Waves in magnetized plasma

→ dispersion relation: $\left(\omega^2 \bar{\varepsilon}(\omega) - c^2 k^2\right)\underline{E} + c^2 \underline{k}(\underline{k}\cdot\underline{E}) = 0$

→ rewrite as: $\underline{M}\cdot\underline{E} = 0$

→ where $\underline{M} = \frac{\omega^2}{c^2}\bar{\varepsilon} + \underline{k}\cdot\underline{k} - k^2\underline{I}$

refractive index

→ keep track of the direction of the k-vector with: $\underline{N} = \frac{c\underline{k}}{\omega}$

we have: $\underline{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$ and $\underline{N} = \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} = \frac{c}{\omega}\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$

$$\Rightarrow \underline{k}\cdot\underline{k} = \begin{pmatrix} k_x k_x & k_x k_y & k_x k_z \\ k_y k_x & k_y k_y & k_y k_z \\ k_z k_x & k_z k_y & k_z k_z \end{pmatrix}$$

and $k^2 = k_x k_x + k_y k_y + k_z k_z$

$\Rightarrow$ now build $\underline{M}$: $\underline{M} = \frac{\omega^2}{c^2}\left(\bar{\varepsilon} + \frac{c^2}{\omega^2}\underline{k}\cdot\underline{k} - \frac{c^2}{\omega^2}k^2\right)$

$$\underline{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 + \frac{c^2}{\omega^2}k_x k_x - \frac{c^2}{\omega^2}k^2 & -i\varepsilon_2 + \frac{c^2}{\omega^2}k_x k_y & \frac{c^2}{\omega^2}k_x k_z \\ i\varepsilon_2 + \frac{c^2}{\omega^2}k_x k_y & \varepsilon_1 + \frac{c^2}{\omega^2}k_y k_y - \frac{c^2}{\omega^2}k^2 & \frac{c^2}{\omega^2}k_y k_z \\ \frac{c^2}{\omega^2}k_x k_z & \frac{c^2}{\omega^2}k_y k_z & \varepsilon_3 + \frac{c^2}{\omega^2}k_z k_z - \frac{c^2}{\omega^2}k^2 \end{pmatrix}$$

$$= \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - \frac{c^2}{\omega^2}k_y k_y - \frac{c^2}{\omega^2}k_z k_z & -i\varepsilon_2 + \frac{c^2}{\omega^2}k_x k_y & \frac{c^2}{\omega^2}k_x k_z \\ i\varepsilon_2 + \frac{c^2}{\omega^2}k_x k_y & \varepsilon_1 - \frac{c^2}{\omega^2}k_x k_x - \frac{c^2}{\omega^2}k_z k_z & \frac{c^2}{\omega^2}k_y k_z \\ \frac{c^2}{\omega^2}k_x k_z & \frac{c^2}{\omega^2}k_y k_z & \varepsilon_3 - \frac{c^2}{\omega^2}k_x k_x - \frac{c^2}{\omega^2}k_y k_y \end{pmatrix}$$

$$= \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_y^2 - N_z^2 & -i\varepsilon_2 + N_x N_y & N_x N_z \\ i\varepsilon_2 + N_x N_y & \varepsilon_1 - N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & \varepsilon_3 - N_x^2 - N_y^2 \end{pmatrix}$$

→ refractive index: $N = \frac{kc}{\omega}$ and φ is the angle between the wave vector k and the magnetic field direction

$\Rightarrow$ general expression thus.

$$M = \frac{\omega^2}{c^2} \cdot \begin{pmatrix} \varepsilon_1 - N^2\cos^2\psi & -i\varepsilon_2 & N^2\cos\psi\sin\psi \\ i\varepsilon_2 & \varepsilon_1 - N^2 & 0 \\ N^2\cos\psi\sin\psi & 0 & \varepsilon_3 - N^2\sin^2\psi \end{pmatrix}$$

where the wave vector in the x-z plane. $\underline{k} = (k\sin\psi, 0, k\cos\psi)$

$\Rightarrow$ see Piel book, page 164

- <u>waves propagating // to B-field</u> (z-direction)
  (k along z-axis)
  $\rightarrow$ i.e $\psi = 0 \Rightarrow N_x, N_y = 0$

① solution $\varepsilon_3 = 0$, $\omega^2 = \omega_p^2$

$\rightarrow$ for all // waves, we must have k-vector along z-axis

$\Rightarrow$ this requires: $E = \begin{pmatrix} 0 \\ 0 \\ E_z \end{pmatrix}$ with z-component only too

