Plasma Physics

TU Dresden

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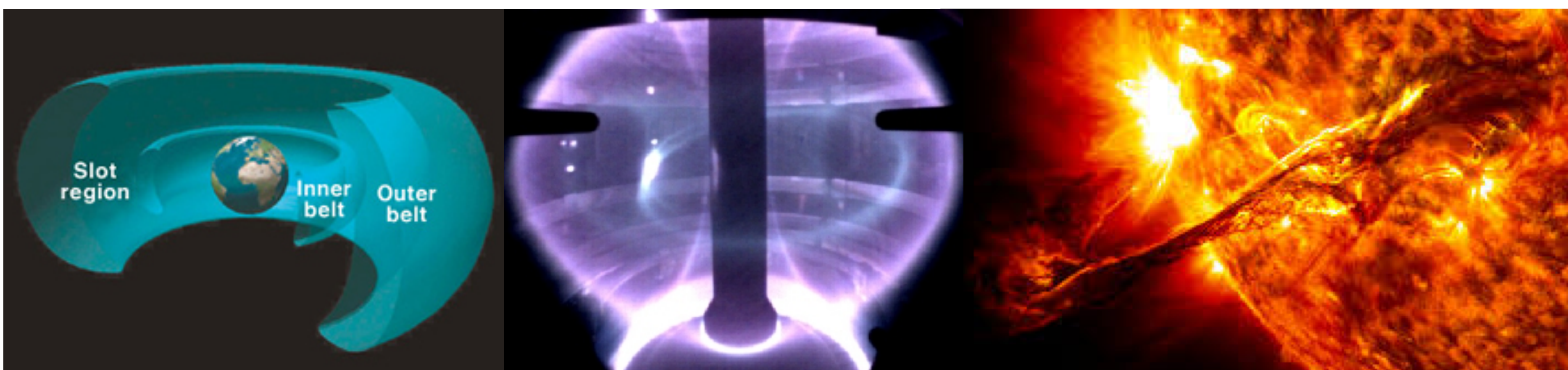
Lecture 3: Collisions and radiation



Plasma Physics: lecture 3

- Electron-ion collisions
- Coulomb logarithm
- Collision time and Spitzer resistivity
- ‘Collisionless’ nature of plasmas
- Radiation losses in plasma
- EM waves in plasma

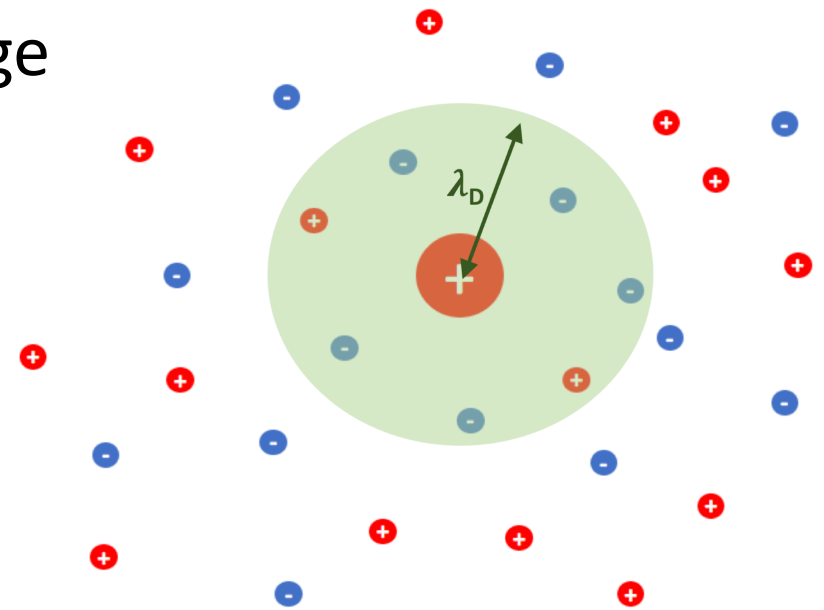
Collisions in plasmas

- Mobile particles in plasma feel electrostatic forces
- They collide with other particles inside the Debye sphere through a series of Coulomb collisions
- A large number of small momentum transfers (small angle deflections due to these collisions) add up to a large deflection over a long rage

