Plasma instabilities:

- 3 different ways to study stability of a system:

→ restoring force: a deflection from an unstable equilibrium leads to a force F_{defl} that increases the initial deflection and is $\propto$ to the angle of deflection and will grow exponentially in time

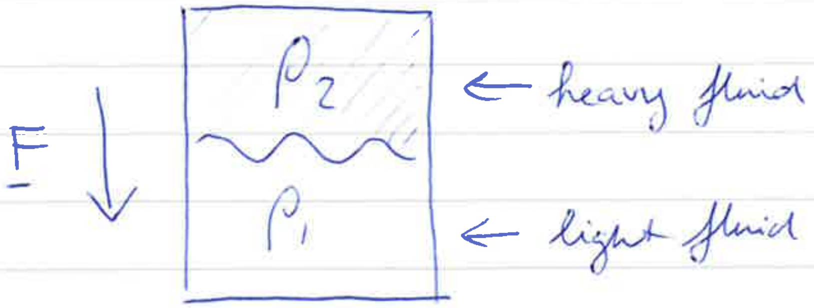
in a stable situation a restoring force F_{res} will drive the system back to stable equilibrium
⇒ harmonic oscillations

→ potential energy: W_{pot}, the system is stable, when the potential energy increases upon any possible perturbation, i.e. stable point is a minimum in W_{pot}

→ mode analysis: decompose small perturbation into Fourier components ⇒ modes $\propto \exp(-i\omega t)$
⇒ if frequencies of all modes are real
→ system stable
⇒ if any mode has a complex frequency with a positive imaginary part
→ this mode will grow in time
→ unstable

Rayleigh-Taylor instability:

→ macroscopic fluid instability



$\rho_1 < \rho_2$

$P_1 > P_2$

↑ opposing gradients

→ fluid description, start from continuity and momentum equations:

$$\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho = 0$$

gravity (in z-direction)

$$\rho \frac{d\underline{u}}{dt} + \rho \underline{u} \cdot \nabla \underline{u} = -\nabla P - \rho g \hat{\underline{z}}$$

~~at~~ equilibrium fluid is at rest: $\underline{u} = 0$

→ z-dependency: $P = P(z)$, $\rho = \rho(z)$

→ introduce perturbation:

$$P \to P + \delta P$$
$$\rho \to \rho + \delta \rho$$
$$\underline{u} \neq 0$$

(remember initial $\underline{u} = 0$ as system starts from rest!)

→ insert into the continuity equation:

$$\frac{\partial(\rho+\delta\rho)}{\partial t} + \underline{u}\cdot\nabla(\rho+\delta\rho) = 0$$

→ linearize and subtract the equilibrium equation:

$$\frac{\partial(\cancel{\rho}+\delta\rho)}{\partial t} + \underset{\text{perturbed} \neq 0}{\underline{u}}\cdot\nabla(\rho+\delta\rho) - \cancel{\frac{\partial\rho}{\partial t}} - \cancel{\underset{=0}{\underline{u}}\cdot\nabla\rho} = 0$$

$\underline{u}\cdot\nabla\delta\rho = 0$ (linearize)

$=0$ (no initial velocity)

(system at rest)

$$\Rightarrow \frac{\partial(\delta\rho)}{\partial t} + u_z\frac{\partial\rho}{\partial z} = 0$$ (only consider z-direction)

→ do the same with the momentum equation

$$\underbrace{(\rho+\delta\rho)\frac{\partial u_z}{\partial t}}_{\text{perturbation products } =0} + \cancel{(\rho+\delta\rho)\,u_z\frac{\partial u_z}{\partial z}}_{=0} + \frac{\partial(\cancel{P}+\delta P)}{\partial z} + (\cancel{\rho}+\delta\rho)\cdot g$$

→ drop perturbation products

$$-\rho\frac{\partial u_z}{\partial t} - \cancel{\rho\, \underset{=0}{u_z}\frac{\partial u_z}{\partial z}}_{=0} - \cancel{\frac{\partial P}{\partial z}} - \cancel{\rho g}$$

(no initial velocity!)