$\rightarrow$ recover electrostatic waves $_{//}$

② solution $\varepsilon_1 - \varepsilon_2 = N_z^2$

and refractive index: $n^2 = N_z^2 = \frac{c^2k^2}{\omega^2}$ (k in z-direction)

$$\Rightarrow \varepsilon_1 - \varepsilon_2 = 1 + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2} - \frac{\omega_{ce}}{\omega} \cdot \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}$$

$$= 1 + \frac{\omega_p^2(\omega - \omega_{ce})}{\omega(\omega_{ce}^2 - \omega^2)} = 1 + \frac{\omega_p^2(\omega - \omega_{ce})}{\omega(\omega_{ce} + \omega)(\omega_{ce} - \omega)}$$

$$= 1 - \frac{\omega_p^2\cancel{(\omega - \omega_{ce})}}{\omega(\omega + \omega_{ce})\cancel{(\omega - \omega_{ce})}} = \frac{c^2k^2}{\omega^2}$$

(3) Solution $\varepsilon_1 + \varepsilon_2 = N_z^2$

$$\Rightarrow \varepsilon_1 + \varepsilon_2 = 1 + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2} + \frac{\omega_{ce}}{\omega} \cdot \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}$$

$$= 1 - \frac{\omega_p^2 \cancel{(\omega + \omega_{ce})}}{\omega \cancel{(\omega_{ce} + \omega)}(\omega - \omega_{ce})}$$

$$= 1 - \frac{\omega_p^2}{\omega(\omega - \omega_{ce})} = \frac{c^2 k^2}{\omega^2}$$

→ for solutions (2) and (3) the E-field must have the form of $E = \begin{pmatrix} E_x \\ \pm E_y \\ 0 \end{pmatrix}$

while the B-field is along the z-axis and the wave propagation (k-vector) is also along z-axis

⇒ transverse wave!

⇒ with no B-field ⇒ $\omega_{ce} = 0$ → reduces to the EM wave solution

→ get circularly polarized transverse waves in B-field:

• "left-handed" wave: $n^2 = 1 - \frac{\omega_p^2}{\omega(\omega + \omega_{ce})}$

→ for $\omega \gg \omega_p, \omega_{ce}$ ⇒ get vacuum dispersion: $\frac{c^2 k^2}{\omega^2} \sim 1$

$\Rightarrow \omega = ck$ //

$\rightarrow$ at low ω, we get a cut off - at $k=0$

$\Rightarrow$ the wave is imaginary, no propagation below $\omega_{LC} = \dfrac{-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_p^2}}{2}$ //

quadratic equation solution: $ax^2 + bx + c = 0$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c^2k^2 = \omega^2 - \frac{\omega^2 \omega_p^2}{\omega(\omega + \omega_{ce})}$$ $\rightarrow$ wave reflected //

- "right-handed" wave: $n^2 = 1 - \dfrac{\omega_p^2}{\omega(\omega - \omega_{ce})}$

$$\Rightarrow c^2k^2 = \omega^2 - \frac{\omega^2 \omega_p^2}{\omega(\omega - \omega_{ce})}$$

$\rightarrow$ for $\omega \gg \omega_p, \omega_{ce}$ $\rightarrow$ recover $\omega = ck$ again

$\rightarrow$ as $\omega \rightarrow 0$ and $\omega \ll \omega_p, \omega_{ce}$

$$c^2k^2 \approx \frac{\omega \omega_p^2}{(\omega_{ce} - \omega)}$$

at low ω $\Longrightarrow$ $k \propto \sqrt{\omega}$!? warning - WRONG!

must consider motion of ions at very low frequencies (next)

$\rightarrow$ Alfvén waves again!

$\rightarrow$ as $\omega \rightarrow \omega_{ce}$ $\rightarrow$ get resonance, there is a pole as $k \rightarrow \infty$ $\rightarrow$ wave absorbed,

(remember cyclotron motion of electrons is also right-handed, that is why we get the resonance here)

→ if ω just above ω_{ce} ⇒ k is imaginary
⇒ no wave propagation

but as ω increases the 1st ω^2 term
in $c^2k^2 = \omega^2 - \frac{\omega^2 \omega_p}{\omega(\omega-\omega_{ce})}$ exceeds the second
term and waves are allowed to propagate again
$\left(\omega^2 > \frac{\omega^2 \omega_p^2}{\omega(\omega-\omega_{ce})}\right)$
⇒ cut-off frequency for $k=0$:
(quadratic equation)

$$\omega^2 - \omega_{ce}\,\omega - \omega_p^2 = 0$$

$$\Rightarrow \omega_{RC} = \frac{\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\,\omega_p^2}}{2}$$

