Shocks in plasma

"Shock" - shocked by an unexpected event

"shock wave" - sudden transition in the properties of a fluid medium involving a difference in flow velocity across a narrow (ideally abrupt) transition

→ in physics: shock wave is a disturbance travelling at speed greater than the local sound speed

⇒ i.e. Mach number: $M = \frac{u_n}{c_s} > 1$

in plasma physics we use the Alfvén speed:

$$M = \frac{u_n}{V_A} > 1$$

→ abrupt change in conditions: $P, \rho, T, \dots$

→ shock → carries energy forward @ shock velocity $> c_s$

→ heats & accelerates medium as it passes

→ shock heats the fluid behind it so the motion of the shockwave to the heated fluid is subsonic

i.e. changes in source of Energy are communicated to the shock at a new (higher) sound speed!

→ produces irreversible change in fluid state (increase in entropy)

Rankine-Hugoniot relations

→ first consider jump conditions with no B-field

→ assume adiabatic shock

→ in frame of the shock (i.e. shock front stationary), upstream fluid stationary

↳ shock at $x = 0$

→ conservation laws: (remember lecture 5)

continuity equation: $\frac{\partial \rho}{\partial t} + \nabla(\rho \underline{u}) = 0$

momentum equation: $\rho \frac{\partial u}{\partial t} + \nabla \rho \underline{u}.\underline{u} + \nabla P = 0$

(no Lorentz force)

energy equation:

internal energy

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho \epsilon\right) + \nabla\left[\rho \underline{u}\left(\epsilon + \frac{u^2}{2}\right) + P\underline{u}\right] = 0$$

→ also known as Euler equations

→ express the Euler equations in the form of: $\frac{\partial}{\partial t}\rho_Q = -\frac{\partial}{\partial x}\Gamma_Q$

for ∞-small interval $[x_1, x_2]$ → $\int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho_Q \, dx \to 0$

assume steady state

→ apply logic + the divergence theorem:

$$\iiint_V \nabla \underline{A} \cdot dV = \oint_S A \cdot d\underline{S}$$

volumetric change in A

flux

→ rewrite the Euler equations in a "conservative form":

$$\underbrace{\frac{\partial}{\partial t}\rho_Q}_{\text{change in density of quanity } Q} + \underbrace{\nabla \cdot T_Q}_{\text{flux of the quantity } Q \text{ [per unit area per time]} \rightarrow \text{flux}} = \underbrace{S_Q}_{\text{volumetric source of quantity } Q}$$

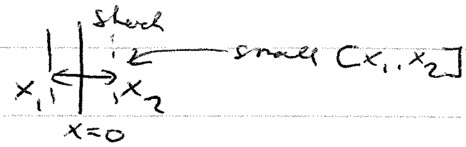
→ integrate the conservative form over some volume with a surface σ and Apply the divergence theorem:

$$\int_V \nabla \cdot T_Q \cdot dV = \oint_\sigma T_Q \cdot dA$$

$$\Rightarrow \int_V \frac{\partial}{\partial t}\rho_Q + \oint_\sigma T_Q \cdot dA = \text{net source in Volume}$$

$$\Rightarrow \frac{\partial}{\partial t} Q + \underbrace{\oint_\sigma T_Q \cdot dA}_{\text{net flow of } Q \text{ thgh surface } \sigma} = \underbrace{\text{net source in volume}}_{=0}$$

→ we have form:



$$\underbrace{\int_{x_1}^{x_2} \frac{\partial}{\partial t}\rho_Q \, dx}_{\text{approaches } 0 \text{ for small } [x_1, x_2]} = -\int_{x_1}^{x_2} \frac{\partial}{\partial x} T_Q(x)\, dx = \underbrace{T_Q(x_2) - T_Q(x_1)}_{=0}$$

approaches 0

fa small $[x_1, x_2]$

or say steady state $\frac{\partial}{\partial t} = 0$

→ rewrite the continuity eq. as:

$$\underbrace{\rho_u u_u}_{\text{upstream}} = \underbrace{\rho_d u_d}_{\text{downstream}}$$

→ momentum equation:

$$\rho u_u^2 + P_u = \rho u_d^2 + P_d$$

→ energy equation:

$$u_u\left(\frac{1}{2}u_u^2 + \epsilon_u + \frac{P_u}{\rho_u}\right) = u_d\left(\frac{1}{2}u_d^2 + \epsilon_d + \frac{P_d}{\rho_d}\right)$$

→ and for ideal gas: $\epsilon = \frac{1}{\gamma-1}\cdot\frac{P}{\rho}$

substitute ⇒ $u_u\left(\frac{1}{2}u_u^2 + \frac{\gamma}{\gamma-1}\cdot\frac{P_u}{\rho_u}\right) = u_d\left(\frac{1}{2}u_d^2 + \frac{\gamma}{\gamma-1}\cdot\frac{P_d}{\rho_d}\right)$

$$\Rightarrow \rho_u u_u\left(\epsilon_u + \frac{u_u^2}{2}\right) + P_u u_u = \rho_d u_d\left(\epsilon_d + \frac{u_d^2}{2}\right) + P_d u_d \quad //$$