<u>NOTE</u>: u_z only a perturbation, i.e. all products of u_z and $\partial u_z = 0$!

$$\Rightarrow 0 = \frac{\partial(\delta P)}{\partial z} + \delta\rho g + \rho\frac{\partial u_z}{\partial t}$$

survives linearization

→ also ripples in x-direction (no g-component):

$$\Rightarrow \rho\frac{\partial u_x}{\partial t} = -\frac{\partial(\delta P)}{\partial x}$$

(same for y-direction, but here we consider only x for simplicity)

→ assume wave-like solution with exponentia growth rate γ:

ripples in x-direction

$$\underline{u} = \underline{u}_0 \exp(ikx)\exp(\gamma t)$$
$$\delta\rho = \delta\rho_0 \exp(ikx)\exp(\gamma t)$$
$$\delta P = \delta P_0 \exp(ikx)\exp(\gamma t)$$

→ substitute wave-like solutions for density into continuity eq.:

derivatives: $\frac{\partial(\delta\rho)}{\partial t} = \gamma\,\delta\rho \Rightarrow \gamma\,\delta\rho = -u_z \frac{d\rho}{dz}$ (exp terms cancel out)

→ substitute wave-like solutions for δP and u into momentum eq:

derivatives: $\frac{\partial u}{\partial t} = \gamma u \Rightarrow \rho \frac{\partial u}{\partial t} = \gamma \rho u$

$$\frac{\partial(\delta P)}{\partial x} = ik\,\delta P \Rightarrow -\frac{\partial(\delta P)}{\partial x} = -ik\,\delta P$$

⇒ for x: $\gamma \rho u_x = -ik\,\delta P \Rightarrow \delta P = -\frac{\gamma\rho u_x}{ik}$ //

⇒ for z: $\gamma \rho u_z = -\frac{\partial(\delta P)}{\partial z} - g\,\delta\rho$

→ amplitude of the perturbations grows in z-direction but velocity has x-component as well

→ for incompressible fluid:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0 \qquad \text{as} \qquad u = u_0 \exp(ikx)\exp(\gamma t)$$

$$\Rightarrow ik u_x + \frac{\partial u_z}{\partial z} = 0 \qquad \Rightarrow u_x = \frac{-\partial u_z}{ik\,\partial z}$$

→ and substitute for u_x in $\delta P = -\frac{\gamma \rho u_x}{ik}$:

$$\Rightarrow \delta P = -\frac{\gamma \rho}{k^2}\frac{\partial u_z}{\partial z} \qquad (\text{as } i \times i = -1)$$

and we had: $\delta \rho = -\frac{u_z}{\gamma}\frac{\partial \rho}{\partial z}$

→ substitute for δP and $\delta \rho$:

$$\gamma \rho u_z = -\frac{\partial(\delta P)}{\partial z} - g\,\delta\rho$$

$$= +\frac{\gamma}{k^2} \cdot \frac{\partial}{\partial z}\left(\rho \frac{\partial u_z}{\partial z}\right) + g\frac{u_z}{\gamma} \cdot \frac{\partial \rho}{\partial z} \qquad / \times \frac{k^2}{\gamma}$$

$$\Rightarrow \frac{\partial}{\partial z}\left(\rho \frac{\partial u_z}{\partial z}\right) = k^2 \rho u_z - \frac{g k^2}{\gamma^2} u_z \frac{\partial \rho}{\partial z}$$

↳ this equation describes the evolution of the fluid after the perturbation

→ since $iku_x + \frac{\partial u_x}{\partial z} = 0$ and the conservation of u_z (continuous across the fluid boundary) also the spatial derivative of $u_z/\partial z$ must be continuous

→ integrate across the boundary:

$$\frac{\partial}{\partial z}\left(\rho \frac{\partial u_z}{\partial z}\right) = k^2 \rho u_z - \frac{gk^2}{\gamma^2} u_z - \frac{gk^2}{\gamma^2} u_z \frac{\partial \rho}{\partial z}$$

$$\Delta\left(\rho \frac{\partial u_z}{\partial z}\right) = -\frac{gk^2}{\gamma^2} u_z \Delta(\rho)$$

where $\Delta f \equiv f(0)_+ - f(0)_-$

jump quantities across the boundary, signs for +ve and -ve sides of the boundary, i.e. jump in ρ !

