Magnetic confinement fusion

~ 300 keV (D,T)

→ ions must overcome the Coulomb barrier to undergo fusion reaction during collisions
   ⇒ need a lot of kinetic energy to overcome the repulsive potential ⇒ high temperature (10^8 K)
   ⇒ at close proximity the nuclear forces dominate

→ in fusion experiments on Earth we use the D-T fusion:

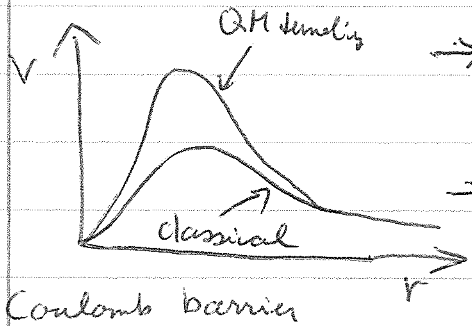
$$D + T \rightarrow \underset{3.5\,MeV}{\alpha} + \underset{14.1\,MeV}{n}$$

→ Deuterium available in sea water
→ Tritium can be manufactured by Lithium fusion:

$$Li^6 + n \rightarrow T + \alpha + 4.8\,MeV$$

→ 1 kg of fuel (D,T) can generate a sustained power output of <u>1 GW</u> !!!

→ to be able to use it as a power source we need an efficient and reliable technological to
   → <u>heat</u> and <u>confine</u> the plasma



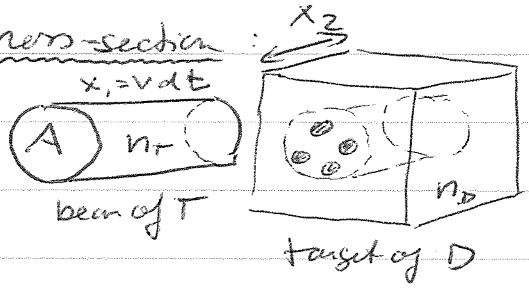
Coulomb barrier
~ 300 keV for DT

→ first need to overcome repulsive electrostatic forces between +ve ions

→ at $r < 10^{-15}$ m strong force dominates and pulls the ions together

→ quantum tunneling increases this distance ⇒ helps fusion, lowers the barrier

⇒ cross-section :



v = velocity

⇒ no. of particles overlapping the beam $= n_D x_2 A$

⇒ probability of single beam particle causing fusion reaction

$$= \frac{\text{area of all particles}}{\text{area of beam}} = \frac{n_D x_2 \cancel{A} \sigma}{\cancel{A}}$$

⇒ in time dt no. of particles passing $= n_1 \underbrace{A v dt}_{V_1}$

⇒ no. of reactions $= n_T A v dt \cdot n_D \underbrace{x_2 V}_{\to \text{volume} = A \cdot x_2} \sigma$

⇒ reaction rate $= \dfrac{\text{no. of reactions}}{\text{volume} \times \text{time}} = n_T n_D v \sigma$

⇒ in fusion, we rely on the energetic tail of the Boltzmann distribution ⇒ need ~100 keV for DT

- gain : $Q = \dfrac{\text{electrical energy produced}}{\text{electrical energy required to generate fusion}}$

⇒ break even: $Q = 1$

- ignition : when α - heating alone balances losses
  ⇒ sustained fusion

The Lawson criterion

→ to achieve energy gain the reacting nuclei need to be confined long enough for a sufficient number of particles to fuse together

→ fusion energy released per volume: $E_f = n_D n_T \sigma W \tau$

energy released per reaction

time

Maxwellian-averaged reaction rate

→ assume equal no. of D and T $\Rightarrow n_D = n_T = \frac{n}{2}$

$$\Rightarrow E_f = \frac{n^2 \sigma W \tau}{4}$$

confinement time τ

→ energy required to heat the fuel:

$$E_t = \frac{3}{2} n_i k_B T_i + \frac{3}{2} n_e k_B T_e = 3 n k_B T \,//$$

(assuming equal temperatures/equilibrium for simplicity)

→ for gain, require: $E_f > E_t$, i.e. more energy out

$\Rightarrow$ the Lawson criterion: $n\tau > \frac{12 k_B T}{\sigma W}$ //

(density-confinement product constant)

→ i.e. can achive fusion by increasing density or the confinement time!

