Plasma Physics

TU Dresden

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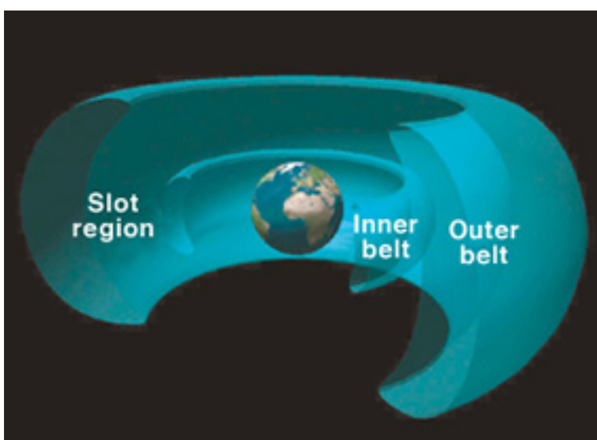
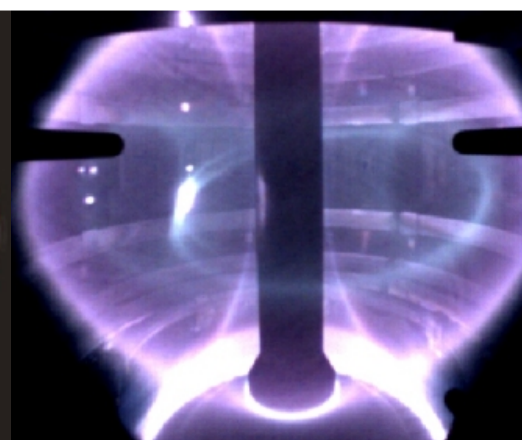
Lecture 2: Single particle motion






Plasma Physics: lecture 2

- Single particle motion
- Larmor radius
- Guiding centre drift
- Confinement with magnetic mirrors
- Adiabatic invariants

Charged particle motion in plasma

- Plasmas made out of mobile positive and negative charges
- The motion of each charged particle is determined by the electric and magnetic fields it experiences
- Those fields can be external (e.g. Earth's magnetic field and ionosphere) and also generated by all the individual moving particles within the plasmas (very messy situation!) → start with simple approximations
- Motion of a charged particle governed by Lorentz force:

$$\boxed{\mathbf{F} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q(\mathbf{E} + [\mathbf{v} \times \mathbf{B}])}$$

Charged particle in uniform B-field

- Reduce the Lorentz equation to:

$$m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q(\mathbf{v}\times\mathbf{B})$$

- Start with the simple example of a particle moving in a constant B-field with no curvature in z-direction:

$$m\dot{\mathbf{v}} = q(\mathbf{v}\times\mathbf{B}) \quad \rightarrow \quad \dot{\mathbf{v}} = \boldsymbol{\omega}_c \times \mathbf{v}$$

- Define the cyclotron (also gyro) frequency:

$$\boxed{\vec{\omega}_c = \frac{-q\mathbf{B}}{m}} \quad \rightarrow \quad \left(\vec{\omega}_{ce} = \frac{e\mathbf{B}}{m},\ \ \vec{\omega}_{ci} = \frac{-Ze\mathbf{B}}{M}\right)$$

electrons *ions*

Charged particle in uniform B-field

- The energy of the particle is not altered as the magnetic field does not do any work on it.
- Note that the force on the particle due the magnetic field is perpendicular to the direction of motion of the particle.
- The motion of particles in the plane perpendicular to the magnetic field is altered → circular.
- The velocity of the particle prior applying the B-field does not change in the direction of the field (z-axis):

$$z = z_0 + v_z t$$

→ **overall helical motion**

Charged particle in uniform B-field

- Changes of velocity only in the perpendicular directions (along x and y axes).
- Consider the speed of the particle in the plane perpendicular to the field:

$$v_\perp = \sqrt{v^2 - v_z^2}$$

$$\mathbf{v}_\perp = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

- Substitute in the gyro frequency into $\dot{\mathbf{v}} = \omega_c \times \mathbf{v}$:

$$\dot{v_x} = -\omega_c v_y \quad \text{and} \quad \dot{v_y} = \omega_c v_x$$

Charged particle in uniform B-field

- Differentiate with respect to time:

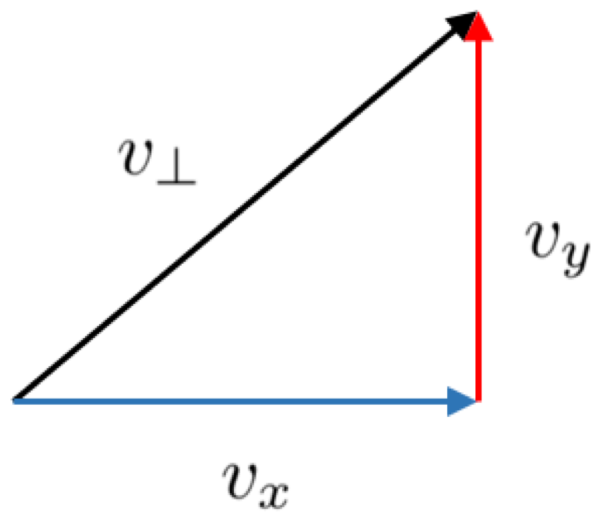
$$\ddot{v}_x = -\omega_c^2 v_x \qquad \text{and} \qquad \ddot{v}_y = -\omega_c^2 v_y$$

- Simple harmonic motion:

$$v_x = v_\perp \cos(\omega_c t + \phi)$$
$$v_y = v_\perp \sin(\omega_c t + \phi)$$

- With v_x and v_y being 2π out of phase.

$$v_\perp = \sqrt{v_x^2 + v_y^2}$$



Charged particle in uniform B-field

- Write out cartesian coordinates:

$$x = x_0 + \frac{v_\perp}{\omega_c} \sin(\omega_c t + \phi)$$
$$y = y_0 - \frac{v_\perp}{\omega_c} \cos(\omega_c t + \phi)$$
$$z = z_0 + v_z t$$