→ i.e. for our fluids 1 and 2:

$$\rho_2 \left(\frac{\partial u_z}{\partial z}\right)_2 - \rho_1 \left(\frac{\partial u_z}{\partial z}\right)_1 = -\frac{gk^2}{\gamma^2} u_z (\rho_2 - \rho_1)$$

→ assuming exponential growth in z:

$$u_z = A \exp(kz) + B \exp(kz)$$

→ u_z continuous at the boundary, but must be zero at $\pm\infty$:

$$u_z = A \exp(kz) \quad \text{for } z<0 \quad (1)$$
$$u_z = A \exp(-kz) \quad \text{for } z>0 \quad (2)$$

→ substitute for u_z:

$$\rho_2 \left(\frac{du_z}{dz}\right)_2 - \rho_1 \left(\frac{du_z}{dz}\right)_1 = -\frac{gk^2}{\gamma^2} u_z (\rho_2 - \rho_1)$$

$$\Rightarrow \left(\frac{du_z}{dz}\right)_2 = -k u_z \quad \text{and} \quad \left(\frac{du_z}{dz}\right)_1 = +k u_z$$

$$\Rightarrow -\rho_2 k \cancel{u_z} + \rho_1 k \cancel{u_z} = -\frac{gk^2}{\gamma^2} \cancel{u_z} (\rho_2 - \rho_1)$$

→ thus R-T instability growth rate:

$$\gamma = \sqrt{\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right) g k}$$

acceleration due to gravity (or different force)

→ in ICF growth rate $= \sqrt{ka}$

acceleration

→ consider any small perturbation in D-T pellet surface smoothness or laser illumination:

→ grows exponentially

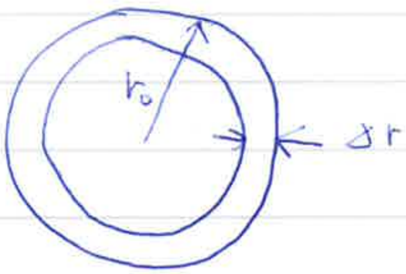
→ fastest growing perturbation mode has largest k ⇒ shortest λ-mode

→ any mode must reach saturation, cannot grow forever

– typically saturation reached, when amplitude ~ λ

⇒ i.e. fastest mode: $k \sim 1/\Delta r$ (most dangerous)

Δr ↑ shell thickness



→ aspect ratio: $\dfrac{\Delta r}{r_0}$

→ cannot let γt exceed some constant of this mode to avoid shell break up

⇒ i.e. for 50 μm shell thickness δr, we can afford 100 Å (typical) surface roughness:

$$\gamma t = \sqrt{ka}\,t = \sqrt{\frac{a}{\delta r}}\,t \leq 8.5$$

and shell acceleration limited by shell radius:

$$a = \frac{2r_0}{t^2} \Rightarrow \sqrt{\frac{2r_0}{\Delta r}} \leq 8.5$$

↑ shell aspect ratio

⇒ need high aspect ratio for strong ICF compression!

Parametric instabilities

→ incident light (EM radiation) can transfer energy to plasma through various processes including driving plasma waves (plasmons, ion acoustic, etc.)

→ eg. resonance absorption

⇒ can result in unstable situations

→ parametric instabilities

– laser beam incident onto plasma can produce real charge separation → generate plasma waves

↳ ω_0

– random fluctuations always exist in plasmas (thermal effects)

↳ ω_1

→ laser can couple to the waves in plasma

↳ $\omega_0 \pm \omega_1$

→ laser light with ω_0 propagates into plasma with wave ω_1:

⇒ laser oscillates with plasma wave @ $\omega_2 = \omega_0 - \omega_1$

$\omega_3 = \omega_0 + \omega_1$

if ω_2, ω_3 not heavily damped, can rebeat with the light and enhance wave @ ω_1

→ +ve feedback ⇒ instability ↗

⇒ parametric instabilities of different kind

→ Common formalism to obtain growth rate for the parametric instabilities:

→ incident light A_0, ω_0, and two different plasmawaves A_1, ω_1 and A_2, ω_2 with damping rates Γ_1 and Γ_2:

coupling to A_1

$$\frac{d^2 A_1}{dt^2} + \Gamma_1 \frac{dA_1}{dt} + \omega_1^2 A_1 = c_1 A_2 A_0$$

constants

$$\frac{d^2 A_2}{dt^2} + \Gamma_2 \frac{dA_2}{dt} + \omega_2^2 A_2 = c_2 A_1 A_0$$

coupling to A_2

→ assume laser produces coupling between the waves and this coupling is proportional to both the laser and coupled waves amplitudes