  → **large effect**

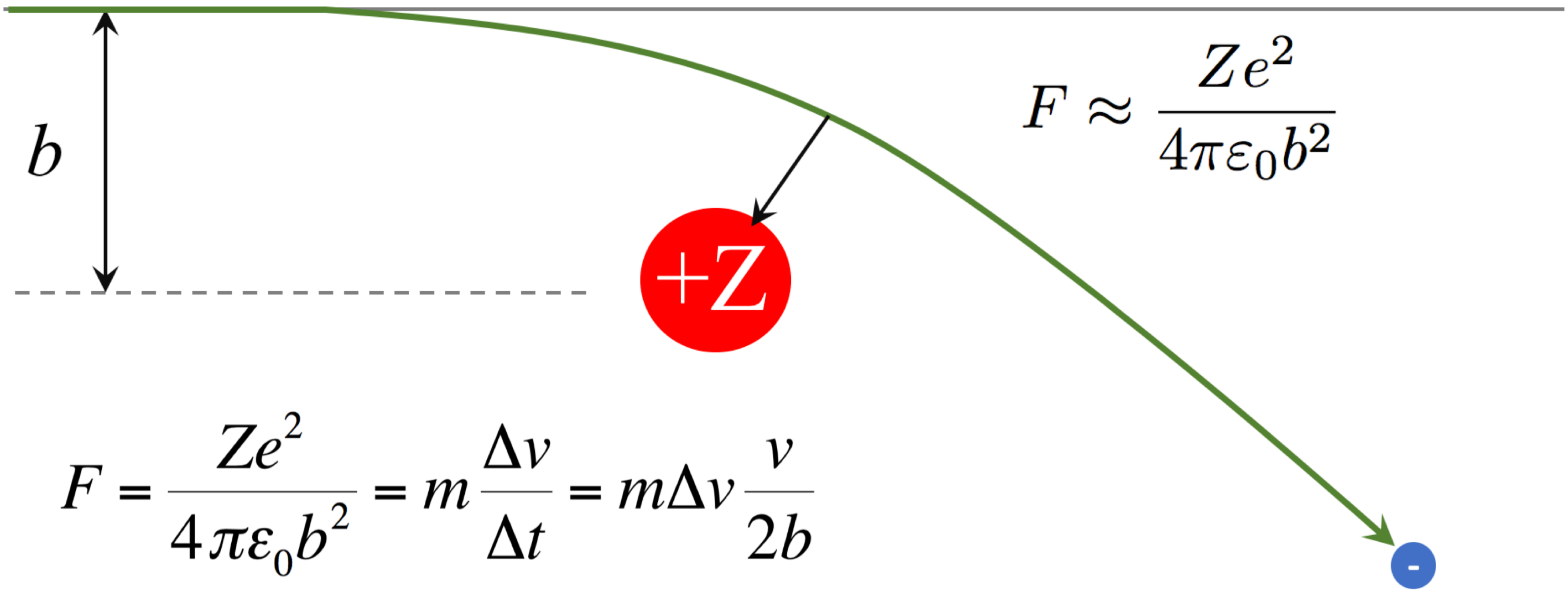
- Describe process by a simple collision model

  → **Rutherford scattering**



Rutherford scattering

- Electron feels an average Coulomb force:



$$F = \frac{Ze^2}{4\pi\varepsilon_0 b^2} = m\frac{\Delta v}{\Delta t} = m\Delta v\frac{v}{2b}$$

- For a time of: $\Delta t \approx \frac{2b}{v}$ $\rightarrow$ $\boxed{\Delta v = \frac{Ze^2}{2\pi\varepsilon_0 m b v}}$

Collision time

- Estimate the rate of change for $(\Delta v)^2$ assuming that a collision has taken place when the rms value of Δv has changed by v leading to a total 90 degree deflection (by adding lots of small angle collisions).

- For a given collision with impact parameter b:

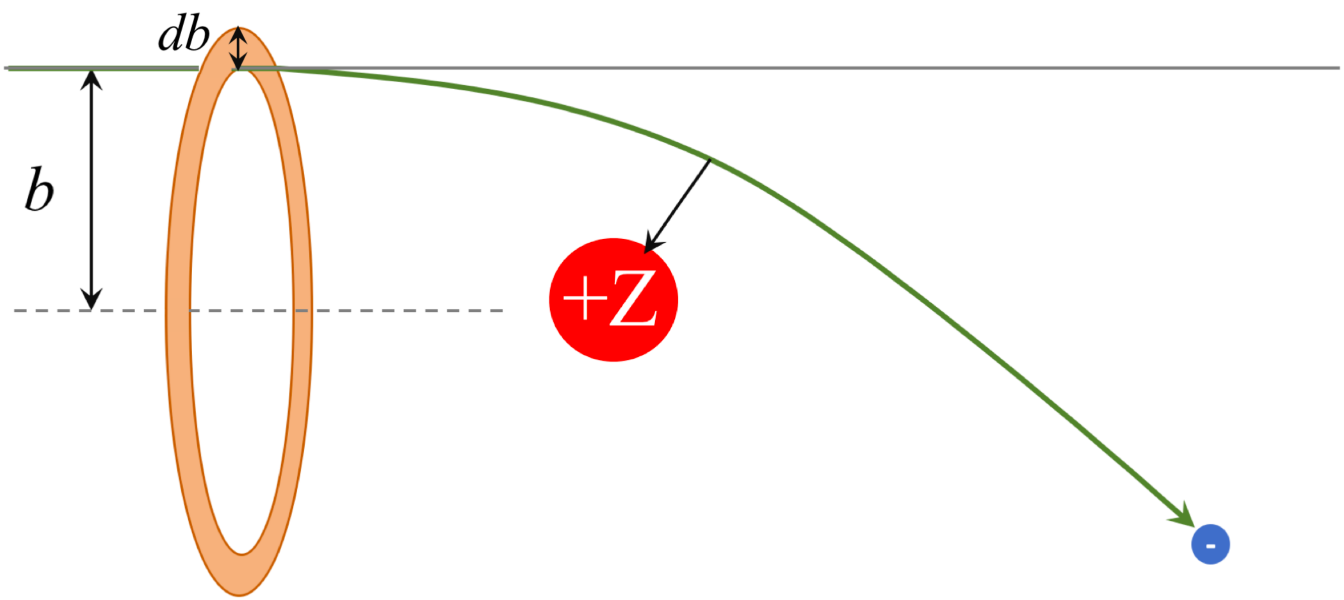
$$(\Delta v)^2 = \frac{Z^2 e^4}{4\pi^2 \varepsilon_0^2 m^2 b^2 v^2}$$

- The rate of particle encounter is: $n_i \sigma v$, where σ is the cross section given by $2\pi b db$

Collision time

cross section

- Thus: $$\frac{\mathrm{d}}{\mathrm{d}t} < (\Delta v)^2 > = \int 2\pi b \ \mathrm{d}b \ n_i v (\Delta v)^2$$
$$= \int 2\pi b \ \mathrm{d}b \ n_i v \frac{Z^2 e^4}{4\pi^2 \varepsilon_0^2 m^2 b^2 v^2}$$
$$= \int \frac{n_i Z^2 e^4 \ \mathrm{d}b}{2\pi \varepsilon_0^2 m^2 v b}$$



Collision time

- The upper limit of the integral is the Debye length λ_D as for any deflection larger than that results in a particle no longer feeling the electrostatic potential due to shielding:

$$b_{\max} = \lambda_D = \sqrt{\frac{\varepsilon_0 k_B T}{n_e e^2}}$$

- Head on collision (zero angle deflection must) have $\Delta v \approx v$, thus:

$$b_{\min} \approx \frac{Ze^2}{4\pi\varepsilon_0 m v^2}$$

- Setting the b_{min} to the deBroglie length (see lecture 1) is the more correct approach

The Coulomb logarithm

- By integration with the correct limits, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t} < (\Delta v)^2 > = \int_{b_{\min}}^{\lambda_{\mathrm{D}}} \frac{n_i Z^2 e^4 \ \mathrm{d}b}{2\pi\varepsilon_0^2 m^2 v b}$$

$$= \frac{n_i Z^2 e^4}{2\pi\varepsilon_0^2 m^2 v} \ln(\Lambda)$$

- Define the **Coulomb logarithm**:

$$\Lambda = \frac{\lambda_{\mathrm{D}}}{b_{\min}}$$

$$= \sqrt{\left(\frac{\varepsilon_0 k_{\mathrm{B}} T}{n_0 e^2}\right)} \frac{2\pi\varepsilon_0 m v^2}{Z e^2}$$

The Coulomb logarithm

- Assuming the thermal velocity:

$$\frac{1}{2}mv^2 \approx \frac{3}{2}k_B T$$

- Thus:

$$\Lambda = \sqrt{\left(\frac{\varepsilon_0 k_\mathrm{B} T}{n_0 e^2}\right)} \frac{2\pi\varepsilon_0 m v^2}{Z e^2} \approx \frac{3\pi n_0 \lambda_\mathrm{D}^3}{Z} \approx \frac{3\pi N_\mathrm{D}}{Z}$$