→ post shock (or particle) velocity u_p: (also piston)

$$u_p = u_u - u_d \quad \Rightarrow u_d = u_u - u_p$$

→ continuity equation:

$$\Rightarrow \rho_u u_u = \rho_d(u_u - u_p)$$

$$\Rightarrow \frac{\rho_d}{\rho_u} = \frac{u_u}{u_u - u_p} \; // \qquad \Rightarrow \rho_d = \frac{u_u \rho_u}{u_u - u_p} \; //$$

→ momentum equation:

$$P_d - P_u = \rho_u u_u^2 - \rho_d u_d^2$$

$$= \rho u_u^2 - \rho_d (u_u - u_p)^2$$

$$= \rho_u u_u^2 - \frac{u_u \rho_u}{\cancel{u_u - u_p}} (u_u - u_p)^{\cancel{2}}$$

as $\rho_d = \frac{u_u \rho_u}{u_u - u_p}$

$$\Rightarrow P_d - P_u = \cancel{\rho_u u_u^2} - \cancel{\rho_u u_u^2} + \rho_u u_u u_p$$

$$= \rho_u u_u u_p \quad //$$

→ energy equation: as $P_u \to 0$ $\quad u_d = u_u - u_p$

$$\rho_u u_u \left(\epsilon_u + \frac{u_u^2}{2}\right) + \overset{0}{\cancel{P_u}} \cdot u_u = \rho_d u_d \left(\epsilon_d + \frac{u_d^2}{2}\right) + P_d u_d$$

$$\rho_u u_u \epsilon_u + \rho_u u_u \cdot \frac{u_u^2}{2} = \frac{u_u \rho_u}{\cancel{u_u - u_p}} \cdot (\cancel{u_u - u_p}) \cdot \epsilon_d$$

$$+ \frac{u_u \rho_u}{\cancel{u_u - u_p}} \cdot (\cancel{u_u - u_p}) \cdot \frac{(u_u - u_p)^2}{2} + P_d (u_u - u_p)$$

$$\rho_u u_u \epsilon_u + \cancel{\rho_u u_u \cdot \frac{u_u^2}{2}} = \rho_u u_u \epsilon_d + \cancel{\rho_u u_u \frac{u_u^2}{2}} - \rho_u u_u \cdot \frac{\cancel{2} u_u u_p}{\cancel{2}}$$

$$+ \rho_u u_u \frac{u_p^2}{2} + P_d (u_u - u_p)$$

$$\Rightarrow \frac{1}{2} \rho_u u_u u_p^2 + \rho_u u_u (\epsilon_d - \epsilon_u) = \overset{= u_u (P_d - \overset{0}{P_u})}{\rho_u u_u^2 u_p} + P_d u_p - P_d u_u$$

$$= \cancel{P_d u_u} - \cancel{P_d u_u} + P_d u_p$$

⇒ initial conditions ρ_0, P_0, ϵ_0 (before shock arrives)

⇒ $\rho_u = \rho_0$ → initial density

⇒ $\rho_d = \rho$ → final shocked density

⇒ $u_u = u_s$ → shock velocity (we work in the frame of the shock)

⇒ $P_u = P_0 = 0$ → pre shock pressure is negligible in comparison

⇒ $P_d = P$ → shock pressure

⇒ rewrite R-H relations:

$$\rho_0 u_s = \rho\,(u_s - u_p)$$

$$P - \cancel{P_0} = \rho_0 u_s = u_p = P$$

$$P u_p = \frac{1}{2}\rho_0 u_s u_p^2 + \rho_0 u_s (\epsilon - \epsilon_0)$$

⇒ measure: u_s and u_p (if use flyer plate)

⇒ know: ρ_0 (initial conditions)

⇒ determine: P, ρ, ϵ (and ϵ_0)

Rankine-Hugoniot relations with B-field

→ Conserved parameters: mass, momentum, energy
AND E and B fields
(across the shock boundary)

→ since notation gets more complicated, use the form:

$$[X] = X_u - X_d \quad \text{for the conservation laws}$$

→ Shock geometry becomes important, define the angle of the upstream B-field to the shock normal as θ_{Bn}

- ∥-shock : $\theta_{Bn} = 0°$
- ⊥-shock : $\theta_{Bn} = 90°$
- oblique shock : $0° < \theta_{Bn} < 90°$
- quasi-∥ shock : $\theta_{Bn} < 45°$
- quasi-⊥ shock : $\theta_{Bn} > 45°$

→ <u>mass conservation</u>: the same as before, from continuity eq.:
(in the <u>shock-normal</u> direction)

$$\rho_u u_{u_n} = \rho_d u_{d_n} \Rightarrow [\rho u_n] = 0$$

⇒ shock compression ratio : $r = \dfrac{\rho_d}{\rho_u} > 1$

maximum value of r can be determined from the conservation laws (see later)

→ **E & B field conservation:**

first consider changes associated with the fluid flow

→ assume a flat shock and position far away from any turbulence ⇒ current can be neglected

Ohm's Law: $j = \sigma(\underline{E} + \underline{u} \times \underline{B})$

Faraday's Law: $\frac{\partial B}{\partial t} = -\nabla \times E = 0$ (have steady state)

$\Rightarrow \underline{E} = -\underline{u} \times \underline{B}$

i.e in a perfectly conducting fluid in a co-moving frame, the induced electric field is:

$\underline{E}' = \underline{E} + \underline{u} \times \underline{B} = 0$ (vanishes)