→ apply Fourier mode solution:

$$A(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t)\, A(t)\, dt$$

→ laser not depleted in interaction (assumption):

$$A_0 = E_0 \cos(\omega_0 t) = \frac{E_0}{2}\left[\exp(i\omega_0 t) + \exp(-i\omega_0 t)\right]$$

→ Fourier transform of mode 1:

$$A_1(\omega_1) = \int_{-\infty}^{\infty} \exp(i\omega_1 t)\, A_1(t)\, dt$$

and $$\frac{d^2A_1}{dt^2} + T_1 \frac{dA_1}{dt} + \omega_1^2 A_1 = c_1 A_2 A_0$$

and $$A_0 = \frac{E_0}{2}\left[\exp(i\omega_0 t) + \exp(-i\omega_0 t)\right]$$

$$\Rightarrow \frac{d^2A_1}{dt^2} = -\omega^2 A_1 \; , \; \frac{dA_1}{dt} = i\omega A_1 \;\Rightarrow T_1 \frac{dA_1}{dt} = i\omega T_1 A_1$$

$\Rightarrow$ collect terms on LHS: $-\omega^2 A_1(\omega) + i\omega T_1 A_1(\omega) + \omega_1^2 A_1(\omega_1)$

$\rightarrow$ on RHS: add up exponential terms:

$$\frac{c_1 E_0}{2} \cdot A_2 \exp(i\omega t) \cdot \left[\exp(i\omega_0 t) + \exp(-i\omega_0 t)\right]$$

$$\Rightarrow \left[\omega^2 - \omega_1^2 + i\omega T_1\right] A_1(\omega) + \frac{c_1 E_0}{2}\left[A_2(\omega + \omega_0) + A_2(\omega - \omega_0)\right] = 0$$

$$\Rightarrow D_1(\omega) A_1(\omega) + \frac{c_1 E_0}{2}\left[A_0(\omega + \omega_0) + A_2(\omega - \omega_0)\right] = 0$$

for $D_i(\omega) = \omega^2 - \omega_i^2 + i\omega T_i$

$\rightarrow$ mode 2 couples at $\omega + \omega_0$ and $\omega - \omega_0$, take Fourier transforms at these frequencies: (same form by analogy)

$$D_2(\omega - \omega_0) A_2(\omega - \omega_0) + \frac{c_2 E_0}{2}\left[A_1(\overbrace{\omega - \omega_0 + \omega_0}^{\omega}) + A_1(\overbrace{\omega - \omega_0 - \omega_0}^{\omega - 2\omega_0})\right] = 0$$

$$D_2(\omega + \omega_0) A_2(\omega + \omega_0) + \frac{c_2 E_0}{2}\left[A_1(\overbrace{\omega + \omega_0 + \omega_0}^{\omega + 2\omega_0}) + A_1(\overbrace{\omega + \omega_0 - \omega_0}^{\omega})\right] = 0$$

→ in reality ω close to the incoming ω_0

⇒ i.e $A_1(\omega) \gg A_1(\omega \pm 2\omega_0)$ → neglect $A_1(\omega \pm 2\omega_0)$

→ so small that SBS satisfies: $\omega_0 \approx \omega \approx \omega_1$

→ for the instability to develop the pump laser must transfer energy to the wave: hence $A_2(\omega-\omega_0)$ finite

neglect $A_2(\omega+\omega_0)$ as mode 2 ω_2 must be less than the incoming light (by energy conservation) $\omega_2 < \omega_0$

⇒ Coupled equations:

$$D_1(\omega) A_1(\omega) + \frac{c_1 E_0}{2}\Big[\underbrace{\cancel{A_0(\omega+\omega_0)}}_{\text{ignore}} + A_2(\omega-\omega_0)\Big] = 0$$

$$D_2(\omega-\omega_0) A_2(\omega-\omega_0) + \frac{c_2 E_0}{2}\Big[A_1(\omega) + \underbrace{\cancel{A_1(\omega-2\omega_0)}}_{\text{neglect}}\Big] = 0$$

$$D_2(\omega+\omega_0)\underbrace{\cancel{A_2(\omega+\omega_0)}}_{\text{ignore}} + \frac{c_2 E_0}{2}\Big[A_1(\omega) + \underbrace{\cancel{A_1(\omega+2\omega_0)}}_{\text{neglect}}\Big] = 0$$

$$\Rightarrow \begin{pmatrix} D_1(\omega) & c_1 E_0/2 \\ c_2 E_0/2 & D_2(\omega-\omega_0) \end{pmatrix} \begin{pmatrix} A_1(\omega) \\ A_2(\omega-\omega_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