→ wave reflected

→ "right-handed" waves result in the
Wistler modes that can be heard on
AM radio at very low frequencies (~100 kHz)

- they originate by a lightning (creates radio waves)
strike in a storm on the opposite
hemisphere and travel along the
B-field lines to the other
pole where detected

• <u>waves propagating $\perp$ to B-field</u>
(k along x-axis)

$\rightarrow$ i.e. $\Psi = \pi/2 \Rightarrow N_y, N_z = 0$

$$\underline{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N_x^2 & 0 \\ 0 & 0 & \varepsilon_3 - N_x^2 \end{pmatrix}$$

and we have $\underline{E}_1 = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

$\Rightarrow$ again get 3 solutions $\underline{M} \cdot \underline{E} = 0$ for $\det(\underline{M}) = 0$:

$$\left(\varepsilon_3 - N_x^2\right) \cdot \left[\varepsilon_1 \cdot (\varepsilon_1 - N_x^2) - \varepsilon_2^2\right] = 0$$

① solution $\varepsilon_3 - N_x^2 = 0$

$$\Rightarrow \varepsilon_3 = N_x^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

$\Rightarrow c^2 k^2 = \omega^2 - \omega_p^2$ $\leftarrow$ electromagnetic wave in plasma (lecture 3)

$\rightarrow$ the E-field is aligned with the static B-field B_0 (along z-direction)

$\Rightarrow$ get "ordinary wave" (a O-wave)

$\rightarrow$ the refractive index is not affected by the B-field

→ as seen before:

cut off in propagation for $k=0$ → cut-off is at $\omega = \omega_p$, i.e. waves cannot propagate for $\omega < \omega_p$ (lecture 3)
⇒ wave reflected //

→ used in interferometry to obtain density of magnetized plasma

② solutions for $\begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 \\ i\varepsilon_2 & \varepsilon_1 - N_x^2 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$

⇒ non-vanishing solutions for $\underline{E}$ when deteminant = 0

$$\Rightarrow (\varepsilon_1 - N_x^2)\,\varepsilon_1 - \varepsilon_2^2 = 0$$

$$\Rightarrow \varepsilon_1^2 - \varepsilon_1 N_x^2 = \varepsilon_2^2 \quad \Rightarrow N_x = \sqrt{\frac{\varepsilon_1^2 - \varepsilon_2^2}{\varepsilon_1}} \rightarrow \text{ref. index}$$

(get dispersion relation)

$$\Rightarrow \varepsilon_1 N_x^2 = \varepsilon_1^2 - \varepsilon_2^2$$

$$\Rightarrow N_x^2 = \varepsilon_1 - \varepsilon_2^2/\varepsilon_1$$

substitute for N_x, ε_1 & ε_2

$$\Rightarrow \frac{c^2k^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2} - \frac{\left(\frac{\omega_{ce}}{\omega} \cdot \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}\right)^2}{1 + \frac{\omega_p^2}{\omega_{ce}^2 - \omega^2}}$$

now define the "upper-hybrid frequency"

$$\boxed{\omega_h^2 \equiv \omega_p^2 + \omega_c^2}$$ and substitute

$$\frac{c^2k^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega_{ce}^2-\omega^2} - \frac{\left(\frac{\omega_{ce}}{\omega}\cdot\frac{\omega_p^2}{\omega_{ce}^2-\omega^2}\right)^2}{\frac{\underbrace{\omega_{ce}^2-\omega_p-\omega}}{\omega_{ce}^2-\omega^2}} \leftarrow \omega_h^2$$

$$= 1 + \frac{\omega_p^2}{\omega_{ce}^2-\omega^2} - \frac{(\omega_{ce}^2-\omega^2)\frac{\omega_{ce}^2}{\omega}\left(\frac{\omega_p^2}{\omega_{ce}^2-\omega^2}\right)^2}{\omega_h^2-\omega^2}$$

$$= 1 + \frac{\omega_p^2}{\omega_{ce}^2-\omega^2} - \frac{\cancel{(\omega_{ce}^2-\omega^2)}\cdot\omega_{ce}^2\cdot\omega_p^4}{\omega^2(\omega_{ce}^2-\omega^2)^{\cancel{2}}(\omega_h^2-\omega^2)}$$

