Electrostatic waves

→ oscillating E-field, but NO oscillating B-field

- start from Maxwell's equations:

Ampère's Law: $\nabla \times \underline{H} = \underline{J} + \frac{\partial D}{\partial t}$ displacement current $(\underline{B} = \mu_0 \underline{H})$

Faraday's Law: $\nabla \times \underline{E} = -\frac{\partial B}{\partial t}$

→ general vector: $\underline{x} = \underline{x}_0 \exp(i[\omega t - \underline{k}\cdot\underline{r}])$

where $\underline{x}_0$ is a constant vector: $\nabla \times \underline{x} = -i\underline{k} \times \underline{x}$

$$\frac{\partial \underline{x}}{\partial t} = i\omega \underline{x}$$

$$\nabla \cdot \underline{x} = -i\underline{k} \cdot \underline{x}$$

and $\nabla(\nabla \cdot \underline{x}) = -\underline{k}(\underline{k} \cdot \underline{x})$

→ assume wave-like solution: $\underline{E} = \underline{E}_0 \exp(i[\omega t - \underline{k}\cdot\underline{r}])$

and find: $-i\underline{k} \times \underline{E} = -\frac{\partial B}{\partial t}$ (from Faraday's Law)

but there is NO oscillating B-field $\Rightarrow \frac{\partial B}{\partial t} = 0$!

(i.e. the k-vector is // to $\underline{E}$)

$$\Rightarrow \underline{k} \times \underline{E} = 0$$

→ also with NO B-field: $\underline{J} = -\frac{\partial D}{\partial t}$ displacement current

Conduction current

$\underline{J} \rightarrow$ moving charges, conduction current

$\frac{\partial \underline{D}}{\partial t} \rightarrow$ displacement current, proportional to the rate of change in the E-field

i.e. in electrostatic wave the conduction and displacement currents are linked, equal and opposite

→ pull electrons away from the ions and let go

→ they rush back (left) $(=\underline{J})$ ⇒ current "right" (opposite sign)

→ E-field pointing "right"

→ $\frac{\partial \underline{D}}{\partial t}$ vector pointing left

→ in plasma: $\underline{J} = -ne\underline{v} = -ne\frac{\partial x}{\partial t}$

→ rate of decrease in E-field? for $E = \frac{nex}{\varepsilon_0}$

$$\Rightarrow \frac{\partial D}{\partial t} = ne\frac{\partial x}{\partial t} = -\underline{J}$$

Sound waves

- Navier-Stokes equation without viscosity:
  (equivalent to the momentum equation)

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v}\right] = -\nabla P = -\frac{\gamma P}{\rho}\nabla\rho \qquad (1)$$

as $\frac{\nabla P}{P} = \gamma \frac{\nabla \rho}{\rho}$ (adiabatic process: $PV^{\gamma} = \text{constant}$)

γ ← adiabatic constant

$$\Rightarrow mn\left(\frac{\partial \underline{v}}{\partial t} - (\underline{v}\cdot\nabla)\underline{v}\right) = -\frac{\gamma n k_B T}{mn}\cdot m\nabla n$$

- continuity equation: $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$ ← mass density ρ (2)

$$\Rightarrow \quad \frac{\partial n}{\partial t} + \nabla(n\underline{v}) = 0$$

→ Sound wave is a result of a small density fluctuation:

$$\underline{v} = \underbrace{\underline{v}_0}_{=0 \text{ is not moving initially}} + \underline{v}_1, \quad \rho = \rho_0 + \rho_1 \quad \text{and} \quad P = P_0 + P_1$$

→ substitute and linearize assuming oscillatory solution in the form of: $\exp[i(\underline{k}\cdot\underline{r} - \omega t)]$ for $\underline{v}, \rho, P$:

(1) → $-i\omega\rho_0\underline{v}_1 = -\frac{\gamma P_0}{\rho_0} i\underline{k}\rho_1$ as $v_0\cdot\nabla v_1 = 0$ small quantity