- The Coulomb logarithm is of the same order as the plasma parameter N_D, i.e. corresponds to the number of particles inside the Debye sphere (typically 1 – 30)

Collision time

- For total of 90 degree deflection after many small angle collisions we expect the Δv^2 to be of the order of v^2:

$$\frac{1}{\tau}v^2 = \frac{n_i Z^2 e^4}{2\pi\varepsilon_0^2 m^2 v}\ln(\Lambda)$$

$$\rightarrow \tau = \frac{2\pi\varepsilon_0^2 m^2 v^3}{n_i Z^2 e^4 \ln(\Lambda)}$$

- Average over the velocities present in the Maxwellian distribution and for electron thermal velocity of $v = \sqrt{k_B T_e / m}$:

$$\boxed{\tau \approx 6.4\frac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)}}$$

Mean free path

- The mean free path is the average distance between two subsequent collisions of the electron (ion) with plasma components:

$$\lambda_{i,e} = \frac{v_{e,i}}{\nu_{e,i}}$$

average velocity of the electron or ion

$$v_{e,i} = \sqrt{k_B T_{e,i}/m_{e,i}}$$

the electron or ion collision rate

- Collision rates for fully ionized non-degenerate plasma:

$$\nu_e = 2{,}91 \times 10^{-6} n_e \ln\Lambda T_e^{-3/2} \mathrm{s}^{-1}$$

$$\nu_i = 4{,}80 \times 10^{-8} Z^4 \left(\frac{m_i}{m_p}\right)^{-1/2} n_e \ln\Lambda T_e^{-3/2} \mathrm{s}^{-1}$$

Collisionless plasmas

- While deriving the plasma frequency ω_{pe} we ignored collisions arguing that there were many oscillations of the electrons on the timescale of a collision

- We can now prove that we can ignore collisions on the time scale of plasma frequency, i.e. $\omega_{pe}\tau \gg 1$

Collisionless plasmas

- Substitute:

$$\omega_{\mathrm{pe}}\tau = \sqrt{\frac{n_e e^2}{\varepsilon_0 m}} 6.4 \frac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)}$$

$$= 6.4\sqrt{\frac{n_e e^2}{\varepsilon_0 k_{\mathrm{B}} T_e}} \frac{2\pi\varepsilon_0^2 (k_{\mathrm{B}} T_e)^2}{n_e Z e^4 \ln(\Lambda)}$$

$$= 6.4 \frac{1}{\lambda_{\mathrm{D}}} 2\pi n_e \frac{\lambda_{\mathrm{D}}^4}{Z \ln\Lambda}$$

$$= 6.4 \quad 2\pi \quad \frac{3}{4\pi} \frac{N_{\mathrm{D}}}{z \ln\Lambda}$$

$$\approx \frac{10 N_{\mathrm{D}}}{Z \ln\Lambda}$$

Collisionless plasmas

- In lecture 1 we defined a 'good' plasma is one with a large number of particles within the Debye sphere:

$$N_D = n_0 \frac{4}{3} \pi \lambda_D^3 >> 1$$

- While typical $\ln \Lambda \sim 10$
- Thus we can treat plasmas as collisionless, i.e. collisions take place on a timescale that is long compared with a plasma period
- This also means that interactions with plasma without collisions are possible!

Effects of collisions

- Collisions heat and ionize plasmas and produce fusion
- The rate of collisions between particles in plasma are related to electrical conductivity/resistivity of plasmas
- Collisions of particles also lead do radiative energy losses in the form of Bremsstrahlung and recombination
- Random motion and collisions in plasma in presence of gradients in thermodynamic conditions also lead to diffusion
- Particles can diffuse across B-fields → energy loss in tokamaks
- This also leads to constraints in confinement time in fusion plasmas

Resistivity in plasmas

- Particles in plasma are repeatedly accelerated and stopped by collisions
- There is some net motion → drift velocity v_D (not guiding centre)
- Motion between collisions in E field in plasma:

$$F = eE = m_e \frac{v_D}{\tau} \quad \rightarrow \quad v_D = \frac{eE\tau}{m_e}$$

- The current density of moving charges in plasma:

$$\mathrm{J} = n_e e v_D = \frac{n_e e^2 E}{m_e}\tau \quad \rightarrow \quad E = \frac{m_e}{n_e e^2 \tau} \cdot \mathrm{J}$$