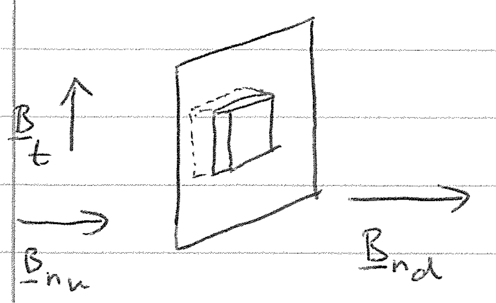
⇒ thus the conservation of the magnetic flux (Faraday's law) gives: $\hat{n} \times (\underline{u} \times \underline{B})$ conserved across the boundary:

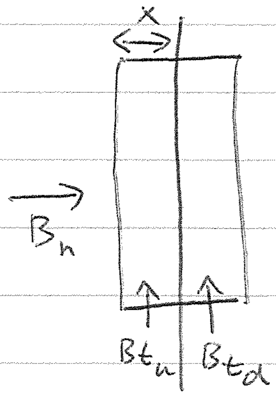
$$[u_n \underline{B}_t - B_n \underline{u}_t] = 0$$

normal transverse

i.e. perpendicular

→ <u>B-field conserved:</u> the same flux in/out





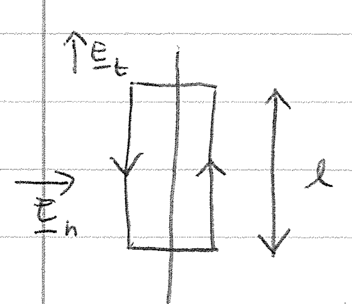
→ the divergence theorem: $\iiint_V \nabla \cdot \underline{B} \cdot dV = \oint \underline{B} \cdot d\underline{S}$

and $\nabla \cdot \underline{B} = 0$

$\Rightarrow B_{nu} = B_{nd} \Rightarrow [\underline{B}_n] = 0$

$\Rightarrow B_t$ is the same ⇒ cancels out

→ <u>E-field conservation:</u>



from Faraday's law $\frac{\partial B}{\partial t} = -\nabla \times E = 0$ steady state

$\oint \underline{E} \cdot dl = \int \underbrace{\nabla \times E}_{=0} \cdot ds$

$\Rightarrow E_t$ continuous $\Rightarrow l E_{tu} = l E_{td}$

$\Rightarrow [\underline{E}_t] = 0$

$\rightarrow \underline{E} = -\underline{u} \times \underline{B} \Rightarrow [u_n \underline{B}_t - B_n \underline{u}_t] = 0$

→ momentum conservation. $\underline{F} = m \cdot \underline{a}$

- ‖ to the shock normal: (normal)

→ $\underbrace{\rho_u u_{u_n}}_{} \cdot \underline{u}_{u_n} - \rho_d u_{d_n} \cdot \underline{u}_{d_n} = m \cdot \underline{a}_n$

mass flux!
per time, per area

→ kinetic pressure (force per area)

$$P_d - P_u$$

→ magnetic pressure: $\frac{B_{tu}^2}{2\mu_0} - \frac{B_{td}^2}{2\mu_0}$ (only normal direction)

and from lecture 5 we had momentum equation:

$$\rho\left[\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}\right] = \underline{j} \times \underline{B} - \nabla P + F \Rightarrow \left[\rho u^2 + P\right] = 0$$

→ integrating and taking both pressure components:
we get the conservation of the momentum flux normal to the shock:

$$\left[\rho u_n^2 + P + \frac{B_t^2}{2\mu_0}\right] = 0$$

• $\perp$ to the shock normal (transverse)

→ momentum conservation across the shock:

$$\rho_u u_{u_t} \underline{u}_{u_t} - \rho_d u_{d_t} \underline{u}_{d_t} = m \cdot \underline{a}_t$$

→ no pressure differential in the transverse direction (thermal, kinetic)

→ no change in B-pressure : $B_n = \text{const.}$

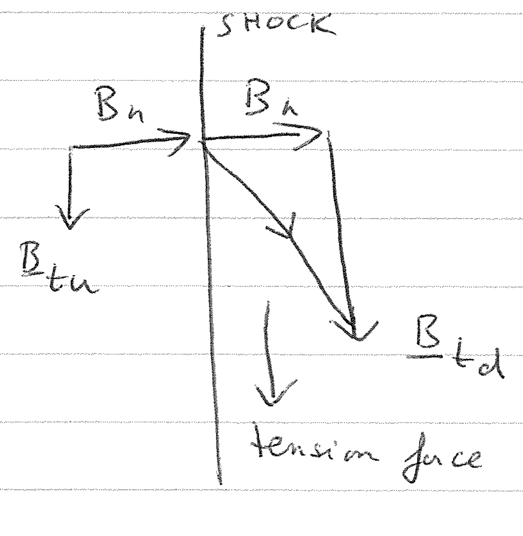
→ but must account for magnetic tension force:

$$\frac{1}{\mu_0}(\underline{B} \cdot \nabla)\underline{B} \Rightarrow \frac{B_n B_{t_u} - B_n B_{td}}{\mu_0}$$

→ plug into the momentum equation in the form of:

$$[\rho u^2 + P] = 0$$

$$\Rightarrow \left[\rho u_t^2 + \frac{B_n B_t}{\mu_0}\right] = 0$$



→ <u>energy conservation with B-field</u>

– include kinetic, internal energy, work done (⇒ enthalpy) and the EM radiation (Poyning vector).