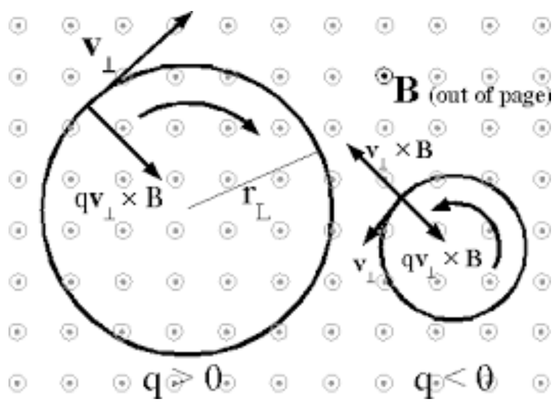
- Notice that:

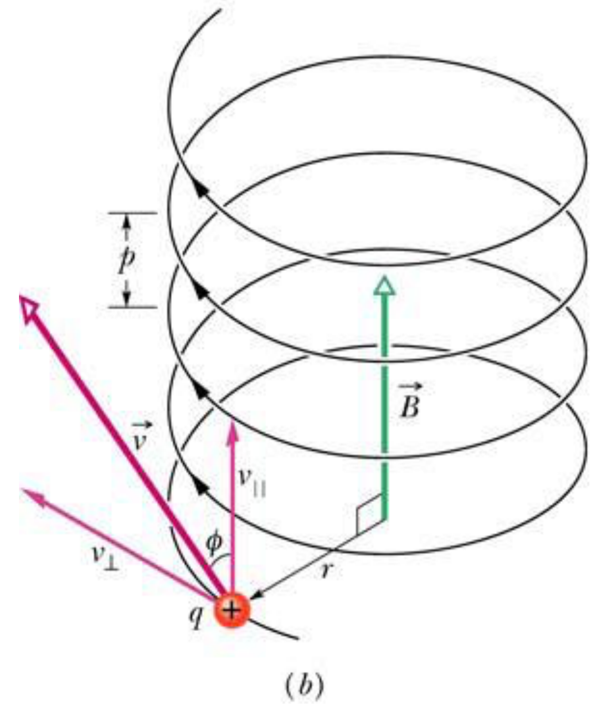
$$(x - x_0)^2 + (y - y_0)^2 = \left(\frac{v_\perp}{\omega_c}\right)^2$$

Charged particle in uniform B-field

- Particle moves along a helix
- Circular component of the motion centered around x_0 and y_0 coordinates → **guiding centre**
- Radius of this motion is the **Larmor radius**:

$$\boxed{r_L = \frac{v_\perp}{|\omega_c|}}$$





The Larmor radius

- In plasmas with finite temperature, by conservation of energy:

degrees of freedom

$n_d = 1$ $n_d = 2$

$$E = \frac{1}{2} m v_{tot}^2 = \frac{n_d}{2} k_B T \quad \text{where} \quad v_{tot}^2 = v_{\parallel}^2 + v_{\perp}^2$$

$$\rightarrow \frac{1}{2} m v_{\perp}^2 = k_B T \quad \rightarrow \quad v_{\perp} = \sqrt{\frac{2 k_B T}{m}}$$

- Thus Larmor radius: $r_L = \frac{v_{\perp}}{\omega_c} = \frac{\sqrt{2 k_B m T}}{qB}$

Guiding centre drift

- Adding another force acting on the particle.
- Consider the **average motion** of the particle, which is simply a movement along the magnetic field with v_z.
- Add another general constant force:
  - In z-direction: the motion in x-y plane is unaltered, but there is additional acceleration along z-direction
  - In x-y plane: more complicated cases. Assume that the force acts in this direction and only velocity in this plane is considered.

General constant force drift

- New equation of motion with $\mathbf{F}$ in x-y plane :

$$\dot{\mathbf{v}} = (\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{v}) + \frac{\mathbf{F}}{m}$$

- And differentiate with respect to time:

$$\ddot{\mathbf{v}} = (\boldsymbol{\omega}_{\mathrm{c}} \times \dot{\mathbf{v}})$$

- And substitute:

$$\begin{aligned}\ddot{\mathbf{v}} &= \boldsymbol{\omega}_{\mathrm{c}} \times \left((\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{v}) + \frac{\mathbf{F}}{m} \right) \\ &= \boldsymbol{\omega}_{\mathrm{c}}(\boldsymbol{\omega}_{\mathrm{c}} \cdot \mathbf{v}) - \mathbf{v}\omega_{\mathrm{c}}^2 + \frac{\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{F}}{m} \\ &= -\mathbf{v}\omega_{\mathrm{c}}^2 + \frac{\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{F}}{m}\end{aligned}$$

General constant force drift

- The motion will be circular, as before, with an additional drift at a constant velocity given by:

$$\mathbf{v}_{\mathrm{d}} = \frac{\boldsymbol{\omega}_{\mathrm{c}} \times \mathbf{F}}{m\omega_{\mathrm{c}}^2} = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}$$

- General expression for any force (e.g. gravity)
- For constant **electric field (**perpendicular to B), we get:

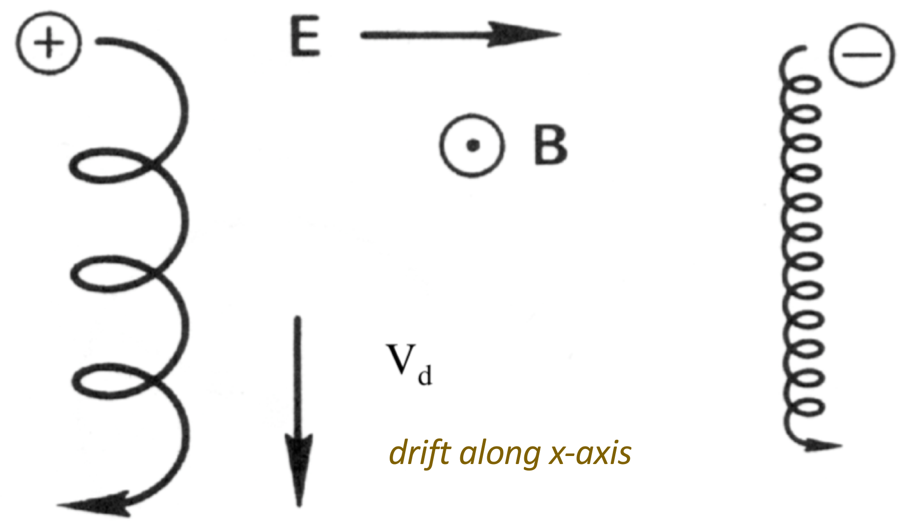
$$\mathbf{v}_{\mathrm{d}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

Note: no charge or mass dependance!

