e \nabla P_i}}_{\text{small}} + m_i \nabla P_e$$ divide by $e\rho_m$

$\Rightarrow$ slowly varying phenomena $\Rightarrow \frac{\partial \underline{j}}{\partial t} = 0$, neglect $\frac{m_e}{m_i}$ terms

$\Rightarrow$ generalized Ohm's law:

Hall effect | thermal pressure

$$\underline{E} + \underline{v}_m \times \underline{B} = \eta \underline{j} + \frac{1}{ne}(\underline{j} \times \underline{B} - \nabla P_e)$$

where plasma resistivity: $\eta = \frac{\nu_{ei} m_e}{ne^2}$ (Coulomb collisions)

Induction equation

- electron/ion velocity components: 1) random thermal motion
  2) bulk average motion
  3) motion in flow of current $\underline{j}$

$\Rightarrow$ start from the simplified Ohm's Law:
(ignore the Hall and thermal pressure terms)

$$\underline{E} + \underline{u} \times \underline{B} = \eta \underline{j} = \frac{\eta}{\mu_0} \nabla \times \underline{B}$$

and substitute Ampère's Law: $\nabla \times \underline{B} = \mu_0 \underline{j}$

$$\Rightarrow \underline{E} = \frac{\eta}{\mu_0} \nabla \times \underline{B} - \underline{u} \times \underline{B}$$

then using the Faraday's Law:

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} = -\nabla \times \left(\frac{\eta}{\mu_0} \nabla \times \underline{B} \right) + \nabla \times (\underline{u} \times \underline{B})$$

$$\therefore \frac{\partial B}{\partial t} = -\nabla \times \underline{E} = -\frac{\eta}{\mu_0} \nabla \times (\nabla \times \underline{B}) + \nabla \times (\underline{V} \times \underline{B}) \quad //$$

Magnetic diffusion

→ induction equation in a static case: $\underline{v} = 0$:

$$+\frac{\partial \underline{B}}{\partial t} + \frac{\eta}{\mu_0} \nabla \times (\nabla \times \underline{B}) = +\frac{\partial \underline{B}}{\partial t} + D_B \underbrace{\Delta}_{\nabla\times\nabla\times} \underline{B} = 0$$

where diffusion coefficient $D_B = \frac{\eta}{\mu_0}$

describes the diffusion of B-field in a conducting medium

→ estimate diffusion time: assume exponential decay of the field $\underline{B}(t) \propto \exp(-t/\underbrace{\tau_B}_{\text{diffusion time scale}})$

$$\Rightarrow -\frac{\partial B}{\partial t} = D_B \cdot \Delta B \rightarrow \frac{\partial B}{\partial t} = -\frac{1}{\tau_B} \cdot \underline{B}$$

for $\underline{B}(t) = B_0 \exp(-t/\tau_B)$

$$\Rightarrow \frac{B}{\tau_B} = D_B \Delta \underline{B} \quad \text{and} \quad \Delta B \approx \frac{B}{l^2} \quad \left(\Delta = \nabla \times \nabla \sim \frac{1}{l^2}\right)$$

where l is the characteristic plasma length

$$\Rightarrow \frac{\cancel{B}}{\tau_B} = D_B \frac{\cancel{B}}{l^2} \Rightarrow \frac{1}{\tau_B} = \frac{\eta}{\mu_0 l^2}$$

$$\therefore \tau_B = \frac{\mu_0 l^2}{\eta} \quad //$$

Frozen-in Magnetic Flux

- for $\eta \to 0$, $R_m \to \infty$, induction reaction becomes:

$$\frac{\partial B}{\partial t} = \nabla \times (\underline{v} \times \underline{B})$$

- continuity equation: $\dfrac{\partial n}{\partial t} + \nabla \cdot (n \underline{u}) = 0$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0$$

$$\Rightarrow -\frac{1}{\rho} \frac{d\rho}{dt} = \frac{1}{\rho} \nabla (\rho \underline{v})$$

$$\Rightarrow \frac{d\rho}{dt} + \rho \, \nabla \cdot \underline{v} = 0$$

$$\Rightarrow \nabla \cdot \underline{V} = -\frac{1}{\rho} \cdot \frac{d\rho}{dt}$$

as $\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u}$

(shown earlier)

$\rightarrow$ vector identity: $\nabla \times (A \times B) = (B \cdot \nabla) A + A (\nabla \cdot B) - (A \cdot \nabla) B - B \cdot (\nabla \cdot A)$

$\Rightarrow$ apply the vector identity to the above expression:

$$\nabla \times (\underline{v} \times \underline{B}) = (\underline{B} \cdot \nabla) \underline{v} - \overset{0 \text{ (as } \underline{v} \perp \underline{B})}{(\underline{v} \cdot \nabla)} \underline{B} + \underline{v} \underbrace{(\nabla \cdot \underline{B})}_{=0 \text{ (no mag. monopoles)}} - \underline{B} (\nabla \cdot \underline{v})$$

$$\Rightarrow \frac{\partial \underline{B}}{\partial t} = (\underline{B} \cdot \nabla)\underline{v} + \frac{\underline{B}}{\rho} \cdot \frac{d\rho}{dt}$$

by substituting for $\nabla \cdot \underline{V}$

→ then use the quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\underline{B}}{\rho_m}\right) = \frac{1}{\rho}\frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2}\frac{\partial\rho}{dt}$$

→ and substitute above: (by dividing all terms by ρ)

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \left(\frac{\underline{B}}{\rho} \cdot \nabla\right)\underline{v}} \leftarrow \text{Truesdell theorem}$$

