Plasma Physics

TU Dresden

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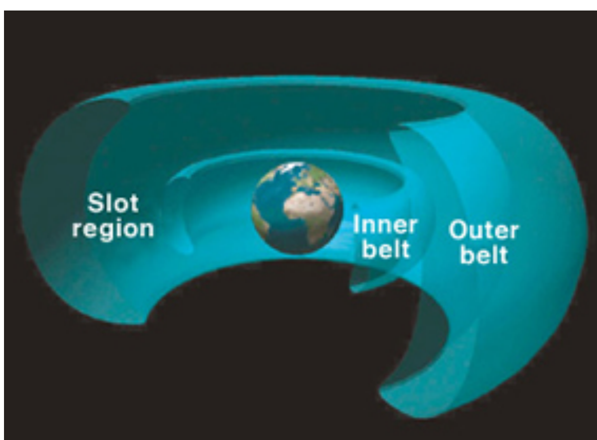
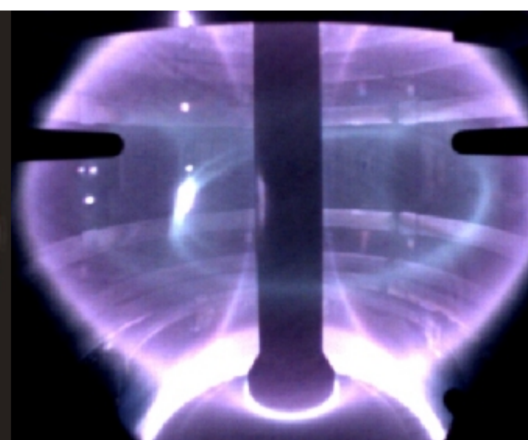
Lecture 4: Kinetic theory






Plasma Physics: lecture 4

- Kinetic description of plasma
- The Vlasov equation
- Langmuir waves
- Bohm-Gross frequency
- Landau damping

The distribution function

- The comprehensive information about the motion of individual particles in plasma is included in the distribution function
- Waves will alter the distribution function
- Maxwell-Boltzmann distribution can be used for homogeneous plasma extending over all space:

$$f(v) = \frac{4}{\sqrt{\pi}} \cdot \frac{v^3}{v_{th}^3} e^{v^2/v_{th}^2} \cdot dv \quad \text{where} \quad v_{th} = \sqrt{\frac{k_B T}{m}}$$

The distribution function

- For 6-dimensional phase space we get 3 velocity and 3 spatial coordinates for each particle.
- Number of particles within small volume $d^3\mathbf{r}$ at position $\mathbf{r}$, with velocity within volume element $d^3\mathbf{v}$ at velocity $\mathbf{v}$:

$$\mathrm{d}N(\mathbf{v},\mathbf{r},t) = f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{v}$$

- Integrate over all real space → distribution function:

$$f(\mathbf{v},t) = \int f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r}$$

- Integrate whole phase space → total no. of particles:

$$N = \int\int f(\mathbf{v},\mathbf{r},t)\mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{v}$$

Continuity equation

- Particles cannot be created or destroyed
- Particle density n and flow velocity $\mathbf{u}$ are connected through conservation of mass:

$$\frac{\partial}{\partial t}\iiint_V n\cdot dV = -\oiint_S n\cdot\mathbf{u}\cdot dS \quad \text{and} \quad \iiint_V \nabla\mathbf{A}\cdot dV = \oiint_S \mathbf{A}\cdot dS$$

Rate of decrease of total charge *Total current* *the divergence theorem*

$$\Rightarrow \quad \iiint_V \nabla(n\mathbf{u})\cdot dV + \frac{\partial}{\partial t}\iiint_V n\cdot dV = \iiint_V\left[\nabla(n\mathbf{u}) + \frac{\partial n}{\partial t}\right]\cdot dV = 0$$

- Thus the **continuity equation**: $\therefore \boxed{\nabla(n\mathbf{u}) + \frac{\partial n}{\partial t} = 0}$

The Vlasov equation

- Particles/mass are conserved
- The distribution function obeys the continuity equation:

$$\frac{\partial f}{\partial t} + \nabla_r(f\mathbf{u}) + \nabla_u(f\boldsymbol{a}) = 0$$

- Simplify using the product rule:

$$\nabla_r(f\mathbf{u}) = f\overset{=0}{\cancel{\nabla_r \mathbf{u}}} + \mathbf{u}\nabla_r f$$

$$\nabla_u(f\boldsymbol{a}) = f\overset{=0}{\cancel{\nabla_u \mathbf{a}}} + \mathbf{a}\nabla_u f = \frac{\mathbf{F}}{m}\cdot\nabla_u f$$

The Vlasov equation

- Get the collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{u}\nabla_r f + \frac{\mathbf{F}}{m} \cdot \nabla_u f = 0$$