(2) → $-i\omega\rho_1 + \rho_0 i\underline{k}\cdot\underline{v}_1 = 0$

as $\frac{\partial \rho_0}{\partial t} = 0, \ \frac{\partial v_0}{\partial \underline{r}} = 0, \ \frac{\partial P_0}{\partial \underline{r}} = 0$ → stationary before the wave

→ for a plane wave with $\underline{k} = k\hat{\underline{x}}$ and $\underline{V} = V\hat{\underline{x}}$
(moving in the x-direction)

$$\rho_1 = \frac{\rho_0 i k \underline{v}_1}{\omega}$$

$i \cdot i = -1$

and $$-i\omega\rho_0 v_1 = -\frac{\gamma P_0}{\cancel{\rho_0}} ik \frac{\cancel{\rho_0}}{\omega} ik v_1$$

$$\Rightarrow \omega^2 \rho_0 v_1 = \gamma P_0 k^2 v_1$$

$$\Rightarrow \omega^2 \cancel{v_1} = k^2 \frac{\gamma P_0}{\rho_0} \cancel{v_1}$$

→ phase velocity of sound wave:

$$\boxed{\frac{\omega}{k} = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k_B T}{M}\right)^{1/2} \equiv c_s}$$

↑ sound speed

Ion acoustic waves

→ continuity equation: $\frac{\partial n_i}{\partial t} + \frac{\partial (n_i u_i)}{\partial x} = 0$

↳ take temporal derivative $\partial/\partial t$

→ momentum equation for $B=0$ (for ions):

$$\frac{\partial}{\partial t}(n_i u_i) + \frac{\partial}{\partial x}(n_i u_i^2) = \frac{n_i Z e E}{M} - \frac{1}{M}\frac{\partial P_i}{\partial x}$$

↳ take spatial derivative $\partial/\partial x$

→ subtract equations and eliminate term $\partial^2(n_i u_i)$:

$$0 = \frac{\partial^2 n_i}{\partial t^2} + \cancel{\frac{\partial^2(n_i u_i)}{\partial t \partial x}} - \cancel{\frac{\partial^2(n_i u_i)}{\partial t \partial x}} - \frac{\partial^2(n_i u_i^2)}{\partial x^2} + \frac{Ze\,\partial(n_i E)}{M\,\partial x} - \frac{1}{M}\frac{\partial^2 P_i}{\partial x^2}$$

→ waves are fluctuations in density at any point is the mean density n_0 plus some fluctuation $\tilde{n}$:

$$n_i = n_0 + \tilde{n}$$

→ mean ion velocity is zero with no fluctuation, thus.

$$u_i = \tilde{u}_i$$

→ the electric field is also just a result of the fluctuation (no overall E-field that oscillates without fluctuations)

$$E = \tilde{E}$$

→ total pressure, like density, is the mean value <u>plus</u> the oscillation:

$$P_i = P_0 + \tilde{P}$$

→ substitute for these quantities and linearize, i.e. ignore products of small (perturbed) quantities:

$$\Rightarrow \frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_0 Ze}{M} \cdot \frac{\partial \tilde{E}}{\partial x} - \frac{1}{M} \cdot \frac{\partial^2 \tilde{P}_i}{\partial x^2} = 0$$

as $\frac{\partial n_0}{\partial t} = 0$, $\frac{\partial u_0}{\partial x} = 0$, $\frac{\partial P_0}{\partial x} = 0$

and quantities like $n_0 \tilde{u}_i$ are small → ignore

→ the Gauss Law: $\nabla E = -\frac{\rho}{\varepsilon_0} \Rightarrow \frac{\partial \tilde{E}}{\partial x} = -\frac{nq}{\varepsilon_0}$

→ pressure variation depends on the density and the relationship of the two is given by the equation of state, which in the classical regime is given by the ideal gas equation: $pV^{\gamma} = \text{const.}$ (adiabatic), where adiabatic index: $\gamma = \frac{c_p}{c_V}$ (ratio of the specific heats)