⇒ kinetic energy: $E_K = \frac{1}{2}\rho_u u_{n_u} u_{n_u}^2 = \frac{1}{2}\rho_d u_{n_d} u_{n_d}^2$

→ internal energy: $\epsilon = \frac{\alpha}{2} k_B T$ and for ideal gas: $PV = n k_B T$

$$\Rightarrow \frac{P}{\rho} = k_B T \Rightarrow \epsilon = \frac{\alpha}{2}\cdot\frac{P}{\rho} = \frac{1}{\gamma-1}\cdot\frac{P}{\rho}$$

→ work done: $W = P\,dV = \left[\frac{P}{\rho}\right]$

→ enthalpy: $\epsilon + p\,dV = \epsilon + \frac{P}{\rho} = \frac{\gamma}{\gamma-1}\cdot\frac{P}{\rho}$ (remember lecture 8)

monatomic ideal gas: $\gamma = \frac{c_P}{c_V} = \frac{\alpha+2}{\alpha} = \frac{5}{3}$ (adiabatic)

$$\Rightarrow \frac{\gamma-1+1}{\gamma-1} = \frac{\gamma}{\gamma-1}$$

$$\Rightarrow H = \epsilon + \frac{P}{\rho} = \frac{1}{\gamma-1}\cdot\frac{P}{\rho} + \frac{P}{\rho} = \frac{\gamma}{\gamma-1}\cdot\frac{P}{\rho} \;//$$

→ the Poynting flux: $\underline{S} = \frac{\underline{E}\times\underline{B}}{\mu_0}$

and since $\underline{E} = \underline{u}\times\underline{B} \Rightarrow \underline{S} = \frac{(\underline{u}\times\underline{B})\times\underline{B}}{\mu_0}$

using the vector identity: $\underline{A}\times\underline{B}\times\underline{B} = \underline{A}B^2 - (\underline{A}\cdot\underline{B})\underline{B}$

$$\Rightarrow \underline{S} = \frac{\underline{u}B^2}{\mu_0} - \frac{(\underline{u}\cdot\underline{B})\cdot\underline{B}}{\mu_0}$$

$\Rightarrow$ thus the complete expression for the energy conservation across the shock front including the kinetic, internal and electromagnetic energy flux is:

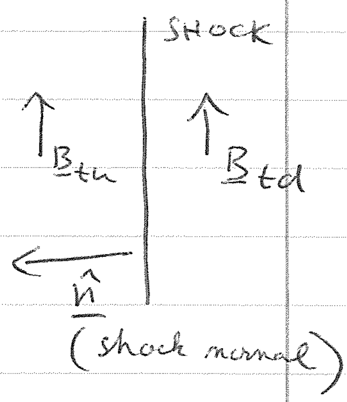
$$\left[\underbrace{\rho u_n}_{\text{mass flux}}\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\cdot\frac{P}{\rho}\right) + \underbrace{u_n\frac{B^2}{\mu_0}}_{\text{mag. pressure}} - \underline{u}\cdot\underline{B}\,\frac{B_n}{\mu_0}\right] = 0$$

remember the energy equation (MHD):

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon\right) = -\nabla\left[\rho\underline{u}\left(\epsilon + \frac{u^2}{2}\right) + \underbrace{P\underline{u}}_{\text{mag. pressure}}\right]$$

$\Rightarrow$ solution for perpendicular shocks:

$\underline{B} \perp \underline{n}$ and $\underline{u} \parallel \underline{n}$



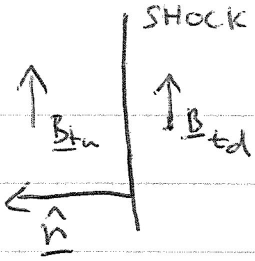
$\rightarrow$ magnetic field lines compressed by the same factor as density

$$[B_n] = 0,\; B_{nu} = 0 \;\Rightarrow B_{nd} = 0$$

$\rightarrow$ <u>mass conservation</u>: $\rho_u u_{nu} = \rho_d u_{nd}$

$$\Rightarrow r = \frac{\rho_d}{\rho_u} = \frac{u_{nu}}{u_{nd}}$$

for transverse shock then: $\left[\rho u_n u_t - \overbrace{\frac{B_n}{\mu_0}}^{0}\cdot B_t\right] = 0$



Solutions for $\perp$ shocks $(\underline{B} \perp \hat{\underline{n}})$ $(\underline{u} \parallel \hat{\underline{n}})$

→ **momentum conservation:**

$$\rho_u u_u^2 + P_u + \frac{B_u^2}{2\mu_0} = \rho_d u_d^2 + P_d + \frac{B_d^2}{2\mu_0}$$

and $\rho_d = r \cdot \rho_u$, $u_d = \frac{u_u}{r}$, $B_d = r \cdot B_u$

as: $r = \frac{\rho_d}{\rho_u} = \frac{u_u}{u_d}$

→ substitute:

$$\rho_u u_u^2 + P_u + \frac{B_u^2}{2\mu_0} = \rho_u \cdot \cancel{r} \cdot \left(\frac{u_u}{r^{\cancel{2}}}\right)^2 + \frac{r^2 B_u^2}{2\mu_0} + P_d$$

$$\Rightarrow \rho_u u_u^2 \left(1 - \frac{1}{r}\right) + \left(P_u - P_d\right) + \frac{B_u^2}{2\mu_0}\left(1 - r^2\right) = 0 \;//$$