Magnetic confinement fusion

→ good for low density plasmas @ ~ $10^{20}\ m^{-3}$

→ from the Lawson criterion get : τ ~ seconds

→ hot plasma : 10 keV ~ 10^8 K

→ tends to expand (thermal pressure!)

→ given B-fields that can be generated in a laboratory what plasma (max. density) can we confine? (few Tesla)

⇒ the magnetic pressure must balance the thermal pressure of the plasma trying to expand:

$$\frac{B^2}{2\mu_0} \approx n k_B T$$

lecture 5

⇒ for 10^8 K and few Tesla the maximum density n that can be confined is $10^{20}\ m^{-3}$

→ Order of atmosphere (Pressure ~ $10^5\ N m^{-2}$)

→ in reality plasmas are diamagnetic and the B-field penetrates the plasma, thus we can only state:

$$\frac{B^2}{2\mu_0} + \sum n k_B T = \text{constant}$$

Magnetic mirrors (lecture 2)

→ magnetic moment conserved. $\mu = -\frac{1}{2} m v_\perp^2 \frac{\underline{B}}{B^2}$ = constant

(1st adiabatic invariant)

→ the highest point of the field: B_m

$$\sin^2\theta = \frac{V_{\perp 0}^2}{V_0^2} = \frac{B_0}{B} \quad \rightarrow \theta = \text{pitch angle}$$

⇒ loss cone: $\sin^2\theta_m = \frac{B_0}{B_m}$, if angle is too small the particle is lost

⇒ particles lost due to collisions, even if trapped before (collisions change the pitch)

⇒ very BAD confinement vessel

Z-pinch

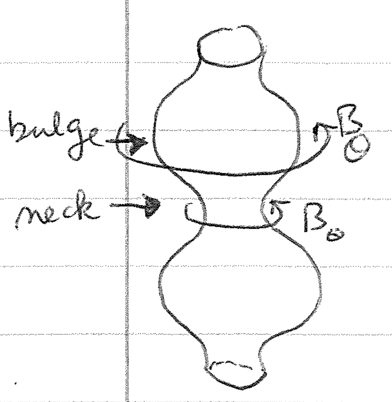
→ Cylindrical configuration of B-field & plasma
→ current flowing in z-direction:
  → heats plasma (Ohmic heating)
  → generates B-field ⇒ $\underline{j} \times \underline{B}$ force ($\underline{v} \times \underline{B}$)
⇒ plasma pushed towards the axis
  - electrons accelerated onto the axis and are very fast
  - ions are dragged with the electrons (electrostatic force)

⇒ $B_\theta = \frac{\mu_0 I}{2\pi r}$ ← current

only need 0.1 MA to confine plasma
BUT Z-pinch unstable

↑ mag. field

- Sausage instability ($m=0$ mode)
  - no instability structure in θ-direction
  - due to random fluctuations



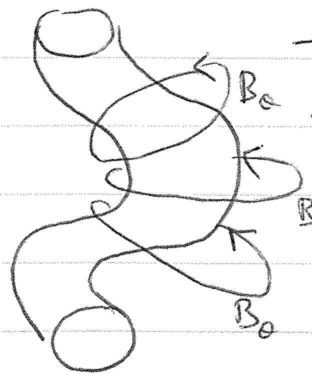
$\Rightarrow$ neck smaller radius $\rightarrow$ larger $\dfrac{B^2}{2\mu_0}$ pressure

and $\dfrac{B^2}{\mu_0 r}$ tension

$\Rightarrow$ as neck contracts, bulge expands

$\Rightarrow$ looks like a string of sausages

- kink instability ($m=1$ mode)
  - period in θ-direction
  - B-field lines get closer on one side of z-pinch
  - mag. pressure acts to increase the size of the kink



$$P_B = \frac{B^2}{2\mu_0}$$

$\rightarrow$ confinement eventually lost

- Torus $\rightarrow$ join the ends of z-pinch to stop loss from them

$\rightarrow$ plasma still unstable (sausage, kink)

$\rightarrow$ particle drift (lecture 2) in curved B-field:

$$\underline{v}_{tot} = \frac{m}{qB^2} \cdot \frac{\underline{R} \times \underline{B}}{R^2} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right) \qquad \text{(lecture 2)}$$

$\Rightarrow$ drift $\perp$ to the radius of curvature $\Rightarrow$ electrons and ions drift apart $\Rightarrow$ rise of E field

$\Rightarrow$ this causes further drifts:

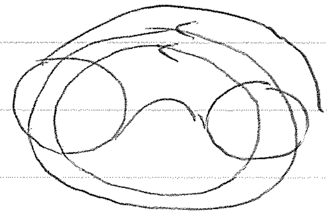
$$v_d = \frac{\underline{E} \times \underline{B}}{B^2} \qquad \text{(lecture 2)}$$

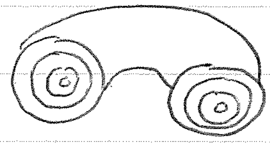
$\leftarrow$ charge independent!