N.B. NOT $P = nk_B T$ (not isothermal process)

as fluctuations are very fast, then the compression and expansion in plasma during the wave motion, i.e. there is no net E transfered by work → adiabatic process
(energy)

- oscillations are so fast that any temperature rise on compression has no time to conduct anyway

⇒ adiabatic equation of state in 1-D: $\boxed{\tilde{P}_i = 3k_B T_i \tilde{n}_i}$

→ substitute: $\frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_0 Ze}{M} \cdot \frac{\partial \tilde{E}}{\partial x} - \frac{1}{M} \cdot \frac{\partial^2 \tilde{P}_i}{\partial x^2} = 0$

→ consider gradient in E-field with care:

- ions much heavier than electrons
- electrons move much faster than ions

⇒ electrons move quickly to eliminate any E-field fluctuations set up by the ions, before ions have time to get there

for electrons we have:

$$\underset{0}{\underset{\|}{\frac{F}{m}}} = \overbrace{\underset{0}{\underset{\|}{\frac{\partial(n_e u_e)}{\partial t}}} + \underset{0}{\underset{\|}{\frac{\partial}{\partial x}(n_e u_e^2)}}}^{\text{mass flow} \to 0} = -\frac{n_e \overset{q=-e}{e} E}{m} - \frac{1}{m} \cdot \frac{\partial P_e}{\partial x}$$

as electrons are almost massless compared to ions

→ thus the expression reduces to: $n_e e E = -\frac{\partial P_e}{\partial x}$

→ now what is the link of P_e (electron pressure!) to the electron density?

→ be careful again!

$$\Rightarrow n_e e \tilde{E} = -k_B T_e \frac{\partial \tilde{n}_e}{\partial x}$$

this time assuming isothermal $\tilde{P} = \tilde{n} k_B T_e$ for electrons, as ion waves are slow in comparison with the electrons, which have plenty of time to equilibrate their temperature during the ion oscillation time

and since $n_e = Z n_0$ and $\tilde{n}_e = Z \tilde{n}_i$ we can differentiate both sides to get the right form to substitute:

$$\Rightarrow n_e e \frac{\partial \tilde{E}}{\partial x} = n_0 Z e \frac{\partial \tilde{E}}{\partial x} = -Z k_B T_e \frac{\partial^2 \tilde{n}_i}{\partial x}$$

$$\rightarrow \text{and substitute: } \frac{n_0 Z e}{M} \cdot \frac{\partial \tilde{E}}{\partial x} = -\frac{Z k_B T_e}{M} \cdot \frac{\partial^2 \tilde{n}_i}{\partial x^2}$$

$$\Rightarrow \frac{\partial^2 \tilde{n}_i}{\partial t^2} - \frac{Z k_B T_e}{M} \cdot \frac{\partial^2 \tilde{n}_i}{\partial x^2} - \frac{3 k_B T_i}{M} \cdot \frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

where $\frac{\partial^2 P_i}{\partial x^2} = 3 k_B T \frac{\partial^2 \tilde{n}_i}{\partial x^2}$ (adiabatic)

and get: $$\frac{\partial^2 \tilde{n}_i}{\partial t^2} - \left(\frac{Z k_B T_e + 3 k_B T_i}{M} \right) \cdot \frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

assume wave-like solution $\tilde{n}_i \propto \exp[i\omega t - kx] \Rightarrow \omega = \pm k V_{ia}$

$$\Rightarrow \boxed{V_{ia} = \sqrt{\left(\frac{Z k_B T_e + 3 k_B T_i}{M} \right)}}$$ ion-acoustic frequency (phase velocity)