→ **energy conservation:** $\theta_{Bn} = 90° \Rightarrow$ both $B_{n_d} = 0$, $B_{n_u} = 0 \;//$ (no $\parallel$ B-field component)

$$\rho_u u_u \left(\frac{1}{2} u_u^2 + \frac{\gamma}{\gamma-1} \cdot \frac{P_u}{\rho_u}\right) + u_u \frac{B_u^2}{\mu_0} - \cancel{u \cdot B \frac{B_n}{\mu_0}}$$

$$= \rho_d u_d \left(\frac{1}{2} u_d^2 + \frac{\gamma}{\gamma-1} \cdot \frac{P_d}{\rho_d}\right) + u_d \frac{B_d^2}{\mu_0} - \cancel{0}$$

$$= \rho_u \cdot \cancel{r} \cdot \frac{u_u}{\cancel{r}} \left(\frac{1}{2} \cdot \frac{u_u^2}{r^2} + \frac{\gamma}{\gamma-1} \cdot \frac{P_d}{\rho_u \cdot r}\right) + \frac{u_u}{\cancel{r}} \frac{r^{\cancel{2}} B_u^2}{2\mu_0}$$

$$\Rightarrow \frac{1}{2} \rho_u u_u^{\cancel{3}\,2} + \left(\frac{\gamma}{\gamma-1}\right) \cancel{u_u} \cdot P_u + \cancel{u_u} \frac{B_u^2}{\mu_0}$$

$$= \frac{1}{2} \rho_u \frac{u_u^{\cancel{3}\,2}}{r^2} + \left(\frac{\gamma}{\gamma-1}\right) \cancel{u_u} \cdot \frac{P_d}{r} + \frac{\cancel{u_u} \cdot r B_u^2}{\mu_0}$$

$$\Rightarrow \frac{1}{2}\rho_u u_u^2\left(1-\frac{1}{r^2}\right)+\frac{\gamma}{\gamma-1}\left(P_u-\frac{P_d}{r}\right)+\frac{B_u^2}{\mu_0}(1-r)=0$$

$\rightarrow$ <u>eliminate P_d by substitution:</u>

$$P_d=\rho_u u_u^2\left(1-\frac{1}{r}\right)+P_u+\frac{B_u^2}{2\mu_0}\left(1-r^2\right)$$

$$\Rightarrow \frac{1}{2}\rho_u u_u^2\left(1-\frac{1}{r^2}\right)+\frac{\gamma}{\gamma-1}\left(P_u-\frac{\rho_u u_u^2}{r}\left(1-\frac{1}{r}\right)-\frac{P_u}{r}-\frac{B_u^2}{2\mu_0}\frac{(1-r^2)}{r}\right)$$
$$+\frac{B_u^2}{\mu_0}(1-r)=0 \qquad /\times r$$

$$\frac{1}{2}\rho_u u_u^2\left(r-\frac{1}{r}\right)+\frac{\gamma}{\gamma-1}\left(rP_u-\rho_u u_u^2\left(1-\frac{1}{r}\right)-P_u-\frac{B_u^2}{2\mu_0}(1-r^2)\right)+r\frac{B_u^2}{\mu_0}(1-r)=0$$
$$/\times \rho_u u_u^2$$

$$\frac{1}{2}\left(r-\frac{1}{r}\right)+\frac{\gamma}{(\gamma-1)}\left(r\frac{P_u}{\rho_u u_u^2}-\frac{P_u}{u_u^2\rho_u}-\left(1-\frac{1}{r}\right)-\frac{B_u^2}{2\mu_0}\frac{(1-r^2)}{\rho_u u_u^2}\right)+\frac{rB_u^2}{\rho_u u_u^2\mu_0}(1-r)=0$$

$$/\times 2(\gamma-1)$$

$$(\gamma-1)\left(r-\frac{1}{r}\right)+2\gamma\frac{P_u}{u_u^2\rho_u}(r-1)-2\gamma\left(1-\frac{1}{r}\right)-\cancel{\frac{2}{2}}\frac{\gamma B_u^2}{u_u^2\mu_0\rho_u}(1-r^2)$$
$$+\frac{2rB_u^2}{u_u^2\mu_0\rho_u}(1-r)(\gamma-1)=0 \qquad /\times r$$

$$(\gamma-1)\underbrace{(r^2-1)}_{=(r-1)(r+1)}+\frac{2\gamma rP_u}{u_u^2\rho_u}\cdot(r-1)-2\gamma(r-1)+\frac{\gamma B_u^2}{\mu_0\rho_u u_u^2}r\underbrace{(r^2-1)}_{(r-1)(r+1)}+\frac{2r^2B_u^2}{u_u^2\mu_0\rho_u}(1-r)(\gamma-1)=0$$

$$\Rightarrow (r-1)\left[(\gamma-1)(r+1)+r\cdot\frac{2\gamma P_u}{u_u^2\rho_u}-2\gamma+\frac{\gamma B_u^2}{u_u^2\mu_0\rho_u}\cdot r(r+1)-r^2\frac{2B_u^2}{u_u^2\mu_0\rho_u}(\gamma-1)\right]=0$$