Guiding centre drift in constant E field

- Electrons and ions gyrate at opposite directions with ions having bigger Larmor radius.
- Ion gains energy from electric field (y-direction) increasing $v_\perp$ and with it r_L, but looses energy in the next half cycle and r_L decreases again
- Electron does the same, but in the opposite direction
- Overall drift in x-direction for **both electrons and ions** as these effects cancel out

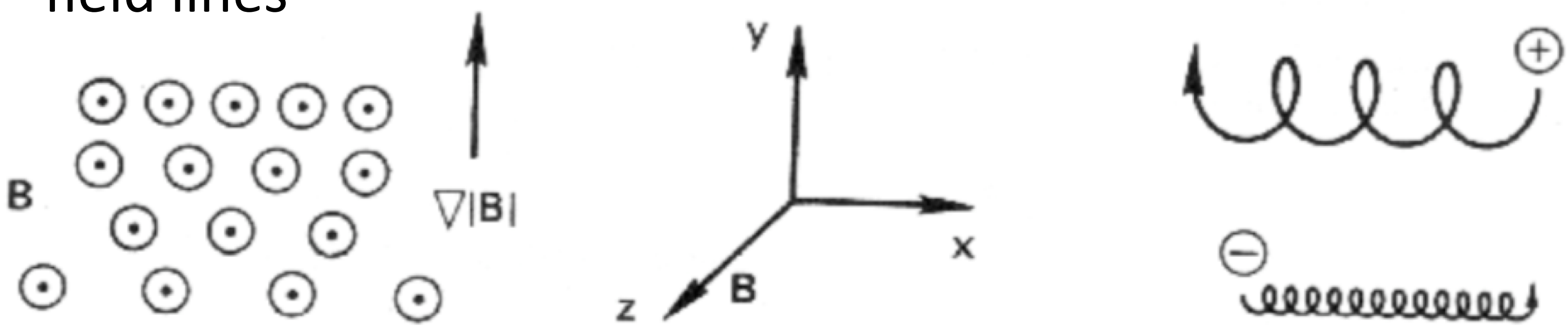
E field along y-axis



Note: z-axis out of page

Gradient drift in non-uniform B field

- Also called grad-B drift
- Consider straight B-field, but with gradient in density of the field lines



Note: z-axis out of page

- The Larmor radius is inversely proportional to the B field
- There is an effective force in the y-direction drift in x-direction
- Electrons and ions drift in opposite directions as the gyration is opposite

Gradient drift in non-uniform B field

- For uniform B-field: $\dot{v}_y = \omega_c v_x$

$$\rightarrow \quad \begin{aligned} F_y &= -q v_x B_z \\ &= -q v_\perp (\cos \omega_c t) B_z \end{aligned}$$

- Assume B-field varies slowly in y-direction:

$$B_z = B_0 + y \left(\frac{\partial B_z}{\partial y} \right)$$

- Substitute:

$$F_y = -q v_\perp (\cos \omega_c t) \left[B_0 + y \left(\frac{\partial B_z}{\partial y} \right) \right]$$

Note: *Larmor radius small compared to the scale length of the gradient in the B-field*

Gradient drift in non-uniform B field

- Substitute for y-coordinate with $y_0 = 0$:

$$F_y = -qv_\perp(\cos\omega_c t)\left[B_0 - \frac{v_\perp}{\omega_c}(\cos\omega_c t)\left(\frac{\partial B_z}{\partial y}\right)\right]$$

Note: The first term averages to 0 and the second to ½.

- Averaged force:

$$\bar{F}_y = qv_\perp^2 \frac{1}{2\omega_c}\left(\frac{\partial B_z}{\partial y}\right)$$

- Substitute to find drift velocity:

$$\boxed{\mathbf{v}_\mathrm{d} = v_\perp^2 \frac{1}{2\omega_c}\frac{\nabla \mathbf{B} \times \mathbf{B}}{B^2}}$$

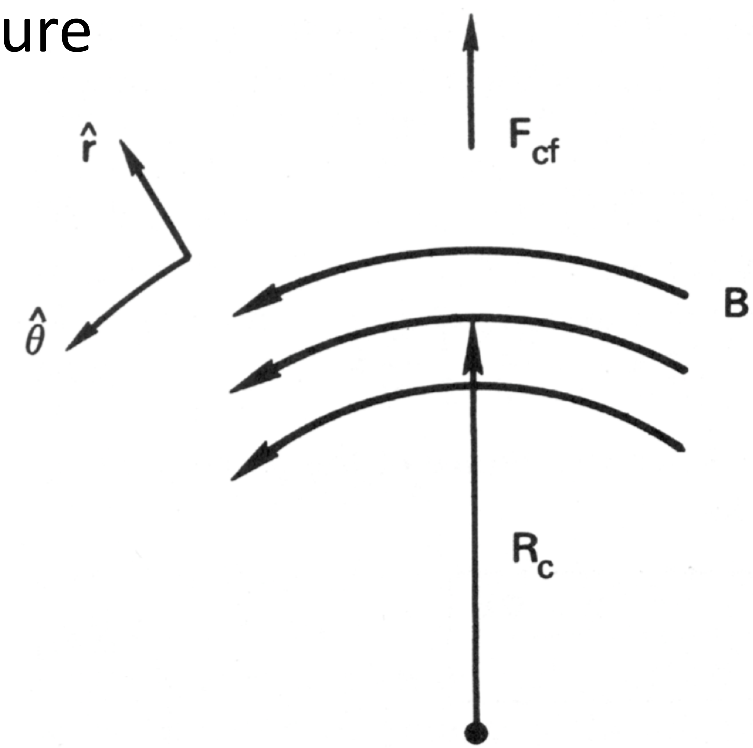
Curvature drift in non-uniform B field

- Consider circular B-field with radius of curvature R, particle experiences an effective centrifugal force:

$$\mathbf{F}_{\mathrm{cf}} = m v_{\parallel}^2 \frac{\mathbf{R}}{R^2}$$

- Thus drift velocity due to the curvature of the field is simply:

$$\boxed{\mathbf{v}_{\mathrm{d}} = \frac{m v_{\parallel}^2}{q B^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2}}$$



But that is not the full story …

Curvature drift in non-uniform B field

- There is also an associated grad-B drift that arises directly from the requirements set by the Maxwell's equations, i.e. curved field in vacuum cannot be uniform!
- B-field has to be inversely proportional to R:

$$|B| \propto \frac{1}{R} \rightarrow \quad \frac{\nabla |B|}{|B|} = -\frac{\mathbf{R}}{R^2}$$

- Additional grad-B drift: $$\mathbf{v}_{\nabla \mathbf{B}} = -v_\perp^2 \frac{|B|}{2\omega_c} \frac{\mathbf{R} \times \mathbf{B}}{R^2 B^2}$$

$$= \frac{m v_\perp^2}{2qB^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2}$$

- Full drift thus: $$\boxed{\mathbf{v}_{\mathrm{total}} = \frac{m}{qB^2} \frac{\mathbf{R} \times \mathbf{B}}{R^2} \left(v_\parallel^2 + \frac{v_\perp^2}{2} \right)}$$

Magnetic moment

- Magnetic moment defined as: $\mu = \text{area} \times \text{current}$

- Current : $I = \frac{dQ}{dt} = \frac{q}{T} = \frac{q}{2\pi/\omega_c} = \frac{q\omega_c}{2\pi}$

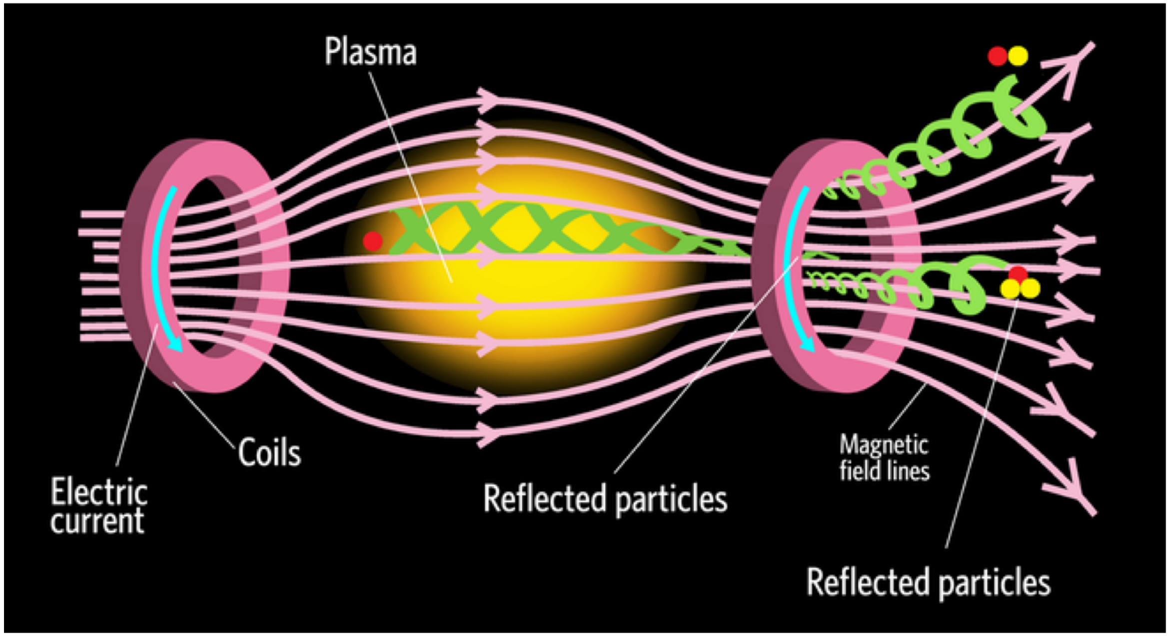
- Thus: $\mu = \pi r_L^2 \cdot \frac{q\omega_c}{2\pi} = \frac{v_\perp^2}{2\omega_c^2} \cdot q\omega_c = \frac{1}{2}\frac{mv_\perp^2}{B}$ as $\omega_c = \frac{qB}{m}$

- Magnetic moment: $$\boxed{\boldsymbol{\mu} = \frac{1}{2}mv_\perp^2 \frac{\mathbf{B}}{B^2}}$$