→ by analogy from the sound wave equation:

$$\frac{\partial^2 \rho}{\partial t^2} - C_s^2 \frac{\partial^2 \rho}{\partial x^2} = 0$$

$$\left.\begin{aligned} \rightarrow \frac{\partial^2 \tilde{n}_i}{\partial t^2} &= -\omega^2 n_0 \exp(i\omega t - kx) \\ \frac{\partial^2 \tilde{n}_i}{\partial x^2} &= -k^2 n_0 \exp(i\omega t - kx) \end{aligned}\right\} \omega^2 - \frac{(Z k_B T_e + 3 k_B T_i)}{M} \cdot k^2 = 0$$

$$\Rightarrow \frac{\omega^2}{k^2} = \frac{Z k_B T_e + 3 k_B T_i}{M}$$

→ phase velocity: $\frac{\omega}{k} = V_{ia}$ //

⇒ ion-acoustic waves are the equivalent of sound waves in plasma

Alfvén waves

→ MHD momentum equation:

$$\rho \frac{du}{dt} = \underbrace{-\nabla P - \frac{\nabla B^2}{2\mu_0}}_{\text{magnetosonic waves}} + \underbrace{\frac{1}{\mu_0}(\underline{B}\cdot\nabla)\underline{B}}_{\text{Alfvén waves (transverse)}}$$

• <u>transverse waves</u>: from magnetic tension force

$$\frac{1}{\mu_0}(\underline{B}\cdot\nabla)\underline{B}$$

Start from the sound wave differential equation derived earlier: to get the dispersion relation; Alfvén speed:

$$\left[\frac{\partial^2 n}{\partial t^2} - v_A^2 \frac{\partial^2 n}{\partial z^2}\right] = 0$$

phase velocity of the wave

→ wave solution: $n(x,t) = \hat{n}\sin[k(x \pm v_A t)]$

$\pm$ ⇒ waves can propagate $\pm x$-direction as v_A^2 dependency

→ Start from ideal MHD ($\eta = 0$):

• momentum equation:

$$\underbrace{nm}_{\rho_m \text{ (mass density)}}\Big[\underbrace{\frac{\partial u}{\partial t} + (\underline{u}\cdot\nabla)\underline{u}}_{=\frac{du}{dt}}\Big] = nq(\underbrace{\underline{E}}_{\text{ignore}} + v\times\underline{B}) - \underbrace{\nabla P}_{\text{ignore}}$$

→ only care about the B-field contribution and ignore the spatial disturbance

$\Rightarrow$ simplify: $\boxed{\rho_m \frac{\partial \underline{v}_m}{\partial t} = \underline{j} \times \underline{B}}$

- induction equation ($\eta = 0$, ideal MHD)

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) - \nabla \times \Big(\overset{\parallel 0}{\frac{\eta}{\mu_0}} \cdot \nabla \times \underline{B}\Big)$$

$\Rightarrow$ simplify: $\boxed{\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v}_m \times \underline{B})}$

$\Rightarrow$ apply vector identity (as in lecture 5):

$$\nabla \times (\underline{v}_m \times \underline{B}) = (\underline{B} \cdot \nabla)\underline{v}_m - (\underline{v}_m \cdot \nabla)\underline{B} + \underline{v}_m \underbrace{(\nabla \cdot \underline{B})}_{=0 \text{ no magnetic monopoles}} - \underline{B} \underbrace{(\nabla \cdot \underline{v}_m)}_{=0}$$

$\Rightarrow$ since we are not interested in sound wave, we make additional assumption of an incompressible flow: $\nabla \cdot \underline{v}_m = 0$ which gives ρ_m = constant (no density fluctuations)

$$\Rightarrow \boxed{\frac{\partial \underline{B}}{\partial t} = (\underline{B} \cdot \nabla)\underline{v}_m - (\underline{v}_m \cdot \nabla)\underline{B}}$$

no compression $\Leftrightarrow$ frozen-in flux (keep same distance between B-field lines)

$\Rightarrow$ wave-like perturbation: $\underline{B} = \underline{B}_0 + \underline{B}_1$ where $\underline{B}_0 = (0, 0, B_0)$ z-direction