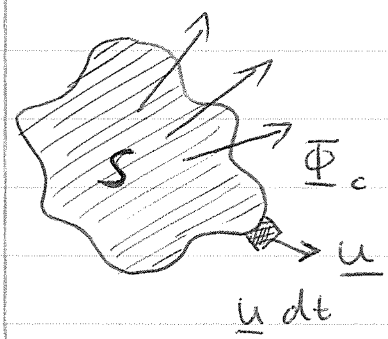
$\frac{\underline{B}}{\rho}$ ← number of field lines per unit mass of the plasma

- when mass flow ⊥ to B-field RHS vanishes
  - ⇒ $\frac{B}{\rho}$ becomes conserved quantity
  - ⇒ i.e. mass motion can only occur together with the B-field
  - → magnetic flux is "frozen in" the plasma
  - → mass flow <u>along</u> magnetic field lines

(in highly conductive plasmas, e.g. solar prominences)

→ alternative derivation from Faraday's Law:

$$\mathcal{E} = -\frac{d\Phi}{dt} = V \quad \text{(induced voltage)}$$
$$= -\oint \underline{E}\cdot d\underline{l}$$



$$\Rightarrow d\underline{S} = (\underline{u}\,dt)\times d\underline{l}$$

magnetic flux: $\Phi_B = \iint_S \underline{B}\cdot d\underline{S}$

$$\Rightarrow \frac{d\Phi_c}{dt} = \int_S \frac{\partial B}{\partial t}\cdot dS + \int_S B\cdot\frac{dS}{dt} = \text{change in } \underline{B} + \text{change in } \underline{S}$$

$$= \int_S (-\nabla\times\underline{E})\cdot dS + \oint(\underline{B}\cdot\underline{u})\times d\underline{l}$$

$$\boxed{-\frac{\partial B}{\partial t} = \nabla\times\underline{E}}$$

Faraday's Law

→ using Stokes theorem: $\oint_l \underline{A}\cdot dl = \iint_S \nabla\times\underline{A}\cdot d\underline{S}$

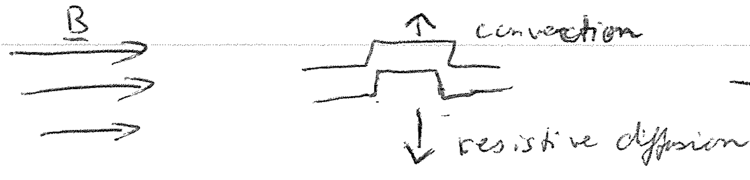
$$= -\oint_c \underline{E}\cdot dl - \oint(\underline{u}\times\underline{B})\cdot dl$$

$$\Rightarrow \boxed{\frac{d\Phi}{dt} = -\oint(\underline{E} + \underline{u}\times\underline{B})\cdot dl}$$

from Ohm's Law: $\underline{E} + \underline{u}\times\underline{B} = \eta j$

$$\Rightarrow \text{if } \eta\to 0 \Rightarrow \frac{d\Phi}{dt} = 0$$

i.e. flux through surface dS moving with plasma is conserved if $\eta\to 0$



→ Rm!

Magnetic pressure

vector identities:

$$\nabla(\underline{B}\cdot\underline{B}) = 2\left[\underline{B}\times(\nabla\times\underline{B}) + (\underline{B}\cdot\nabla)\underline{B}\right]$$

$$-(\nabla\times\underline{B})\times\underline{B} = \frac{1}{2}\nabla(\underline{B}\cdot\underline{B}) - (\underline{B}\cdot\nabla)\underline{B} = \underline{B}\times(\nabla\times\underline{B})$$

→ start from Ampère's Law: $\nabla\times\underline{B} = \mu_0 j$

→ take curl: $j\times\underline{B} = \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B}$

$$= -\frac{1}{\mu_0}\underline{B}\times(\nabla\times\underline{B})$$

then employ vector identity: $A\times(\nabla\times B) = (\nabla B)A_c - (A\cdot\nabla)B$

constant in differentiation by ∇

$$\Rightarrow j\times\underline{B} = -\frac{1}{\mu_0}\left[(\nabla\cdot\underline{B})\underline{B} - (\underline{B}\cdot\nabla)\underline{B}\right]$$

$$= -\frac{1}{2\mu_0}\nabla(B^2) + \frac{1}{\mu_0}(\underline{B}\cdot\nabla)\underline{B}$$

analogy to convective derivative

since $(\nabla\underline{B})\cdot\underline{B} = \frac{1}{2}\nabla(\underline{B}\cdot\underline{B})$

→ substitute into momentum equation: $\rho\frac{du}{dt} = j\times\underline{B} - \nabla P$

$$\therefore \boxed{\rho\frac{du}{dt} = -\nabla P - \frac{\nabla B^2}{2\mu_0} + \frac{1}{\mu_0}(\underline{B}\cdot\nabla)\underline{B}}$$

thermal pressure gradient — magnetic pressure gradient — magnetic tension force

⇒ magnetic pressure $= \dfrac{B^2}{2\mu_0}$

(magnetic energy density)

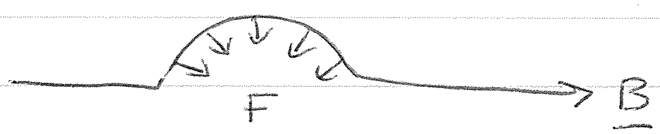
→ used to compress Z-pinch

implosion stops when $\rho u^2 = \dfrac{B^2}{2\mu_0}$

⇒ magnetic tension force $\sim \dfrac{1}{\mu_0} (\underline{B} \cdot \nabla) \underline{B}$

In direction of curvature of the field

⇒ opposes bending of mag. field lines through tension, i.e. restoring force ⇒ wave motion (magnetic Alfvén waves – next lecture)



→ ignore if $\eta \to 0$