- Velocity distribution constantly changed by collisions
- Add collision term for completeness
- Collisional Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{u}(\nabla_r f) + \frac{\mathbf{F}}{m} (\nabla_u f) = \left(\frac{\partial f}{\partial t}\right)_{collisions}$$

The Vlasov equation

- Plasmas are subject to the Lorentz force:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

- Substitute to the collisional Boltzmann equation
- The full **Vlasov equation**:

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{u}(\nabla_r f) + \frac{q}{m}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot (\nabla_u f) = \left(\frac{\partial f}{\partial t}\right)_{collisions}}$$

- Ignoring collisions:

$$\frac{\partial f}{\partial t} + \mathbf{u}(\nabla_r f) + \frac{q}{m}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot (\nabla_u f) = 0$$

The Vlasov equation

- The Vlasov equation is used to describe and study the kinetic theory of plasmas.
- The electric and magnetic fields can be:
  - External acting on a whole group of particles
  - Generated by collective effects in plasma, i.e. waves
- It is used to model waves in plasma, transport and collisions.
- We will use it to get a complete description of Langmuir waves in plasma and recover the Bohm-Gross frequency.
- We will also study the damping rate of the Langmuir waves (Landau damping).

Langmuir waves

- Assume static ions (no change in distribution function)
- Electron distribution function perturbed by $f_1(\mathbf{r}, \mathbf{v}, t)$
- The total distribution function:

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{r}, \mathbf{v}, t) + f_1(\mathbf{r}, \mathbf{v}, t)$$

- Thus, electric field present due to f_1 perturbation only, no net electric fields
- Following from the Gauss law:

$$\nabla \cdot \mathbf{E}(\mathbf{r}, t) = -\frac{e}{\varepsilon_0} \int f_1(\mathbf{r}, \mathbf{v}, t) \mathrm{d}^3\mathbf{v}$$

Langmuir waves

- Vlasov equation before perturbation: $\frac{\partial f_0}{\partial t} + \mathbf{v} \cdot \frac{\partial f_0}{\partial \mathbf{r}} = 0$
- After perturbation:

$$\frac{\partial (f_0 + f_1)}{\partial t} + \mathbf{v} \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{v}} = 0$$

- Subtract equations:

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial (f_1)}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial (f_0 + f_1)}{\partial \mathbf{v}} = 0$$

- And linearize (ignore terms $\mathbf{E}\, f_1$): ← $\mathbf{E}\, f_1$ *is small*

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{r}} - \frac{e}{m}(\mathbf{E}) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

Langmuir waves

- For wave motion, we assume that the electric field and perturbation to the distribution function vary periodically:

$$\mathbf{E} = \mathbf{E}_{\mathbf{k},\omega} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$$

$$f_1 = f_{1(\mathbf{k},\omega)} \exp(i[\mathbf{k} \cdot \mathbf{r} - \omega t])$$

- Substituting into the Vlasov equation:

$$-i\omega f_{1(\mathbf{k},\omega)} + i\mathbf{k} \cdot \mathbf{v} f_{1(\mathbf{k},\omega)} - \frac{e}{m}\mathbf{E}_{\mathbf{k},\omega} \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$$

- Simplify: $$f_{1(\mathbf{k},\omega)} = \frac{ie}{m} \frac{1}{(\omega - \mathbf{k} \cdot \mathbf{v})} \mathbf{E}_{\mathbf{k},\omega} \cdot \frac{\partial f_0}{\partial \mathbf{v}}$$

Langmuir waves

- Gauss law again: $$\nabla \cdot \mathbf{E}(\mathbf{r},t) = -\frac{e}{\varepsilon_0}\int f_1(\mathbf{r},\mathbf{v},t)\mathrm{d}^3\mathbf{v}$$

$$i\mathbf{k} \cdot \mathbf{E}_{\mathbf{k},\omega} = -\frac{e}{\varepsilon_0}\int f_1(\mathbf{r},\mathbf{v},t)\mathrm{d}^3\mathbf{v}$$

- Substitute for f_1 :

$$\mathbf{k} \cdot \mathbf{E}_{\mathbf{k},\omega} = -\frac{e^2}{\varepsilon_0 m}\mathbf{E}_{\mathbf{k},\omega} \cdot \int \frac{\frac{\partial f_0}{\partial \mathbf{v}}}{\omega - \mathbf{k}\cdot\mathbf{v}}\mathrm{d}^3\mathbf{v}$$

$$\mathbf{k} = -\frac{e^2}{\varepsilon_0 m}\int \frac{1}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial f_0}{\partial \mathbf{v}}\mathrm{d}^3\mathbf{v}$$