$\underline{v}_m = \underline{v}_0 + \underline{v}_1$ and $\underline{v}_0 = 0$ initially at rest

and from Ampère's Law: $\nabla \times \underline{B} = \mu_0 \underline{j}$

$\underline{j}$ due to $\underline{v}_1$ $\Rightarrow \underline{j} = ne\underline{v}_1$

$$\Rightarrow \rho_m \frac{d v_1}{dt} = j \times \underbrace{B_0}_{\hookrightarrow \text{ ignore small perturbation}} \quad \text{but since } j \text{ due to } B_1$$

$$\Rightarrow j = \frac{1}{\mu_0} (\nabla \times \underline{B}_1) \quad \text{(Ampère's Law)}$$

$$\Rightarrow \rho_m \frac{d \underline{v}}{dt} = \rho_m \frac{\partial \underline{v}}{\partial t} = \underbrace{\frac{1}{\mu_0} (\nabla \times \underline{B}_1)}_{j} \times B_0$$

then perturbed field $\underline{B}_1$:

$$\frac{\partial B_1}{\partial t} = (\underline{B}_0 \cdot \nabla) \underline{v}_1 - \overbrace{(\underline{v}_1 \cdot \nabla)(\underline{B}_0 + \underline{B}_1)}^{=0}$$

but remember $\underset{x\text{-direction}}{\underline{v}_1} \cdot \underset{z\text{-direction}}{B_0} = 0$ (orthogonal vector)

and $(\underline{v}_1 \cdot \nabla) B_1$ - second order term, too small (product of two small perturbations)

since $\underline{B}_1 = (B_{1x}, 0, 0) \leftarrow$ x-direction

vector identity:

$$(\nabla \times \underline{B}_1) \times \underline{B}_0 = (\underline{B}_0 \cdot \nabla) \underline{B}_1 - \underbrace{(\nabla \underset{x\text{-direction}}{\underline{B}_1})}_{=0} \cdot \underset{z\text{-direction}}{B_0}$$

coupled equations

$$\Rightarrow \rho_m \frac{\partial \overset{x\text{-direction}}{v_{1x}}}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_{1x}}{\partial z} \quad \text{as } (\underline{B}_0 \cdot \nabla) \underline{B}_1 = \overset{z\text{-dir}}{B_0} \frac{\partial B_{1x}}{\partial z} \cdot \hat{x}$$

$$\Rightarrow \frac{\partial B_{1x}}{\partial t} = B_0 \frac{\partial v_{1x}}{\partial z}$$

(acceleration of mass in x-direction)

→ magnetic tension waves

→ transverse waves: → perpendicular perturbations of the B-field
→ local <u>transverse</u> displacement of the field in the x-direction (pick)

→ multiply (combine) the two coupled wave equation to get the Alfvén speed:

$$\left[\frac{\partial^2}{\partial t} - \underbrace{\frac{B_0^2}{\mu_0 \rho_m}}_{V_a^2}\frac{\partial^2}{\partial z^2}\right] V_{1x}\, B_{1x} = 0$$

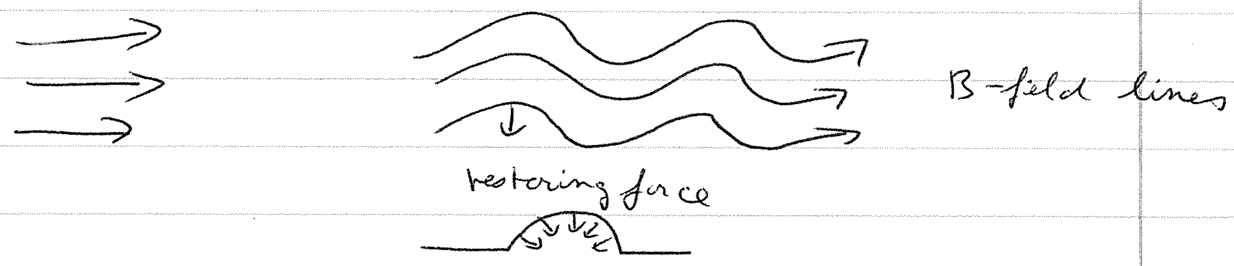
same format as sound waves

$$\Rightarrow V_A = \left(\frac{B_0^2}{\mu_0 \rho_m}\right)^{1/2} \quad //$$

propagation velocity of Alfvén waves

not associated with magnetic pressure

the shear Alfvén wave (transverse)