Principle of magnetic confinement

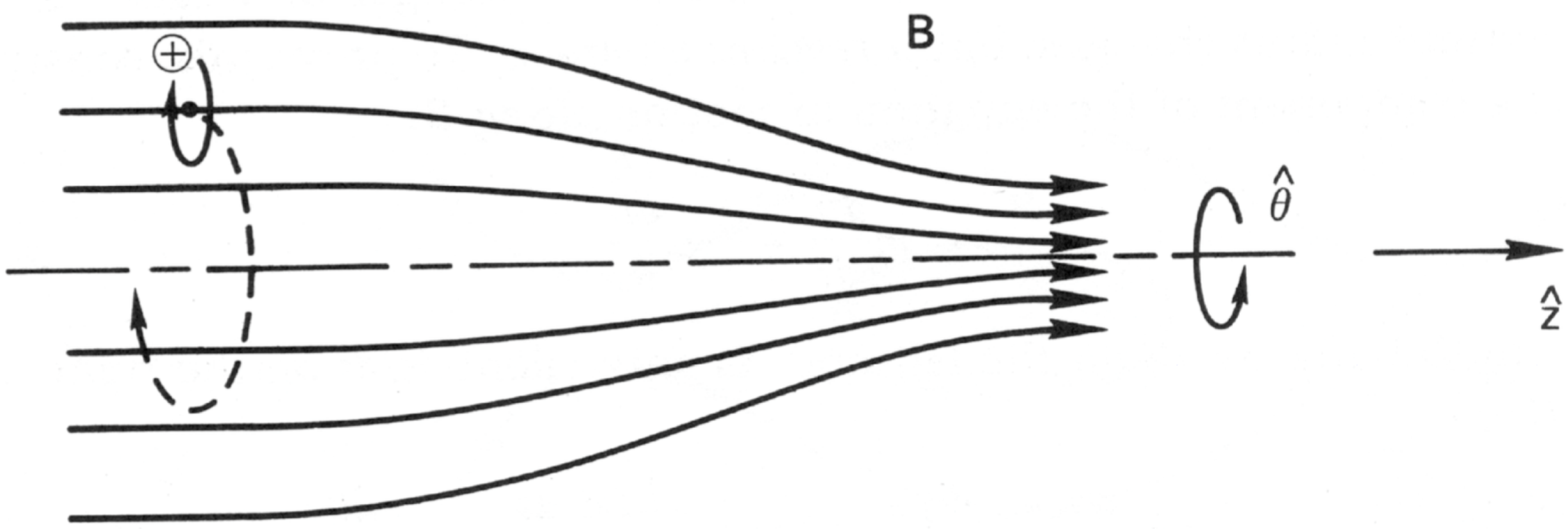
- Particles constrained to move along the B field lines
- Adding circular motion perpendicular to B to motion along B-field → **helical motion**
- Ions rotate in opposite direction to electrons
- Particle motion induces further B fields

→ **plasmas are diamagnetic**



Magnetic mirrors

- Non-uniform B-field with cylindrically symmetric field lines along z-axis (or r), no component in θ-direction
- $v \times \mathbf{B}_r$ force confines particles in the z-direction



Magnetic mirrors

- $\mathbf{B}_r$ field value must be approximated
- Starting from Maxwell's equations: $\nabla \cdot \mathbf{B} = 0$
- Thus: $$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0$$
- Assume that $\partial B_z / \partial z$ is roughly constant, and equal to its value on axis:

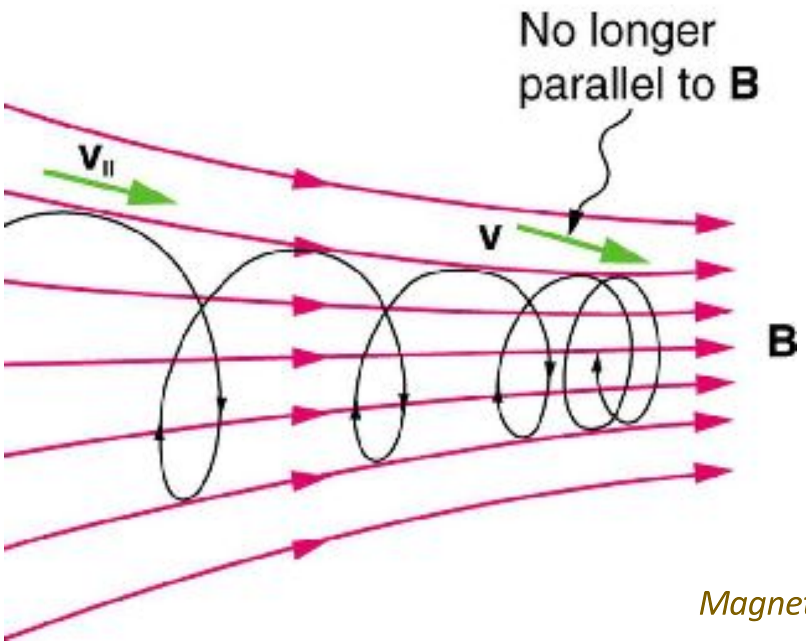
$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) = -\frac{\partial B_z}{\partial z}$$

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}\mathrm{d}r \approx -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

$$B_r = -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

Magnetic mirrors

- Consider particle moving along z-axis with Larmor radius r_L, motion perpendicular to B_r, thus the particle experiences $\mathbf{v}\times\mathbf{B}$ force in the z-direction:



$$F_z = qv_\perp\left(-\frac{1}{2}r_\mathrm{L}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}\right)$$