Langmuir waves

- Assuming plane wave propagating along z-direction
- $\mathbf{E}$ and $\mathbf{k}$ along z-axis for electrostatic waves:
- Thus, the velocity component v_z also along z and the non-zero electric field yields dispersion relation:

$$1 + \frac{e^2}{\varepsilon_0 m k} \int \frac{\frac{\partial f_0}{\partial v_z}}{\omega - k v_z} \mathrm{d}v_x \mathrm{d}v_y \mathrm{d}v_z = 0$$

- This leads to a dispersion relation links k and ω
- Note: there is a pole in the integral when the velocity of the electron equals the phase velocity of the wave, i.e. for $v_z = \omega / k$ (we ignore it for $v_z \gg v_{th}$)

Langmuir waves

- For Maxwell-Boltzmann distribution function:

$$f_0 = \frac{n_0}{\pi^{3/2}}\left(\frac{m}{2k_{\mathrm{B}}T}\right)^{3/2}\exp\{-m(v_x^2+v_y^2+v_z^2)/(2k_{\mathrm{B}}T)\}$$

- And substitute for f_0:

$$1-\frac{2n_0e^2}{\pi^{3/2}\varepsilon_0 mk}\left(\frac{m}{2k_{\mathrm{B}}T}\right)^{5/2}\int\frac{v_z e^{-\frac{mv_z^2}{2k_{\mathrm{B}}T}}}{\omega-kv_z}\mathrm{d}v_z\int e^{-\frac{mv_x^2}{2k_{\mathrm{B}}T}}\mathrm{d}v_x\int e^{-\frac{mv_y^2}{2k_{\mathrm{B}}T}}\mathrm{d}v_y=0$$

- And:

$$\int e^{-\frac{mv_{x,y}^2}{2k_BT}}dv_{x,y}=\sqrt{\frac{2\pi k_BT}{m}}$$

Langmuir waves

- And simplify:

$$1 - \frac{2n_0 e^2}{\pi^{1/2}\varepsilon_0 m k}\left(\frac{m}{2k_\mathrm{B}T}\right)^{3/2}\int \frac{v_z e^{-\frac{m v_z^2}{2k_\mathrm{B}T}}}{\omega - k v_z}\mathrm{d}v_z = 0$$

- We make the assumption that v_z is large compared to the thermal velocity $(k_B T/m)^{1/2}$ and thus $k v_z \ll \omega$

→ ignore the pole

- Binomial expansion:

$$\frac{1}{\omega - k v_z} = \frac{1}{\omega}\left(1 + \frac{k v_z}{\omega} + \left(\frac{k v_z}{\omega}\right)^2 + \ldots\right)$$

Langmuir waves

- Substitute and simplify:

$$1-\frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_\mathrm{B}T}\right)^{3/2}\frac{1}{\omega}\int v_z\left(1+\frac{kv_z}{\omega}+\left(\frac{kv_z}{\omega}\right)^2+\dots\right)e^{-\frac{mv_z^2}{2k_\mathrm{B}T}}\mathrm{d}v_z=0$$

- Integrate from $-\infty$ to ∞, odd functions go to zero:

$$1-\frac{2n_0e^2}{\pi^{1/2}\varepsilon_0 mk}\left(\frac{m}{2k_\mathrm{B}T}\right)^{3/2}\frac{1}{\omega}\int v_z\left(\frac{kv_z}{\omega}+\left(\frac{kv_z}{\omega}\right)^3+\dots\right)e^{-\frac{mv_z^2}{2k_\mathrm{B}T}}\mathrm{d}v_z=0$$

$$\rightarrow\; 1-\frac{n_0e^2}{\varepsilon_0 m}\frac{1}{\omega^2}\left(1+3\frac{k^2}{\omega^2}\frac{k_\mathrm{B}T}{m}+\dots\right)=0$$

Langmuir waves

- Rewrite with thermal velocity $v_e = (k_B T/m)^{1/2}$ and substitute for the plasma frequency:

$$1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_e^2 + \ldots\right) = 0$$

- Complete result taking in account the pole provides the exact solution:

$$\omega^2 = \omega_{pe}^2 + \frac{\omega_{pe}^2}{\omega^2} \cdot \frac{3k_B T}{m} \cdot k^2 - \frac{i\pi\omega_{pe}^2\omega^2}{k^2 n}\left(\frac{\partial f_0}{\partial v}\right)_{v=\omega/k}$$

Langmuir oscillations

Effect of temperature

The damping term (effect of the singularity)