→ particles drift together (no q or m dependence) towards the outside of the torus until they hit the wall of the confinement vessel

• Tokamak – by the Russian scientist Artzimovich
(Tamm, Sakharov, Lavrentiev (letter))
(toroidna kamera s magnitnymi katuskami)
(toroidna kamera s aksialnym mag. polem)

⇒ devised to solve the drift & confinement problem of the torus (still unstable)

→ toroidal field:  by external coils

→ poloidal field:  by current in plasma

due to inner poloidal coils
→ like a large transformer
generate $\partial B / \partial t$ through the middle of the torus
⇒ electromotive force around the torus ⇒ current

→ resultant field is helical

⇒ connects the B-field top-bottom
⇒ drifts cancel out, no charge separation

→ stability in tokamaks: tokamaks offer some resistance to the sausage instability

→ consider B_z and B_θ

→ plasma radius changed by small amount δa

from Ampère's Law → $I = \text{const.} = \frac{B_\theta \cdot r \cdot 2\pi}{\mu_0}$ (conservation of current) as $B_\theta = \frac{\mu_0 I}{2\pi r}$

→ change both in B_z and area $= 0$

mag. flux conserved and current too

$$\rightarrow \delta(B_z \pi a^2) = 0$$

$$\delta B_z \pi a^2 + B_z \pi 2a \delta a = 0$$

$$\Rightarrow \delta B_z = -B_z \frac{2\delta a}{a}$$

$$\delta B_\theta = -B_\theta \frac{\delta a}{a}$$

→ difference of magnetic pressures:

$$\delta p_m = \delta\left(\frac{B_z^2}{2\mu_0} - \frac{B_\theta^2}{2\mu_0}\right)$$ – take derivatives

$$= \frac{\cancel{2} B_z \delta B_z}{\cancel{2}\mu_0} - \frac{\cancel{2} B_\theta \delta B_\theta}{\cancel{2}\mu_0}$$ substitute for δB_z and δB_θ

$$= -\frac{B_z^2 2\delta a}{\mu_0 a} + \frac{B_\theta^2 \delta a}{\mu_0 a}$$

⇒ thus stable against sausage instability as long the condition is met: $B_z^2 > \frac{B_\theta^2}{2}$ // for $\delta p_m \to 0$ // pressure balance

- tokamak also stable against the kink instability.
(derivation is complicated)

→ Kruskal - Shafranov limit for stability:

$$\left|\frac{B_\theta}{B_z}\right| < \frac{2\pi a}{\lambda} \qquad \text{stabilized by } B_z$$

→ λ is the wavelength of the instability mode:
⇒ the longest possible mode in tokamak:

$$\lambda = 2\pi R$$

ring radius

$$\Rightarrow \frac{B_\theta}{B_z} < \frac{2\pi a}{2\pi R} \Rightarrow \frac{B_\theta R}{B_z a} < 1$$

B_p (polar)

B_t (toroidal)

⇒ safety factor:

$$q \equiv \frac{B_t \cdot a}{B_p R} > 1 \quad //$$

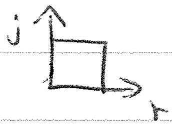
tension due to B_z acts as a "restoring" force against instability growth

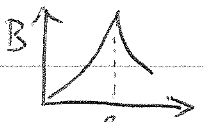
→ reality more complicated as usual though

Z-pinches for fusion (also known as "k-mod" or a "Bennet pinch")