$$\Rightarrow (r-1)\left[r^2\frac{\gamma B_u^2}{u_u^2\mu_0\rho_u}-r^2\frac{2\gamma B_u^2}{u_u^2\mu_0\rho_u}+r^2\frac{2B_u^2}{u_u^2\mu_0\rho_u}+r\frac{2\gamma P_u}{u_u^2\rho_u}+r\frac{\gamma B_u^2}{u_u^2\mu_0\rho_u}+r(\gamma-1)\right.$$
$$\left.+\gamma-1-2\gamma\right]=0$$

$$\Rightarrow (r-1)\left[r^2 \frac{(2-\gamma) B_u^2}{u_u^2 \mu_0 \rho_u} + r\left(\frac{\gamma B_u^2}{u_u^2 \mu_0 \rho_u} + \frac{2\gamma P_u}{u_u^2 \rho_u} + (\gamma - 1)\right) - (\gamma+1)\right] = 0$$

//

$\Rightarrow$ for strong shocks at high velocities the Mach number $M \rightarrow \infty$ and $u_u \rightarrow \infty$, thus the approximation can be made: $\frac{1}{u_u^2}$ terms $\rightarrow 0$

$$\Rightarrow (r-1)\left[r^2 \cdot 0 + r(0+0+\gamma-1) - (\gamma+1)\right] = 0$$

$$\Rightarrow (r-1)\left[r(\gamma-1) - (\gamma+1)\right] = 0$$

$\rightarrow$ 2 solutions: ① $r = 1 \Rightarrow$ not a shock (not a solution)

② $r(\gamma-1) - (\gamma+1) = 0$

$\Rightarrow r = \frac{\gamma+1}{\gamma-1}$ and for ideal gas $\gamma = 5/3$

$$\Rightarrow r = \frac{5/3 + 3/3}{5/3 - 3/3} = 4 \,//$$

i.e. the maximum compression ratio

$$r = \frac{\rho_d}{\rho_u} = \frac{u_{nu}}{u_{nd}} = 4 \,//$$

Solution for ∥ shocks ($\underline{B} \parallel \hat{n}$)

→ i.e. no ⊥ B-field component, $\underline{B}_t = 0$ $(\theta_{Bn} = 90°)$

⇒ B-field does not change across the shock

⇒ particles can move along B-field lines upstream into the shock

⇒ simple hydrodynamic shock

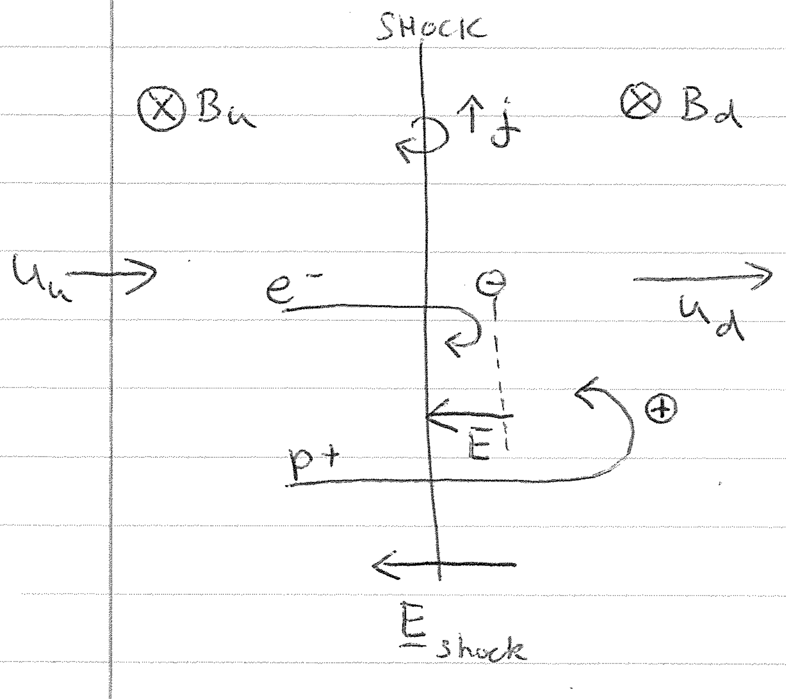
Collisionless shocks

→ bulk behaviour by interactions of charged particles via EM fields

⇒ particles can travel against the bulk flow

⇒ non-linear interactions upstream

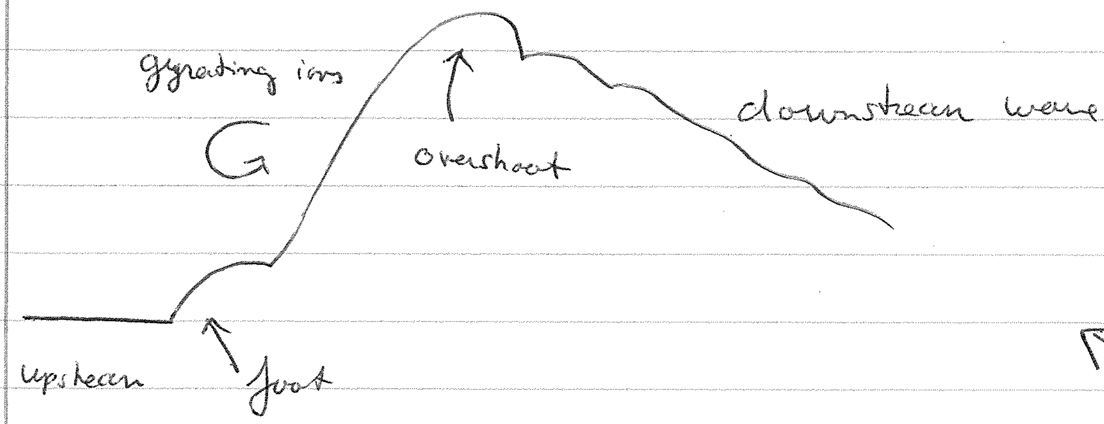
→ for ⊥ collisionless shock:



- e^- ⇒ small gyro-radius
- change in B_t ⇒ current layer J

$$\nabla \times \underline{B} = \mu_0 j$$

- ions / electrons decelerated by E-field
- E-field induced due to different radius of gyration between electrons and ions / protons (protons / ions gyrate further into the shock)



→ shock variability: particle reflection process unstable ⇒ leads to ripples in shock front (shock reformation) and particle acceleration

→ particles escape upstream if θ_{Bn} not too large ⇒ <u>foreshock</u> created

– unstable generation of waves

• quasi-⊥ shock ($\theta_{Bn} > 45°$) – stable up-down stream transition
– steep rise in B-field ⇒ "ramp"

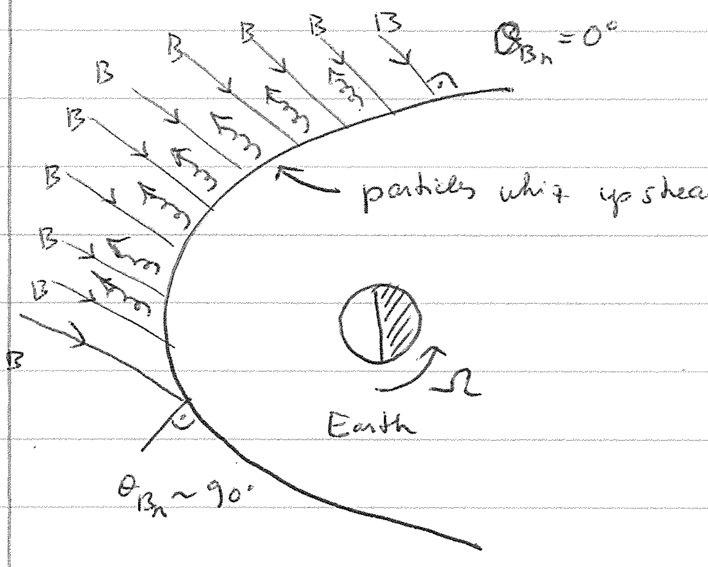
• quasi-∥ shock ($\theta < 45°$) – transition from up-stream to downstream is broad
(1-2 Earth radii for bow shock on Earth)
⇒ turbulent region
⇒ unsteady shock

The Bow shock (example of a collisionless shock)

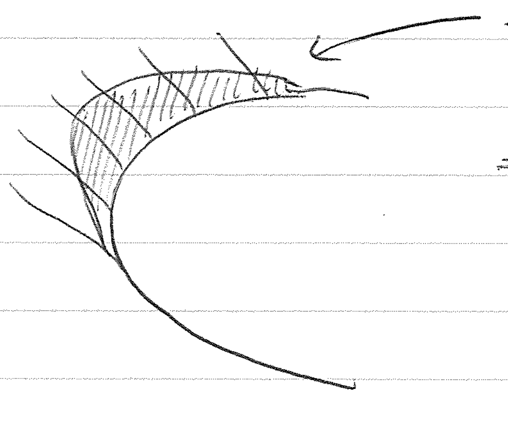
→ solar wind is super-Alfvénic
⇒ bow shock generated

⇒ generates magnetopause : boundary between the solar wind and Earth's magnetosphere
- Solar wind pressure (components : ram, magnetic and plasma/thermal) compresses the Earth's B-field into cavity
- solar wind deflected around magnetosphere

→ magneto sheath = shocked, deflected solar wind



- Mach number varies with position
⇒ range of shock conditions



fore shock
near quasi-// shock
⇒ particles move upstream

Radiative shocks

→ radiative relaxation layer, ahead of shock material is preheated generating a new set of condions:

$$\rho_3, u_3, P_3, T_3 \quad \text{further downstream}$$

→ within the radiative relaxation layer the temperature drops and the gas is squeezed to higher density

→ to get the jump conditions for a radiative shock, we include the cooling term $L(\rho, T)$