→ non-trivial solution: $D_1(\omega) D_2(\omega-\omega_0) - \dfrac{c_1 c_2 E_0^2}{4} = 0$

make approximations: $\omega_0 - \omega \approx \omega_2$, $\omega + \omega_1 \approx 2\omega_1$
(close frequencies)

we had: $\omega_2 = \omega_0 - \omega_1 \leftarrow$ mode grows due to beading

$\omega_3 = \omega_0 + \omega_1$

and $D_i(\omega) = \omega^2 - \omega_i^2 + i\omega \Gamma_i$

$$\Rightarrow D_1(\omega) = \omega^2 - \omega_1^2 + i\omega\Gamma_1 = (\overbrace{\omega + \omega_1}^{\approx 2\omega}) \cdot (\omega - \omega_1) + i\omega\Gamma_1$$
$$\approx 2\omega_1\omega - 2\omega_1^2 + i\underbrace{\omega}_{\approx\omega_1}\Gamma_1 = 2\omega_1\left(\omega - \omega_1 + i\frac{\Gamma_1}{2}\right)$$

$$\Rightarrow D_2(\omega - \omega_0) = (\omega - \omega_0)^2 - \omega_2^2 + i(\omega - \omega_0)\Gamma_2$$
$$\approx \cancel{(-\omega_2)^2 - \omega_2^2} - i\,\omega_2\Gamma_2$$
$$\approx -2\omega_2\left(\underbrace{\omega - \omega_0}_{\approx\omega_2} + \omega_2 + \frac{i\Gamma_2}{2}\right)$$

to keep the same form

$\Rightarrow$ Substitute:

$$-4\omega_1\omega_2\left(\omega - \omega_1 + i\frac{\Gamma_1}{2}\right)\left(\omega - \omega_0 + \omega_2 + i\frac{\Gamma_2}{2}\right) - \frac{c_1 c_2 E_0^2}{4} = 0$$

$\rightarrow$ for growing (instability) mode, need complex frequency:

$$\omega = \underbrace{\omega_R}_{\text{real}} + i\underbrace{\Gamma}_{\text{growth/decay} \rightarrow \text{imaginary}}$$

$\rightarrow$ Substitute:

$$-4\omega_1\omega_2\left[\omega_R + i\Gamma - \omega_1 + i\frac{\Gamma_1}{2}\right]\left[\omega_R + i\Gamma - \omega_0 + \omega_2 + i\frac{\Gamma_2}{2}\right] - \frac{c_1 c_2 E_0^2}{4} = 0$$

$$\Rightarrow \left[\omega_R - \omega_1 + i\left(\Gamma + \frac{\Gamma_1}{2}\right)\right]\left[\omega_R - \omega_0 + \omega_2 + i\left(\Gamma + \frac{\Gamma_2}{2}\right)\right] - \underbrace{\frac{c_1 c_2 E_0^2}{16\omega_1\omega_2}}_{=\Psi E_0^2} = 0$$

$\Rightarrow$ only take the imaginary product and equating the real part to zero:

real parts $\rightarrow$

$$\Rightarrow (\omega_R - \omega_1)\cdot(\omega_R - \omega_0 + \omega_2) + \Psi E_0^2 = \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)$$

$$\Rightarrow (\omega_R - \omega_1)\cdot(\omega_R - \omega_1 - \Delta) + \Psi E_0^2 = \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)$$

where : $\Delta \equiv \omega_0 - \omega_1 - \omega_2$ and $\Psi \equiv \dfrac{c_1 c_2}{16\omega_1\omega_2}$

$\Rightarrow$ now equate imaginary parts:

Starting from: $4\omega_1\omega_2\left(\omega - \omega_1 + i\frac{\Gamma_1}{2}\right)\left(\omega - \omega_0 + \omega_2 + i\frac{\Gamma_2}{2}\right) + \Psi E_0^2 = 0$

but complex frequency: $\omega = \omega_R + i\Gamma \rightarrow$ substitute

$$\Rightarrow \cancel{4\omega_1\omega_2}\left(\omega_R - \omega_1 + i\Gamma + i\frac{\Gamma_1}{2}\right)\left(\overbrace{\omega_R - \omega_0 + \omega_2}^{\omega_R - \omega_1 - \Delta} + i\Gamma + i\frac{\Gamma_2}{2}\right) + \underbrace{\cancel{\Psi E_0^2}}_{\text{real}} = 0$$

$$\Rightarrow \underbrace{\cancel{(\omega_R - \omega_1)(\omega_R - \omega_0 + \omega_2)}}_{\text{real}} + (\omega_R - \omega_1 - \Delta)\left(i\Gamma + i\frac{\Gamma_1}{2}\right) + (\omega_R - \omega_1)\left(i\Gamma + \frac{\Gamma_2}{2}\right) + \underbrace{i^2}_{=-1}\underbrace{\cancel{\left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)}}_{\text{real}} = 0$$

$$\Rightarrow (\omega_R - \omega_1)\cdot\left(i\Gamma + i\frac{\Gamma_2}{2}\right) + (\omega_R - \omega_1)\left(i\Gamma + \frac{i\Gamma_1}{2}\right) - \Delta\left(i\Gamma + i\frac{\Gamma_1}{2}\right) = 0$$

Only imaginary parts left $\rightarrow$ drop the i's:

$$\Rightarrow (\omega_R - \omega_1)\cdot\left(2\Gamma + \frac{\Gamma_2}{2} + \frac{\Gamma_1}{2}\right) - \Delta\left(\Gamma + \frac{\Gamma_1}{2}\right) = 0$$

$$\Rightarrow \omega_R - \omega_1 = \frac{\left(\Gamma + \frac{\Gamma_1}{2}\right)\Delta}{2\Gamma + \frac{\Gamma_1}{2} + \frac{\Gamma_2}{2}}$$

$$\therefore \omega_R = \omega_1 + \frac{\left(\Gamma + \frac{1}{2}\Gamma_1\right)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2} \,//$$

← now have expression for ω_R

→ eliminate ω_R by substitution into:

$$(\omega_R - \omega_1)(\omega_R - \omega_1 - \Delta) + \psi E_0^2 = \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)$$

$$\Rightarrow E_0^2 = -\frac{1}{\psi}\left\{\left[\frac{\left(\Gamma + \frac{1}{2}\Gamma_1\right)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2}\right]\cdot\left[\frac{\left(\Gamma + \frac{1}{2}\Gamma_1\right)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2} - \Delta\right] - \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)\right\}$$

$$= \frac{1}{\psi}\left\{\left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right) - \left(\frac{\left(\Gamma + \frac{\Gamma_1}{2}\right)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2} \cdot \frac{\Delta\Gamma + \cancel{\frac{\Gamma_1}{2}\Delta} - \Delta 2\Gamma - \cancel{\Delta\frac{\Gamma_1}{2}} - \Delta\frac{\Gamma_2}{2}}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2}\right)\right\}$$

$$= \frac{1}{\psi}\left\{\left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right) - \left[\frac{\left(\Gamma + \frac{\Gamma_1}{2}\right)\cdot\Delta}{2\Gamma + \frac{\Gamma_1}{2} + \frac{\Gamma_2}{2}} \cdot \frac{-\Delta\Gamma - \Delta\frac{\Gamma_2}{2}}{2\Gamma + \frac{\Gamma_1}{2} + \frac{\Gamma_2}{2}}\right]\right\}$$

$$= \frac{1}{\psi}\left\{\left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)\cdot\left[1 + \frac{\Delta^2}{\left(2\Gamma + \frac{\Gamma_1}{2} + \frac{\Gamma_2}{2}\right)^2}\right]\right\}$$

→ for instability to grow $\Gamma > 0$, ⇒ threshold @ $\Gamma = 0$:

$$\text{i.e} \Rightarrow E_{0,th}^2 = \frac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1 c_2}\left[1 + \frac{4\Delta^2}{(\Gamma_1 + \Gamma_2)^2}\right] \qquad \div 4$$

→ minimum intensity when $\Delta = 0$ // ⇒ $E_{0,min}^2 = \dfrac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1 c_2}$

⇒ resonance $\Gamma^2 = E_0^2\psi$ ⇒ growth rate: $\Gamma_{max} = E_0\sqrt{\dfrac{c_1 c_2}{16\omega_1\omega_2}}$ //

$$\psi = \frac{c_1 c_2}{16\omega_1\omega_2}$$