→ force balance: cylindrical symmetry: $\frac{d}{d\theta} \to 0$

and $\frac{d}{dz} \to 0$

uniform $j(r)$:  $j = \frac{I}{\pi a^2}$

B-field 

$$\left.\begin{array}{l} B \propto r \text{ for } r<a \\ B \propto \frac{1}{r} \text{ for } r>a \end{array}\right\} B(r) = \frac{B(a)r}{a}$$

$$B(r=a) = \frac{\mu_0 I}{2\pi a}$$

from momentum equation:

$$\nabla P = j \times \underline{B}(r) = \underbrace{\frac{I}{\pi a^2}}_{j} \cdot \frac{\mu_0 I}{2\pi a} \cdot \frac{r}{a} \quad \text{and } \nabla P = \frac{dP}{dr}$$

$$(P \propto n k_B T)$$

integrate: $P(r) = -\frac{\mu_0 I^2}{2\pi^2 a^4} \cdot \frac{r^2}{2} + \text{const}$

in limit of a: $P = \frac{\mu_0 I^2}{4\pi^2 a^4}\left[a^2 - r^2\right] = \frac{\mu_0 I^2}{4\pi a^2}\left(1 - \frac{r^2}{a^2}\right)$

↳ parabolic //

since $P \to 0$ as $r \to a$

for large η (no skin current) i.e. no penetration into plasma //

Bennett relation

→ start from the momentum equation (MHD):

$$\rho \frac{du}{dt} = \underline{j} \times \underline{B} - \nabla P$$

and for equilibrium we have: $\frac{du}{dt} = 0$

→ in cylindrical geometry:

$$\frac{dP}{dr} = -j_z B_\theta \quad \text{and} \quad \nabla \times B = \mu_0 j$$

(Ampère's law)

$$\Rightarrow \mu_0 j_z = \frac{1}{r} \cdot \frac{d}{dr}(r B_\theta)$$ in cylindrical coordinates

$$\Rightarrow j_z = \frac{1}{\mu_0 r} \cdot \frac{d}{dr}(r B_\theta)$$

→ Substitute:

$$\frac{dP}{dr} = -\frac{B_\theta}{\mu_0 r} \cdot \frac{d}{dr}(r B_\theta)$$

and $\frac{d}{dr}(r B_\theta)^2 = 2r B_\theta \frac{d}{dr}(r B_\theta)$

$$\Rightarrow \frac{d(r B_\theta)}{dr} = \frac{1}{2r B_\theta} \cdot \frac{d}{dr}(r B_\theta)^2$$

$\Rightarrow$ substitute:

$$\frac{dP}{dr} = -\frac{\cancel{B_\theta}}{2\mu_0 r^2 \cancel{B_\theta}} \frac{d}{dr}(r B_\theta)^2$$

magnetic pressure

$$\Rightarrow r^2 \frac{dP}{dr} = -\frac{d}{dr}\left(\frac{r^2 B_\theta^2}{2\mu_0}\right)$$

$\Rightarrow$ integrate (LHS by parts): $\boxed{\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b \frac{du}{dx} v\, dx}$

$$\int_0^a r^2 \frac{dP}{dr} dr = -\int_0^a \frac{d}{dr}\left(\frac{r^2 B_\theta^2}{2\mu_0}\right) dr \quad \leftarrow \begin{matrix} u = r^2 \\ v = P \end{matrix}$$

$$[r^2 P]_0^a - \int_0^a 2r P\, dr = -\int_0^a \frac{d}{dr}\left(\frac{r^2 B_\theta^2}{2\mu_0}\right) dr$$

$$= -\left[\frac{r^2 B_\theta^2}{2\mu_0}\right]_0^a$$

note: at $r = a \Rightarrow P = 0 \Rightarrow [r^2 P]_0^a = 0$

and: $B_\theta = \frac{\mu_0 I_z}{2\pi a} \Rightarrow \text{RHS} = \frac{\mu_0 I^2}{8\pi^2} = \frac{a^2}{2\mu_0} \cdot \frac{\mu_0^2 I^2}{4\pi^2 \cancel{a^2}}$

$\Rightarrow$ for ideal gas: $P = (Z+1) n_i k_B T = n_i k_B T + n_e k_B T$

as $n = n_i + n_e$

where $n_e = Z n_i$

→ substituting:

$$\therefore \frac{\mu_0 I^2}{8\pi^2} = (Z+1) k_B \int_0^a 2r\, n_i\, dr$$

$$\frac{\mu_0 I^2}{8\pi} = (Z+1) k_B \int_0^a 2\pi r\, n_i\, dr$$

$n_i \times V$

N = ion line density

as $\pi r^2 = A$

$$\therefore \mu_0 I_z^2 = 8\pi (Z+1) N k_B T \quad //$$

the Bennett relation

where $N = n_i \times \pi a^2$

can be used for z-pinch or tokamak