$$= 1 + \frac{\omega_p^2\cdot\omega^2(\omega_h^2-\omega^2) - \omega_{ce}^2\omega_p^4}{(\omega_{ce}^2-\omega^2)\omega^2(\omega_h^2-\omega^2)} \qquad \begin{matrix}\times(-1)\\ \times(-1)\end{matrix}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{\omega^2(\omega^2-\omega_h^2)+\omega_p^2\omega_{ce}^2}{(\omega^2-\omega_{ce}^2)(\omega^2-\omega_h^2)}$$ (see Chen book page 125)

as $\omega_h^2 = \omega_p^2+\omega_c^2$
substitute

$$= 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{\omega^4-\omega^2\omega_p^2-\omega^2\omega_{ce}^2+\omega_p^2\omega_{ce}^2}{(\omega^2-\omega_{ce}^2)(\omega^2-\omega_h^2)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{\omega^2\cancel{(\omega^2-\omega_{ce}^2)}-\omega_p^2\cancel{(\omega^2-\omega_{ce}^2)}}{\cancel{(\omega^2-\omega_{ce}^2)}(\omega^2-\omega_h^2)}$$

$$\therefore \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\cdot\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2} \quad //$$ (Chen, page 125)

↳ "extraordinary wave" (x-wave) //

→ resonance occurs when $k \to \infty$
when $\omega \to \omega_h$!

i.e. $\omega_h^2 = \omega_p^2 + \omega_c^2 = \omega^2$ → wave is absorbed at resonance //

→ cut off occurs when $k = 0$ → wave is reflected at cut-off //

→ find cut-off frequencies for X-wave: $k = 0$

→ divide dispersion relation for X-wave by $(\omega^2 - \omega_p^2)$:

$$\frac{c^2k^2}{\omega^2} = 0 = 1 - \frac{\omega_p^2}{\omega^2} \cdot \frac{1}{\frac{\omega^2 - \omega_h^2}{\omega^2 - \omega_p^2}} = 1 - \frac{\omega_p^2}{\omega^2} \cdot \frac{1}{\frac{\omega^2 - \omega_c^2 - \omega_p^2}{\omega^2 - \omega_p^2}}$$

$$\Rightarrow \quad 1 = \frac{\omega_p^2}{\omega^2} \cdot \frac{1}{1 - \frac{\omega_{ce}^2}{\omega^2 - \omega_p^2}}$$

$$\Rightarrow \quad \frac{\omega_p^2}{\omega} = 1 - \frac{\omega_{ce}^2}{\omega^2 - \omega_p^2}$$

$$\Rightarrow \quad 1 - \frac{\omega_p^2}{\omega^2} = \frac{\omega_{ce}^2}{\omega^2 - \omega_p^2} = \frac{\omega_{ce}^2/\omega^2}{1 - \omega_p^2/\omega^2}$$

$$\Rightarrow \quad \left(1 - \frac{\omega_p^2}{\omega^2}\right)^2 = \frac{\omega_{ce}^2}{\omega^2} \qquad / \sqrt{\ } \text{ both sides}$$

$$\Rightarrow \quad 1 - \frac{\omega_p^2}{\omega^2} = \pm \frac{\omega_{ce}}{\omega}$$

$$\Rightarrow \quad \omega^2 \mp \omega\omega_{ce} - \omega_p^2 = 0$$

→ solving the quadratic get 2 cut-off frequencies for each sign:

$$\left.\begin{aligned} \omega_{RX} &= \frac{\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_p^2}}{2} \;/\!/ \\ \omega_{LX} &= \frac{-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_p^2}}{2} \;/\!/ \end{aligned}\right\} \text{wave reflected at the same frequencies as for } \underline{k} \parallel \underline{B}$$

→ Cut-offs and resonances:

- cut-off occurs when refractive index → 0
  ⇒ wave is reflected ($\lambda \to \infty$)
- resonance occurs when refractive index → ∞
  ⇒ wave is absorbed ($\lambda \to 0$)

→ the cut-off and resonance frequencies divide the dispersion diagram into regions of propagation and non propagation

→ as plasma density ↗, ω_L, ω_p, ω_h and ω_R all increase as well

→ at large ω (or low density) the phase velocity $\frac{\omega}{k}$ of the wave → c

→ $V_\phi = \frac{\omega}{k}$ increases until the right-hand cut-off $\omega = \omega_R$