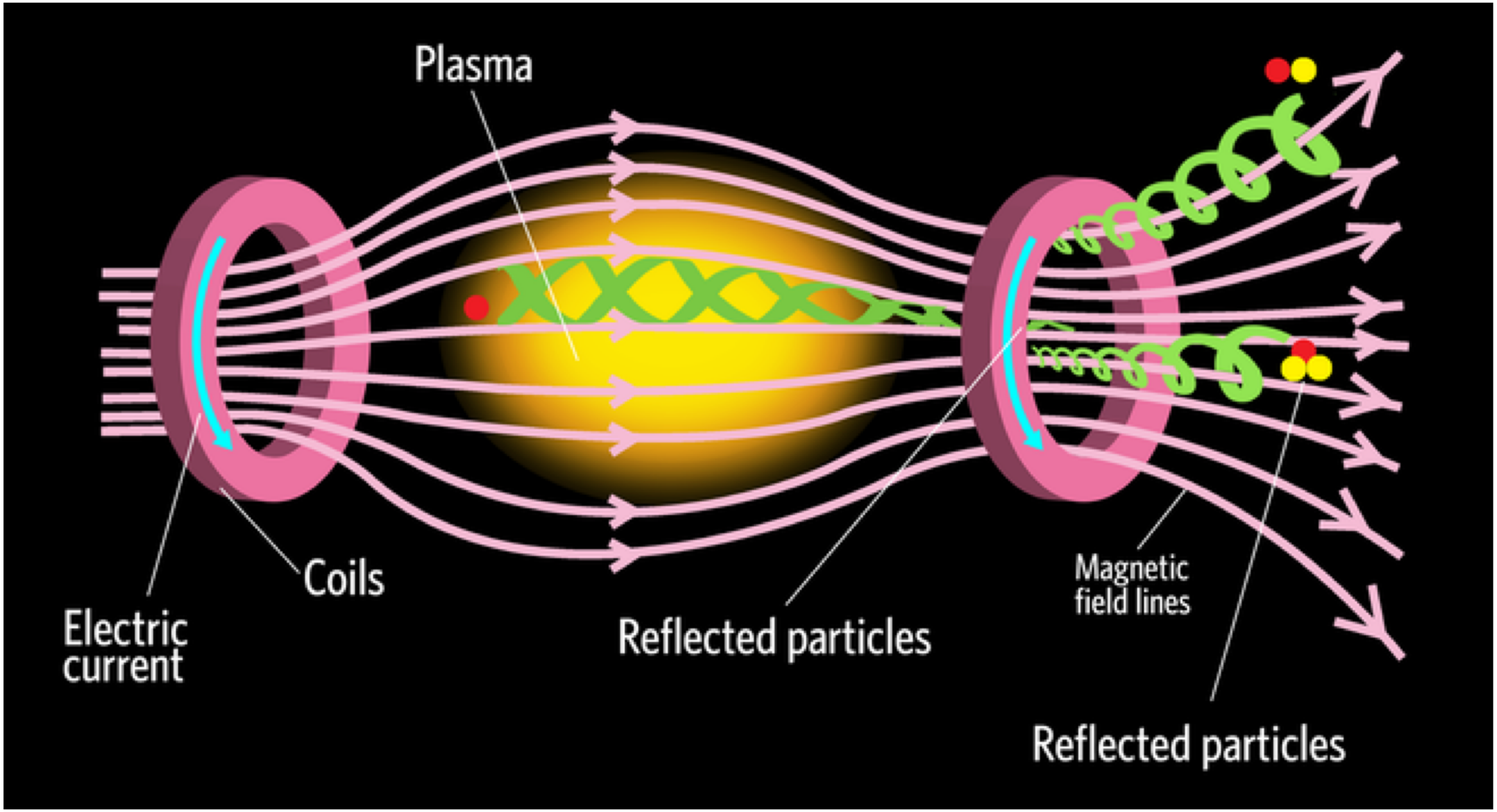
$$= -\frac{1}{2}\frac{mv_\perp^2}{B}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

$$= -\mu\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$

Magnetic moment

- The particle feels a confining force F_z

Magnetic mirrors



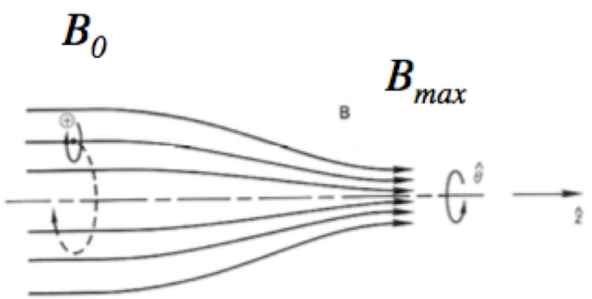
Magnetic mirrors

- Only particles with perpendicular velocity component will be trapped and they have: $$v_0^2 = v_{\perp 0}^2 + v_{\parallel 0}^2$$
- Conservation of the magnetic moment μ or angular momentum $v_\perp r_L = \text{const.}$:
$$v_\perp^2 = \left(\frac{B}{B_0}\right) v_{\perp 0}^2$$
- Conservation of energy:
$$v_\parallel^2 = v_0^2 - v_\perp^2 = v_0^2 \left(1 - \frac{B}{B_0} \frac{v_{\perp 0}^2}{v_0^2}\right)$$
- Particles reflected when $v_\parallel = 0$: $$B_{\text{ref}} = \frac{v_0^2}{v_{\perp 0}^2} B_0$$

Magnetic mirrors

- As **B** increases:
  - $v_\perp$ must increase to keep μ constant
  - $v_\parallel$ must decrease to keep the kinetic energy constant

- At field maximum B_m:
  - Particles with $v_\parallel^2(0) < v_\perp^2(0)\left(\frac{B_m}{B_0} - 1\right)$ are reflected.
  - Particles with $v_\parallel^2(0) > v_\perp^2(0)\left(\frac{B_m}{B_0} - 1\right)$ are lost.



- Trapped particles oscillate between two reflection points

Loss cone

- Particle velocity components:

$$\sin^2\theta = \frac{v_{\perp 0}^2}{v_0^2} = \frac{B_0}{B}$$

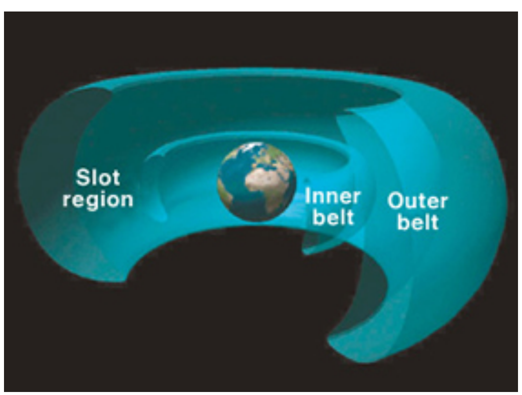
- Thus, particles with the pitch angle of the orbit θ smaller than θ_m will escape from the confinement:

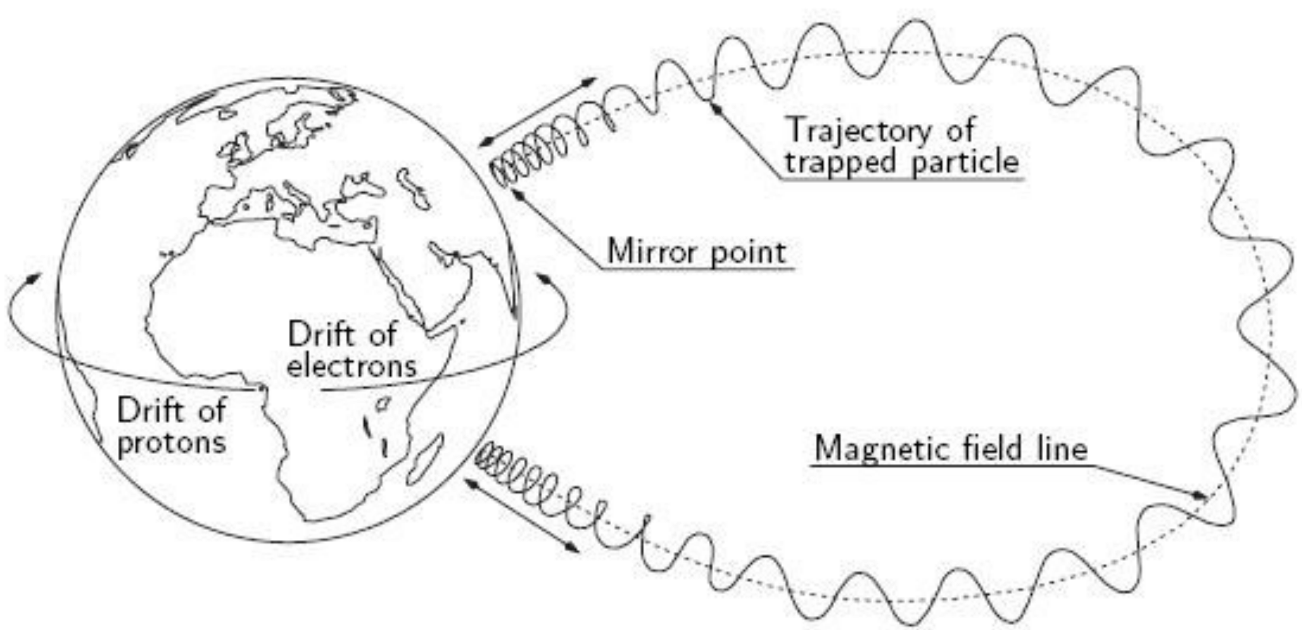
$$\sin^2\theta_{\mathrm{m}} = \frac{B_0}{B_{\mathrm{m}}}$$

- The angle can be changed by particle collisions, thus many particles can be lost → magnetic mirror does not provide good plasma confinement.

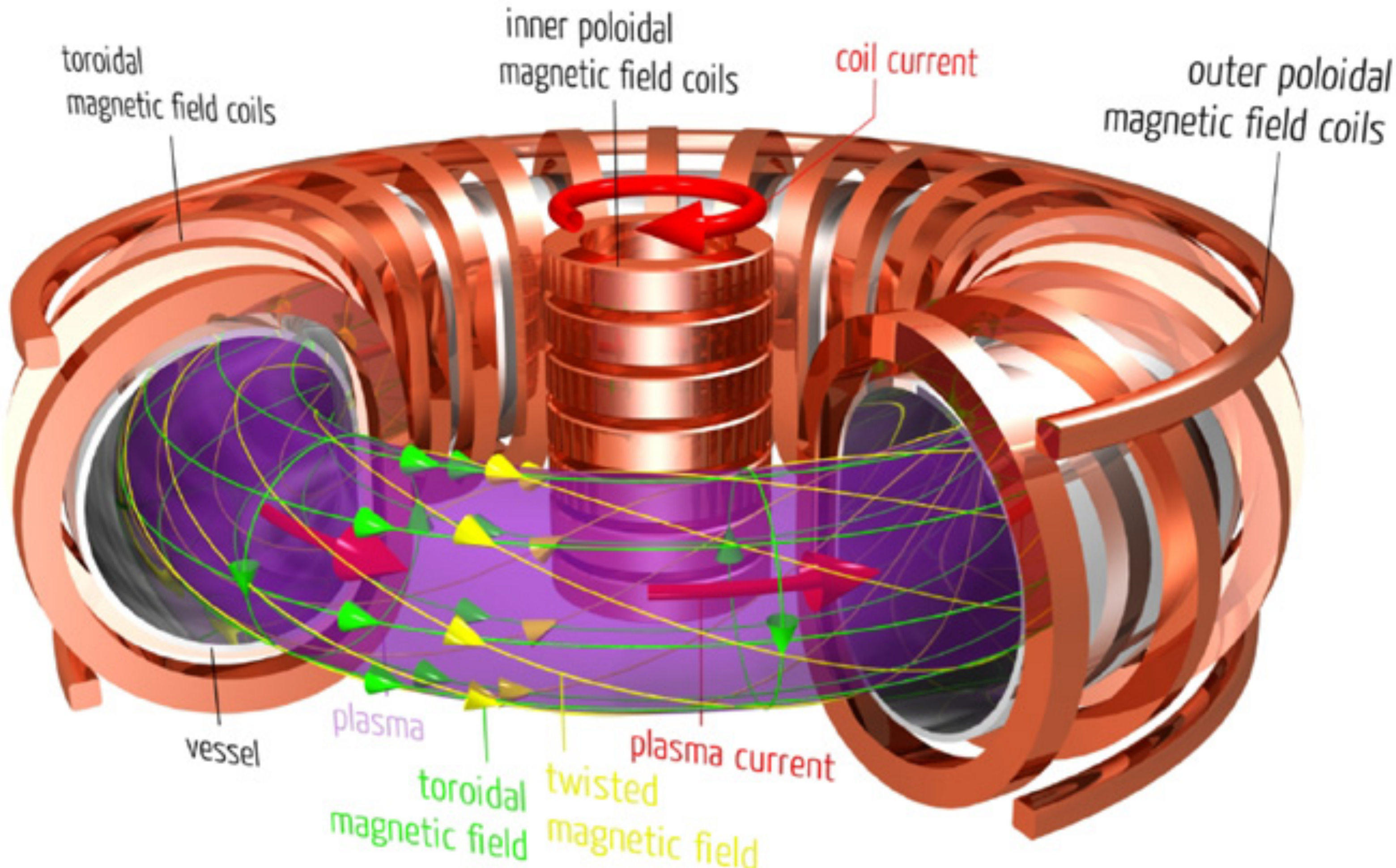
Van Allen belts and the northern lights

- Naturally occurring magnetic mirrors are the Van Ellen radiation belts within Earth's ionosphere
- Particles from the solar wind get trapped within the weaker field region between the poles
- Escaping particles cause the northern lights (aurorae)