Resistivity in plasmas

- From Ohm's law:

Resistivity

$$\mathrm{E} = \eta \cdot \mathrm{J} \qquad \Rightarrow \qquad \eta = \frac{m_e}{n_e e^2 \tau}$$

- Substitute for τ:

$$\tau = \frac{2\pi\varepsilon_0^2 m^2 v^3}{n_i Z^2 e^4 \ln(\Lambda)}$$

$$\Rightarrow \qquad \boxed{\eta \approx \frac{e^2 m_e^{1/2}}{\varepsilon_0^2 (k_B T)^{3/2}} \cdot Z \ln(\Lambda)}$$

- Spitzer resistivity: $\eta_{Spitzer} \approx 5.22\times10^{-5} \cdot \frac{Z \ln(\Lambda)}{T^{\frac{3}{2}}} \, [\Omega m]$

Temperature in eV !!!

Resistivity in plasmas

- Unlike metals, plasmas become better conductors at higher temperatures
- It is not possible to Ohmicaly heat tokamak plasmas to reach fusion temperatures.
- Ohmic heating/resistive heating is the process by which the passage of an electric current through a conductor produces heat.
- Resistivity is independent of electron density for ideal plasmas.

Bremsstrahlung

- Radiation power from accelerating charge:

$$\frac{dW}{dt} = \frac{q^2}{6\pi\varepsilon_0 c^3} \cdot \left(\frac{dv}{dt}\right)^2$$

- Electrostatic force seen by an electron:

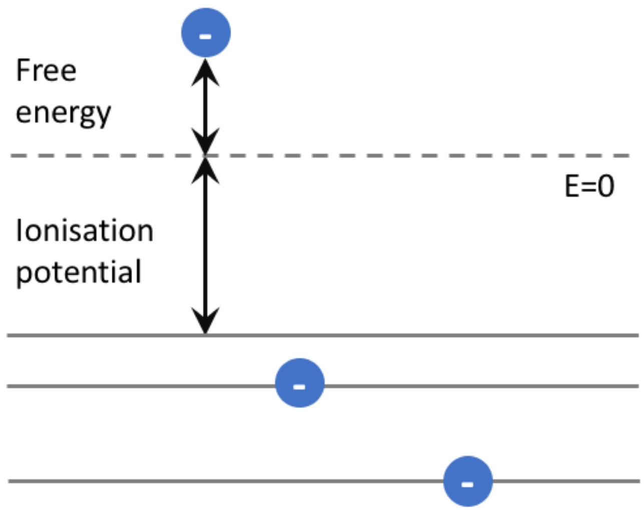
$$m_e \frac{dv}{dt} = \frac{Ze^2}{4\pi\varepsilon_0 b^2}$$

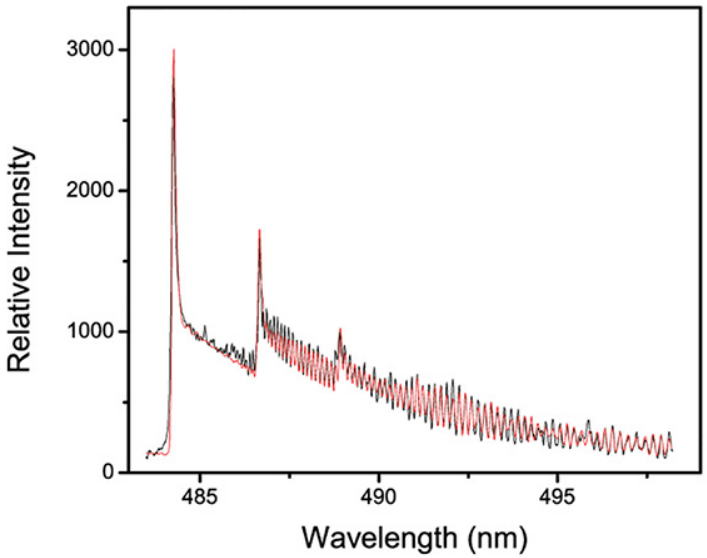
- Integrate over frequency → **Bremsstrahlung power**:

$$P_B = 1.69{\times}10^{-38} Z^2 n_e n_i T^{1/2}\ \mathrm{Wm^{-3}}$$

Other radiation losses in plasma

- Due to recombination, where low energy electrons do not escape the pull of an ion and become bound to it
- Results in a continuum spectrum with a cut-off energy at ionization energy