→ between ω_R and ω_h $\frac{\omega^2}{k^2}$ is negative
⇒ no propagation

→ at $\omega = \omega_h$ ⇒ resonance, $\frac{\omega}{k} \to 0$

→ between ω_h and ω_L propagation again possible, here the wave is faster or slower than c depending whether it is smaller or larger than ω_p

$\Rightarrow$ at $\omega = \omega_p$ the wave travels at c

$\Rightarrow$ for $\omega < \omega_L$ there is another region of nonpropagation

$\Rightarrow$ the X-wave thus has two regions of non propagation separated by a stop band

Alfrén waves & ion waves revisited

→ at very low frequencies must account for the presence of ions

→ the ion cyclotron frequency is lower than electrons due to higher mass

→ opposite sign (+ve charge)

→ ion plasma frequency: $n_i = n_e / Z$

$$\Rightarrow \omega_{pi}^2 = \frac{n Z e^2}{\varepsilon_0 M} \qquad (\omega_{pe} \gg \omega_{pi})$$

M ← ion mass

→ ions move in E-fields just like electrons and thus also contribute to conductivity

⇒ add extra terms to the conductivity tensor

→ for "right-handed" wave we then get:

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega + \omega_{ci})} = \varepsilon_1 + \varepsilon_2$$

$$= 1 + \frac{-\omega_{pe}^2 \times (\omega + \omega_{ci}) - \omega_{pi}^2 \times (\omega - \omega_{ce})}{\omega (\omega - \omega_{ce})(\omega + \omega_{ci})}$$

$$\frac{\omega_{pe}^2}{\omega_{ce}} \approx \frac{\omega_{pi}^2}{\omega_{ci}}$$

$$= 1 + \frac{\underbrace{\cancel{-\omega_{pe}^2 \omega_{ci} + \omega_{pi}^2 \omega_{ce}}}_{\approx 0} + \cancel{\omega}(-\omega_{pe}^2 - \omega_{pi}^2)}{\cancel{\omega}(\omega - \omega_{ce})(\omega + \omega_{ci})}$$

$$= 1 + \frac{\omega_{pe}^2 + \omega_{pi}^2}{(\omega_{ce} - \omega)(\omega_{ci} + \omega)} \quad //$$

$\Rightarrow$ verify: $\omega_{ce} = \dfrac{eB}{m}$ and $\omega_{ci} = \dfrac{ZeB}{M}$

$$\omega_{pe}^2 = \frac{ne^2}{\varepsilon_0 m} \quad \text{and} \quad \omega_{pi}^2 = \frac{nZe^2}{\varepsilon_0 M}$$

$$\Rightarrow \frac{\omega_{pe}^2}{\omega_{ce}} = \frac{\frac{ne^{\cancel{2}}}{\varepsilon_0 \cancel{m}}}{\frac{\cancel{e}B}{\cancel{m}}} = \frac{ne}{\varepsilon_0 B} //$$

$$\Rightarrow \frac{\omega_{pi}^2}{\omega_{ci}} = \frac{\frac{nZe^{\cancel{2}}}{\varepsilon_0 \cancel{M}}}{\frac{Z\cancel{e}B}{\cancel{M}}} = \frac{ne}{\varepsilon_0 B} //$$

$\checkmark$

$\Rightarrow$ low frequency limit: $\omega_{pe} \gg \omega_{pi}$

$\omega \rightarrow 0$

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{\omega_{pe}^2}{\omega_{ce}\,\omega_{ci}} = \frac{\frac{n_e e^{\cancel{2}}}{\varepsilon_0 \cancel{m}}}{\frac{\cancel{e}B}{\cancel{m}} \cdot \frac{Z\cancel{e}B}{M}}$$

$$= 1 + \frac{\overset{n_i}{\widetilde{Zn_e}} \cdot M}{\varepsilon_0 B^2} = 1 + \frac{n_i M}{\varepsilon_0 B^2}$$

and $c^2 = \dfrac{1}{\varepsilon_0 \mu_0}$ $\leftarrow$ speed of light in vacuum

$$\Rightarrow n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{n_i \mu_0 M c^2}{B^2} //$$

$$= 1 + \frac{c^2}{V_A^2} \quad \leftarrow \text{refractive index}$$

where $V_A = \sqrt{\dfrac{B^2}{\mu_0 n_i M}} = \left(\dfrac{B^2}{\mu_0 \rho_m}\right)^{1/2}$ // $\leftarrow$ Alfvén speed !

phase velocity $= \dfrac{\omega}{k} = c' = \dfrac{c}{\sqrt{\varepsilon_r}}$