→ for differential notation, the Euler equations are denoted as:

continuity: $\frac{d}{dx}(\rho u) = 0$

momentum: $u \frac{du}{dx} = -\frac{1}{\rho}\frac{dP}{dx}$

energy: $\frac{d}{dx}(\rho \epsilon u) = \rho u \frac{d\epsilon}{dx} = -P\frac{du}{dx} - \rho L$

and for ideal gas: $\epsilon = \frac{1}{\gamma-1} \cdot \frac{P}{\rho}$

$$\Rightarrow \frac{d\epsilon}{dx} = \frac{1}{\gamma-1}\left(\frac{1}{\rho}\frac{dP}{dx} - \frac{P}{\rho^2}\frac{d\rho}{dx}\right)$$

and we have $\frac{1}{\rho}\frac{d\rho}{dx} = -\frac{1}{u}\frac{du}{dx}$

since $\rho \propto \frac{1}{u}$

$$\Rightarrow \frac{u}{\gamma-1}\left(\frac{dP}{dx} + \frac{P}{u}\cdot\frac{du}{dx}\right) + P\frac{du}{dx} = -\rho L$$

substituting: $\frac{dP}{dx} = -\rho u \frac{du}{dx}$

$$\Rightarrow -\frac{\rho u^2}{\gamma-1}\cdot\frac{du}{dx} + \frac{\gamma}{\gamma-1}P\frac{du}{dx} = -\rho L$$

as $\frac{1}{\gamma-1} + \frac{\gamma-1}{\gamma-1} = \frac{\gamma}{\gamma-1}$

and speed of sound: $u_{ca}^2 = \frac{\gamma P}{\rho}$ (adiabatic)

$$\Rightarrow \text{thus: } \frac{u_{ca}^2 - u^2}{\gamma-1}\frac{du}{dx} = -L$$

Since post-shock flow is sub-sonic ($u < u_{ca}$) and $L > 0$ we conclude $\frac{du}{dx} < 0$, so mass conservation implies $\frac{d\rho}{dx} > 0$

$\Rightarrow$ mass and momentum conservation apply as before:

$$\rho_3 u_3 = \rho_u u_u = \rho_d u_d$$

$$\rho_3 u_3^2 + P_3 = \rho_u u_u^2 + P_u = \rho_d u_d^2 + P_d$$

as u drops in the radiative relaxation layer

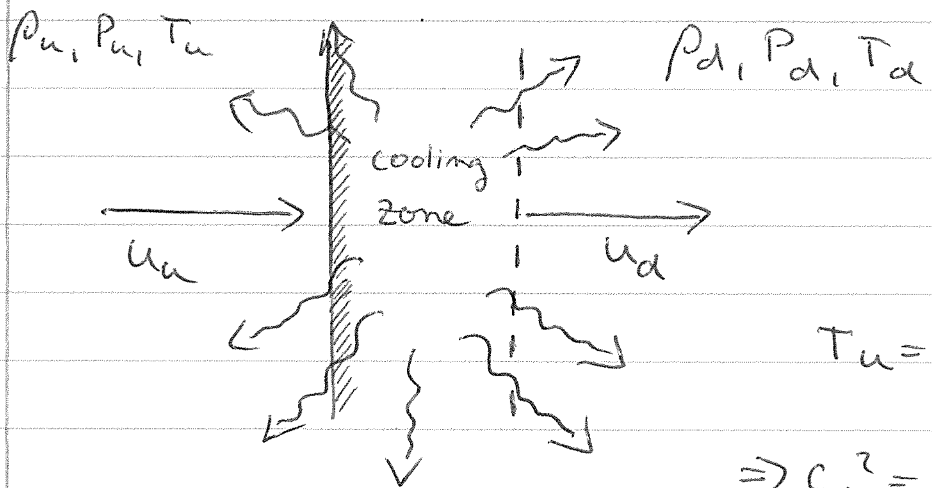
$$\rho u^2 = (\rho u) u \text{ drops}$$

and thus pressure must rise to compensate

→ i.e pressure does not go down, thus the drop in temperature must come with a proportional rise in density to maintain the pressure

→ with efficient post shock cooling, we can get very high compression ratios
(remember the max $r = 4$ for non-radiative shocks)

→ Special case : isothermal shock
(when have long cooling times or for balanced cooling/heating)



$T_u = T_d$

$\Rightarrow c_u^2 = c_d^2 = c_0^2$ (sound speed)

must be isothermal

→ obtain new jump conditions

• <u>Isothermal shocks</u>: $T_u = T_d$

and $c_{iso} = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{k_B T}{m}} \Rightarrow c_u^2 = c_d^2 = c_0^2$

(isothermal sound speed)

→ Speed of sound constant across the shock boundary:

$$c_0^2 = \frac{P_u}{\rho_u} = \frac{P_d}{\rho_d}$$

→ RH1 : $\rho_u u_u = \rho_d u_d$

→ R-H2 : $\rho_u u_u^2 + P_u = \rho_d u_d^2 + P_d$

substitute for P into RH-2

$\Rightarrow \rho_u u_u^2 + \rho_u c_0^2 = \rho_d u_d^2 + \rho_d c_0^2 \qquad / \div \rho_u$

and from RH1 we have: $\frac{\rho_d}{\rho_u} = \frac{u_u}{u_d}$

$\Rightarrow u_u^2 + c_0^2 = \frac{u_u}{u_d} u_d^2 + \frac{u_u}{u_d} c_0^2 \qquad / \times u_d$

$\Rightarrow u_d u_u^2 + u_d c_0^2 + u_u u_d^2 + u_u c_0^2$

$\Rightarrow (u_d - u_u) c_0^2 = u_u u_d (u_d - u_u)$

→ for $u_d \neq u_u \Rightarrow c_0^2 = u_u u_d$

$\therefore u_d = \frac{c_0^2}{u_u}$ //

$\Rightarrow$ compression ratio: $r = \frac{\rho_d}{\rho_u} = \frac{u_u}{u_d} = \frac{u_u^2}{c_0^2} \equiv M_T^2$ (Mach number)

can be arbitrarily high!