- Total power for radiation losses:

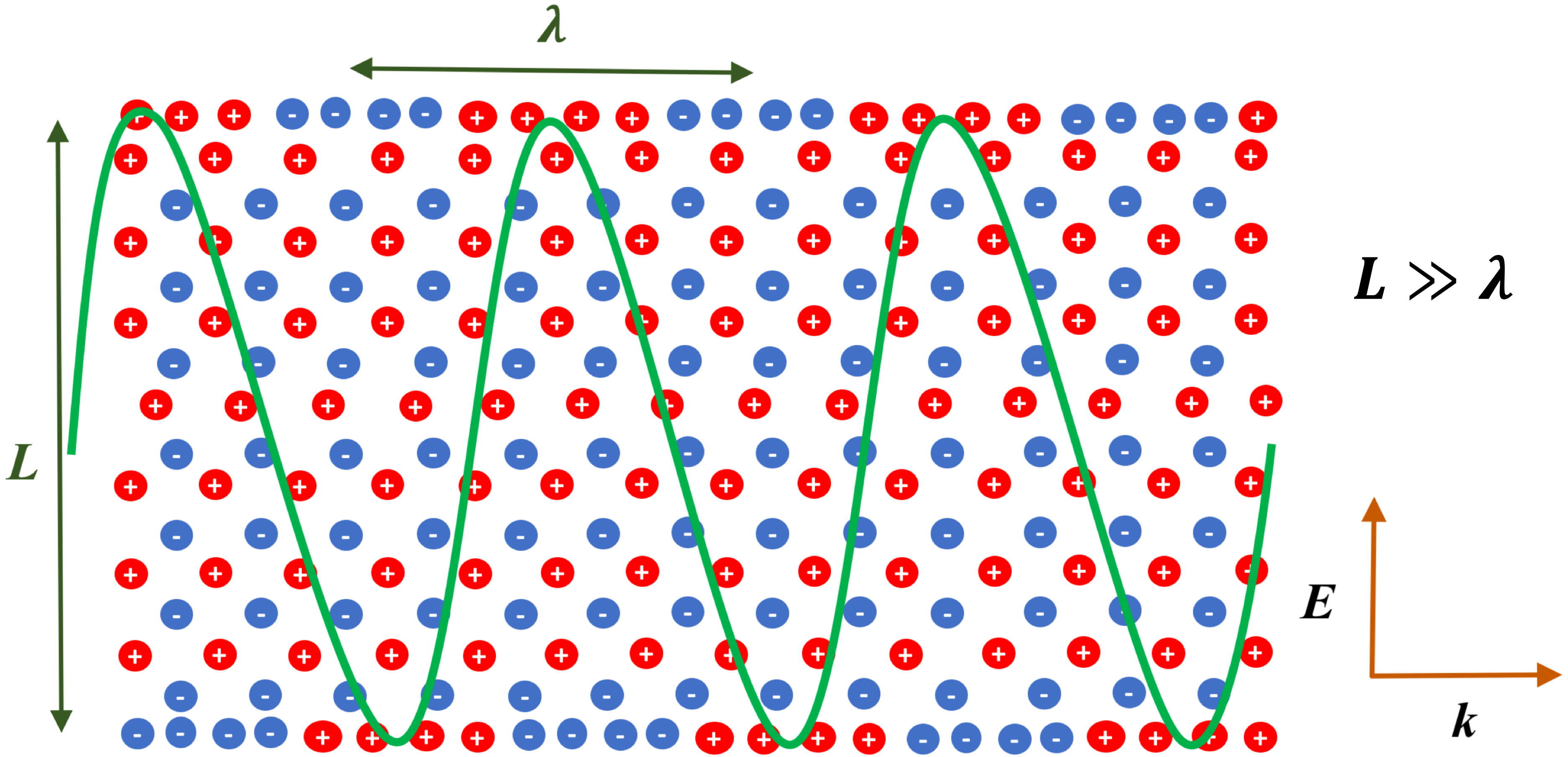
$$P_B = 1.69\times10^{-38} n_e T^{1/2} \sum Z^2 n_z \frac{E_\mathrm{i}}{T} \ \mathrm{Wm^{-3}}$$

Electromagnetic waves in plasma

- What happens when light incident onto plasma?
- Both electric and magnetic fields oscillate
- EM waves are transverse
- Electric field makes electrons move, generating current
- Ohm's law: $$\mathbf{J}(\omega) = \underbrace{\sigma}_{\text{Conductivity}}(\omega)\mathbf{E}$$
- In vacuum (no free charges): $$\nabla \cdot \mathbf{E} = 0$$

$$\rightarrow \quad \mathbf{k} \cdot \mathbf{E} = 0$$

Electromagnetic waves in plasma



A transverse wave does not give rise to charge separation on the scale of a wavelength.