→ no compression of the plasma or of the B-field lines (frozen-in flux, $\eta = 0$)

→ transverse Alfvén waves driven by the magnetic field <u>tension</u> force → plucked string

- longitudinal waves - magneto sonic waves
  - → like sound waves, involve compression of B-field
  - ⇒ propagate across the B-field lines

here magnetic pressure $= \frac{B_0^2}{2\mu_0}$

→ from sound waves: $V_{ph} = \frac{\omega}{k} = \sqrt{\frac{\gamma P}{\rho}}$

here γ = compressibility ratio of specific heatcapacities

$= \frac{n_d + 2}{n_d}$ ← degrees of freedom

→ motion ⊥ to B-field ⇒ $n_d = 2$ ⇒ $\gamma = \frac{2+2}{2} = 2$

$$\Rightarrow V_{ph} = \frac{\sqrt{\gamma}\sqrt{B^2}}{\sqrt{\rho_m 2\mu_0}} = \sqrt{\frac{\cancel{2}\, B_0^2}{\rho\, \cancel{2}\mu_0}} = \frac{B_0}{\sqrt{\mu_0 \rho_m}} \quad \leftarrow \text{Alfvén speed}$$

↑ longitudinal Alfvén waves (same Alfvén speed)

→ complete magnetosonic waves

$$V_\varphi = \frac{\omega}{k} = (V_A^2 + c_s^2)^{1/2} \quad \text{where } V_A = \left(\frac{B_0}{\mu_0 \rho_m}\right)^{1/2}$$

→ when effect from gas pressure is small, can be neglected:

$P_{kin} \ll P_{mag}$ phase velocity becomes $V_\varphi = V_A$

→ for B_0 ⇒ waves reduce to sound waves

Conductivity / dielectric tensor

→ find general expression for the dielectric function $\varepsilon(\omega)$, which includes external E and B fields

→ conductivity scallar in unmagnetized plasma as electrons move $\parallel$ to E-field, but things can get more complicated ⇒ conductivity tensor

→ B-field can cause particles to drift to directions perpendicular both to the E and B-fields

$$\Rightarrow \boxed{\underline{J} = \bar{\sigma} \cdot \underline{E}}$$

conductivity tensor

→ motion in x-y plane ($\perp$ to B-field):

in lecture 2, we had: $\dot{\underline{V}} = (\underline{\omega}_c \times \underline{V}) + \frac{F}{m}$

cyclotron frequency

→ applied field is periodic:

motion $\perp$ to the B-field!

$$\Rightarrow \dot{E}_\perp = i\omega E_{\perp 0} \quad \text{and} \quad \ddot{V}_\perp = -\omega^2 V_{\perp 0}$$

since $\underline{E}_\perp = E_{\perp 0} \exp(i\omega t)$ ← +ve sign, choice!