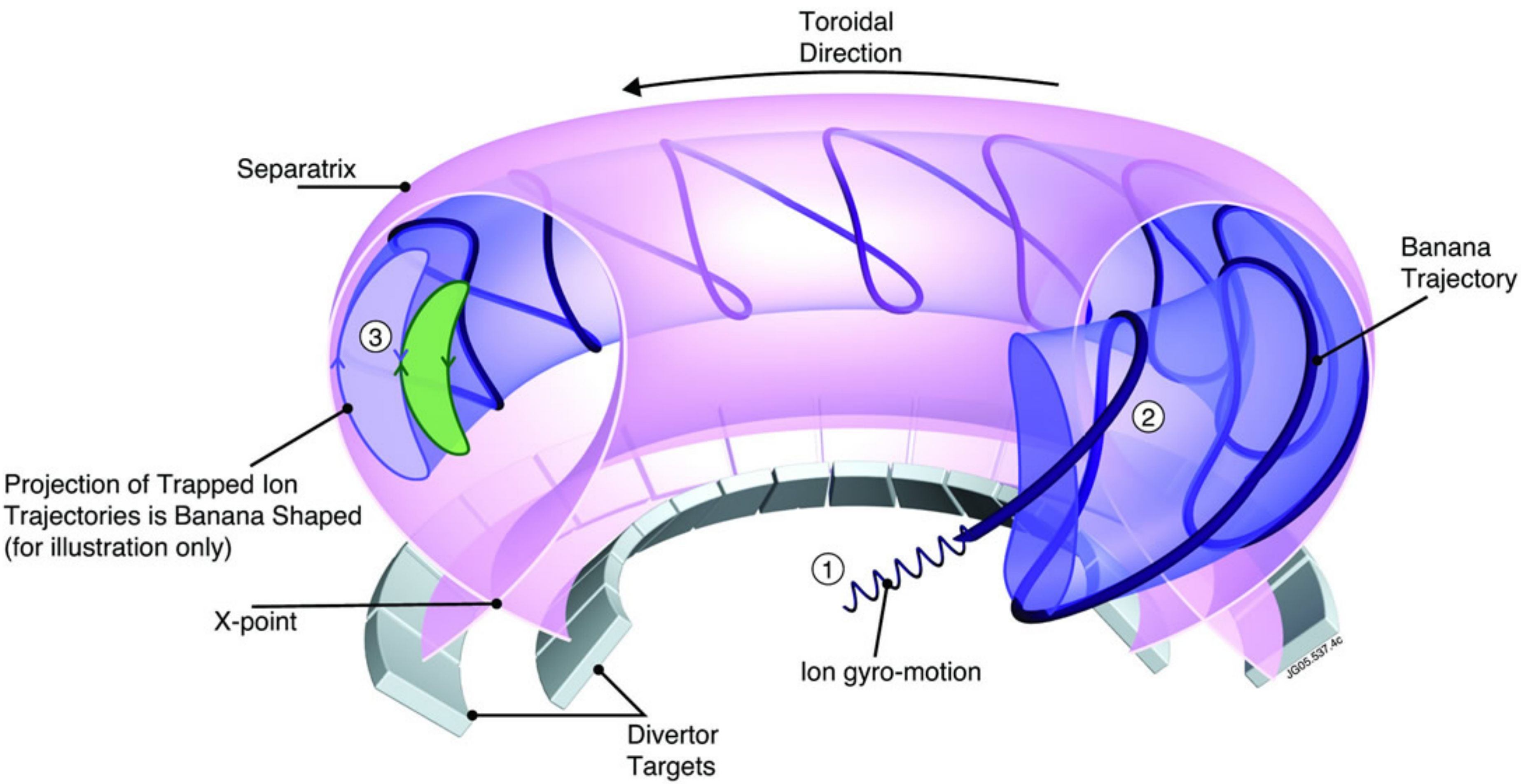


Magnetic confinement fusion



- The tokamak is an example of magnetic confinement

Bananas in tokamaks



- Finite size of ”bananas” due to small finite drift velocity

Adiabatic invariants

1) **The magnetic moment μ is conserved.**

$$\boxed{\boldsymbol{\mu} = \frac{1}{2} m v_\perp^2 \frac{\mathbf{B}}{B^2}} \quad \rightarrow \quad \frac{\mathrm{d}\mu}{\mathrm{d}t} = 0$$

2) **The path integral of $v_\parallel$ is conserved.**

→Particles always follow the field lines.

$$\mathrm{J} = \int_a^b v_\parallel \mathrm{d}s$$

3) **The flux mapped by the 3D surface due to the drift motion in B field curvature and gradients is conserved.**

→Particles drift around the equator returning to the same longitude.

Summary of lecture 2

- In a uniform B field the motion of a charged particle is helical.
- With the addition of another force, perpendicular to B, the guiding centre of the particle drifts along a direction perpendicular to both B and F.
- Electrons and ions drift in the same direction in an electric and magnetic field.
- B-field gradients and curvature also cause drift (this is an issue in tokamaks).
- Magnetic fields can be used to confine plasmas (e.g. magnetic mirrors)
- The magnetic moment of a plasma is called the first plasma/adiabatic invariant.