Electromagnetic waves in plasma

- Start from Maxwell's equations with proper treatment:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Faraday's Law *Ampere's Law*

- Take curl on both sides of the Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t}$$

- Substitute in Ampere's law to obtain:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\boldsymbol{\mu}_\mathbf{0} \frac{\partial}{\partial t}\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

Electromagnetic waves in plasma

- As we have no charge separation $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$:

$$-\nabla^2 \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{B})}{\partial t} = -\mu_0 \frac{\partial}{\partial t}\left(\mathbf{J} + \varepsilon_\mathbf{0} \frac{\partial \mathbf{E}}{\partial \mathbf{t}}\right) = -\mu_0 \frac{\partial}{\partial t}\left(\sigma \mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

- And the transverse EM wave is:

$$\mathbf{E} = E_0 . \exp[i(\omega t - k . \mathbf{r})]$$

- By simple derivation then:

$$k^2 = -\mu_0 \left(i\omega\sigma - \varepsilon_0 \omega^2\right) = \frac{\omega^2}{c^2}\left(1 - \frac{i\sigma}{\varepsilon_0 \omega}\right)$$

- Thus the dielectric function is:

$$\boxed{\varepsilon(\omega) = \left(1 - \frac{i\sigma}{\varepsilon_0 \omega}\right)}$$

Conductivity of plasma

- Ohm's law: $\mathbf{J}(\omega) = \underset{\text{Conductivity}}{\sigma}(\omega)\mathbf{E}$

- Assume oscillating electric field: $\mathbf{E} = \mathbf{E}(\omega)\exp(i\omega t)$

- Thus, equation of motion for electrons:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -e\mathbf{E} = -e\mathbf{E}(\omega)\exp(i\omega t)$$

- Electrons are assumed to also oscillate:

$$\mathbf{v}(t) = \mathbf{v}(\omega)\exp(i\omega t)$$

- Thus: $im\mathbf{v}(\omega)\omega = -e\mathbf{E}(\omega)$

Conductivity of plasma

- Current density also defined as: $\mathbf{J}(\omega) = -ne\mathbf{v}(\omega)$
- Substitute for $\mathbf{v}(\omega)$:

$$\mathbf{J}(\omega) = -ne\mathbf{v}(\omega) = -i\frac{ne^2}{m\omega}\mathbf{E}(\omega)$$

- Thus conductivity of plasma:

$$\boxed{\sigma(\omega) = -i\frac{ne^2}{m\omega}}$$

- And the dielectric function:

Plasma frequency

$$\boxed{\epsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m\omega^2} = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}}$$

Electromagnetic waves in plasma

- The general expression:

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + k^2\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

- Substitute for conductivity $\sigma(\omega)$ as $\mathbf{J}(\omega)= \sigma(\omega)\mathbf{E}$:

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + k^2\mathbf{E} = -\mu_0 \frac{\partial}{\partial t}\left(-i\frac{ne^2}{m\omega}\mathbf{E} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

- Rearrange:

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} = \frac{1}{c^2}\left(\omega_{\mathrm{p}}^2 - \omega^2\right)\mathbf{E}$$

Electromagnetic waves in plasma

- So finally:

$$(\omega^2 - \omega_{\mathrm{p}}^2 - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0$$

- For electrostatic waves $\mathbf{k} \parallel \mathbf{E}$, thus $\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = k^2\mathbf{E}$:

$$\omega = \omega_{\mathrm{p}}$$

Plasmons

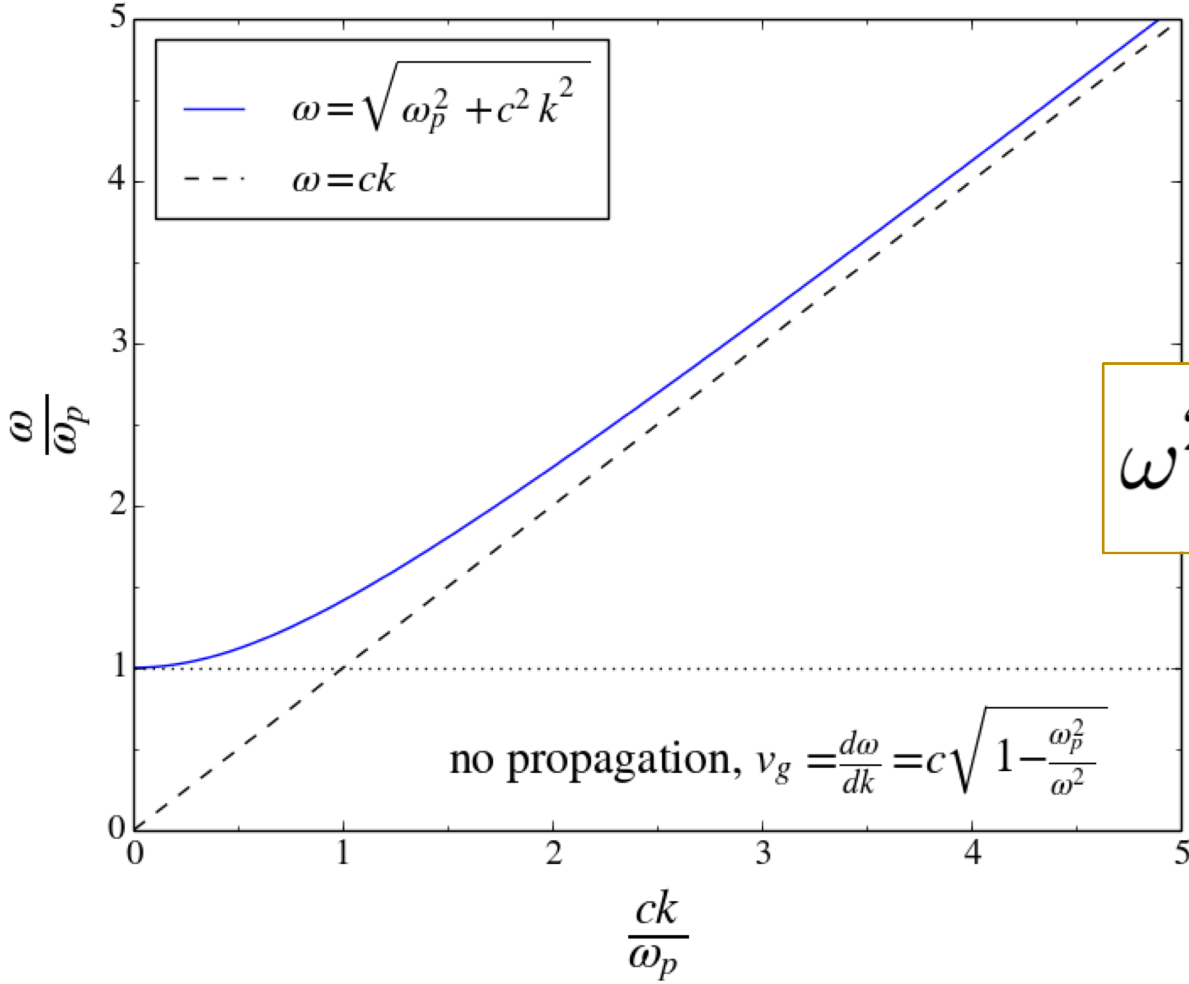
- For electromagnetic waves $\mathbf{k}\cdot\mathbf{E} = 0$:

$$\boxed{\omega^2 = \omega_{\mathrm{p}}^2 + c^2k^2}$$

Dispersion relation for electromagnetic waves in plasma

Electromagnetic waves in plasma

- No propagation for light with $\omega < \omega_p$:



$$\boxed{\omega^2 = \omega_{\mathrm{p}}^2 + c^2 k^2}$$

Summary of lecture 3

- Electrons scatter from ions via Rutherford scattering. A ‘collision’ is usually made up of many small-angle scatters.
- The scattering time: $\tau \approx 6.4 \dfrac{2\pi\varepsilon_0^2 m^2 v_e^3}{n_i Z^2 e^4 \ln(\Lambda)}$
- This time is long compared with a plasma period (the ratio is of order the plasma parameter) - hence ‘good’ plasmas are ‘collisionless’.
- The dielectric function of a plasma: $\varepsilon(\omega) = \left(1 - \dfrac{i\sigma}{\varepsilon_0\omega}\right)$
- Dispersion relation for EM waves in plasma:

$$\omega^2 = \omega_{\mathrm{p}}^2 + c^2k^2$$