$\underline{V}_\perp = \underline{V}_{\perp 0} \exp(i\omega t)$

→ Substituting to the equation of motion:

$$\ddot{\underline{V}}_\perp = -\frac{e}{m}\left(\dot{\underline{E}}_\perp + \underline{\omega}_{ce} \times \underline{E}_\perp\right) - \omega_{ce}^2 V_\perp$$

$$-\omega^2 \underline{V}_{\perp_0} = -\frac{e}{m}\left(i\omega\, \underline{E}_{\perp 0} + \underline{\omega}_{ce} \times \underline{E}_{\perp 0}\right) - \omega_{ce}^2 \underline{V}_\perp$$

$$\Rightarrow V_{\perp 0}\left(\omega_{ce}^2 - \omega^2\right) = -\frac{e}{m}\left(i\omega\, \underline{E}_{\perp 0} + \underline{\omega}_{ce} \times \underline{E}_{\perp 0}\right)$$

→ get individual components J_x and J_y:

$$J_x = -ne\, V_{x0} \qquad \underline{\hat{x}} \cdot \underline{\hat{y}} = 0$$
$$J_y = -ne\, V_{y0} \qquad \underline{\hat{x}} \cdot \underline{\hat{x}} = 1$$

} unit vectors

→ see Chen book (page 124):

$$V_{x0}\left(\omega_{ce}^2 - \omega^2\right) = -\frac{e}{m}\left(i\omega E_{x0} + \omega_{ce} E_{y0}\right)$$
$$V_{y0}\left(\omega_{ce}^2 - \omega^2\right) = -\frac{e}{m}\left(i\omega E_{y0} - \omega_{ce} E_{x0}\right)$$

→ or Piel book (page 162): motion in $\underline{E} \times \underline{B}$ direction!

$$\hat{V}_x = i\frac{q}{\omega m}\left(\hat{E}_x + \hat{V}_y B_0\right) \text{ and } \hat{V}_y = i\frac{q}{\omega m}\left(\hat{E}_y - \hat{V}_x B_0\right)$$

since B field in +ve z-direction! —ve sign!

→ so derive the current components for the conductivity tensor:

$$\underline{V} = \left(\frac{\sigma}{-ne}\right) \cdot \underline{E}$$

$$\Rightarrow J_x = +\frac{ne^2}{m}\cdot\left(\frac{i\omega E_{xo}+\omega_{ce}E_{yo}}{\omega_{ce}^2-\omega^2}\right)$$

$$= \underbrace{\frac{ne^2}{\varepsilon_o m}}_{\omega_p^2}\cdot\varepsilon_o\left(\frac{i\omega E_{xo}+\omega_{ce}E_{yo}}{\omega_{ce}^2-\omega^2}\right)$$

$$= \frac{i\varepsilon_o\omega\omega_p^2}{\omega_{ce}^2-\omega^2}\cdot E_{xo} + \frac{\varepsilon_o\omega_{ce}\omega_p^2}{\omega_{ce}^2-\omega^2}\cdot E_{yo} \quad //$$

$$\Rightarrow J_y = +\frac{ne^2}{m}\cdot\left(\frac{i\omega E_{yo}-\omega_{ce}E_{xo}}{\omega_{ce}^2-\omega^2}\right)$$

$$= \omega_p^2\cdot\varepsilon_o\left(\frac{i\omega E_{yo}-\omega_{ce}E_{xo}}{\omega_{ce}^2-\omega^2}\right)$$

$$= \frac{i\varepsilon_o\omega\omega_p^2}{\omega_{ce}^2-\omega^2}\cdot E_{yo} - \frac{\varepsilon_o\omega_{ce}\omega_p^2}{\omega_{ce}^2-\omega^2}\cdot E_{xo} \quad //$$

and we get:

the conductivity tensor

$$-ne\begin{pmatrix}\hat{v}_x\\ \hat{v}_y\\ \hat{v}_z\end{pmatrix} = \bar{\sigma}\cdot\begin{pmatrix}\hat{E}_x\\ \hat{E}_y\\ \hat{E}_z\end{pmatrix}$$

signs and i swap

$\Rightarrow$ to get the dielectric tensor: multiply $\bar{\sigma}$ by $-\frac{i}{\varepsilon\omega}$

and subtract from identity matrix: $\underline{\underline{I}} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

and get $\varepsilon_o = 1 - \frac{\omega_p^2}{\omega^2}$ (N.B. ε_o and ω cancel)