Langmuir waves

- We assumed a large phase velocity $\frac{\omega}{k}$ compared to thermal velocity $kv_e \ll \omega$
- With no damping term, the effect of temperature is small and we obtain the approximate solution ($\omega \approx \omega_{pe}$) for the **Bohm-Gross frequency**:

$$\boxed{\omega^2 = \omega_{pe}^2 + \frac{3k_B T}{m} \cdot k^2}$$

The dispersion relation for electrostatic waves in warm plasma

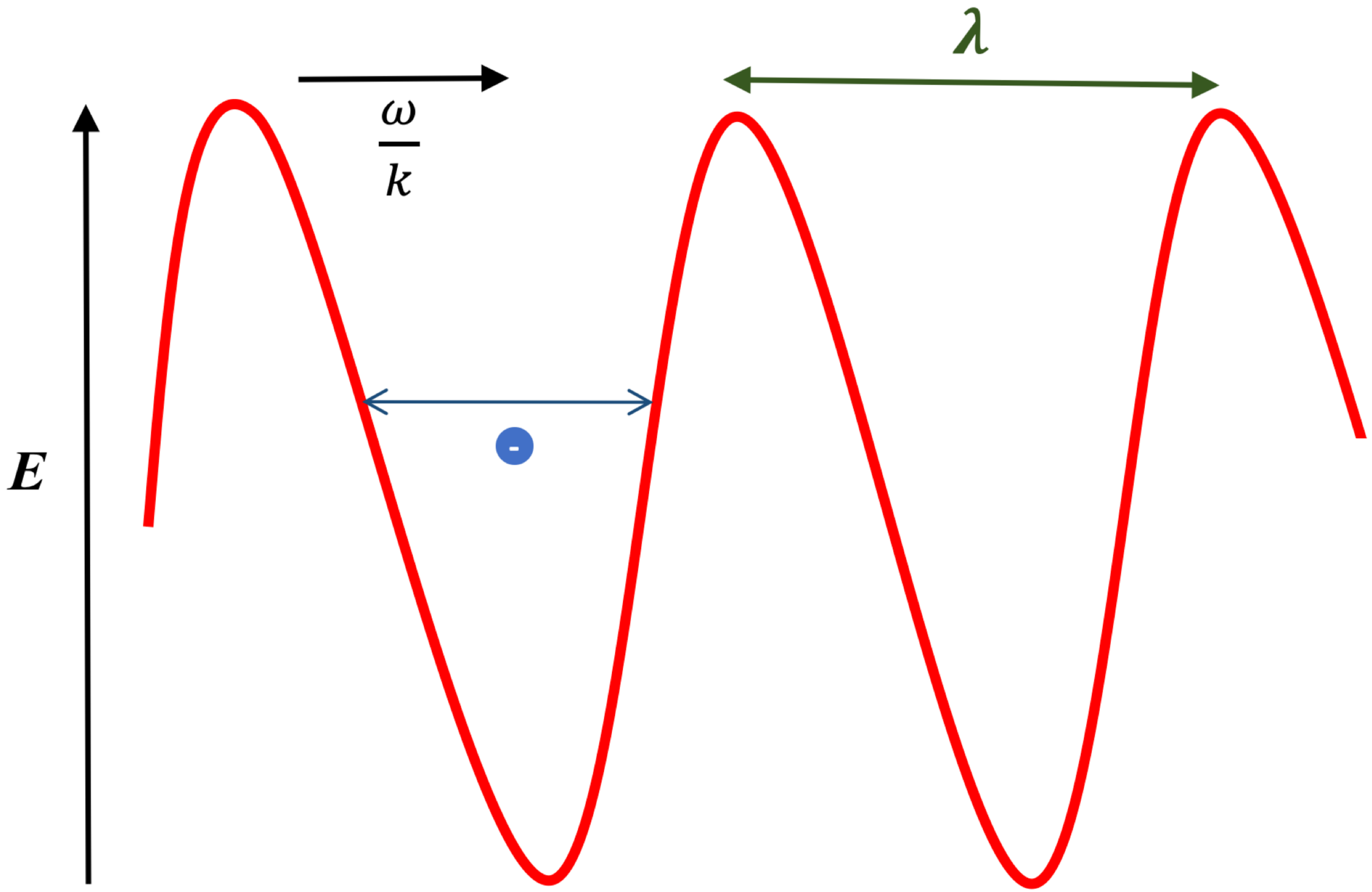
- Can only drive longitudinal waves for $\omega > \omega_{pe}$
- Electron plasma waves similar to sound waves, carry information at roughly the thermal velocity

Landau damping

- The correct solution of the Vlasov equation for electrostatic waves (electrons moving parallel to the k-vector) give a rise to a damping term.
- The pole in the integral occurs when electrons travel with a velocity equal to the phase velocity $\frac{\omega}{k}$ of the wave → **resonant phenomenon**. Electrons with velocity close to $\frac{\omega}{k}$ travel <u>with</u> the wave and get trapped in it and oscillate (potential well).
- Trapped electrons thus see an almost static field → they can be accelerated (if v_z slightly less than ω/k) or decelerated (if v_z slightly more than ω/k) by the wave.

Landau damping

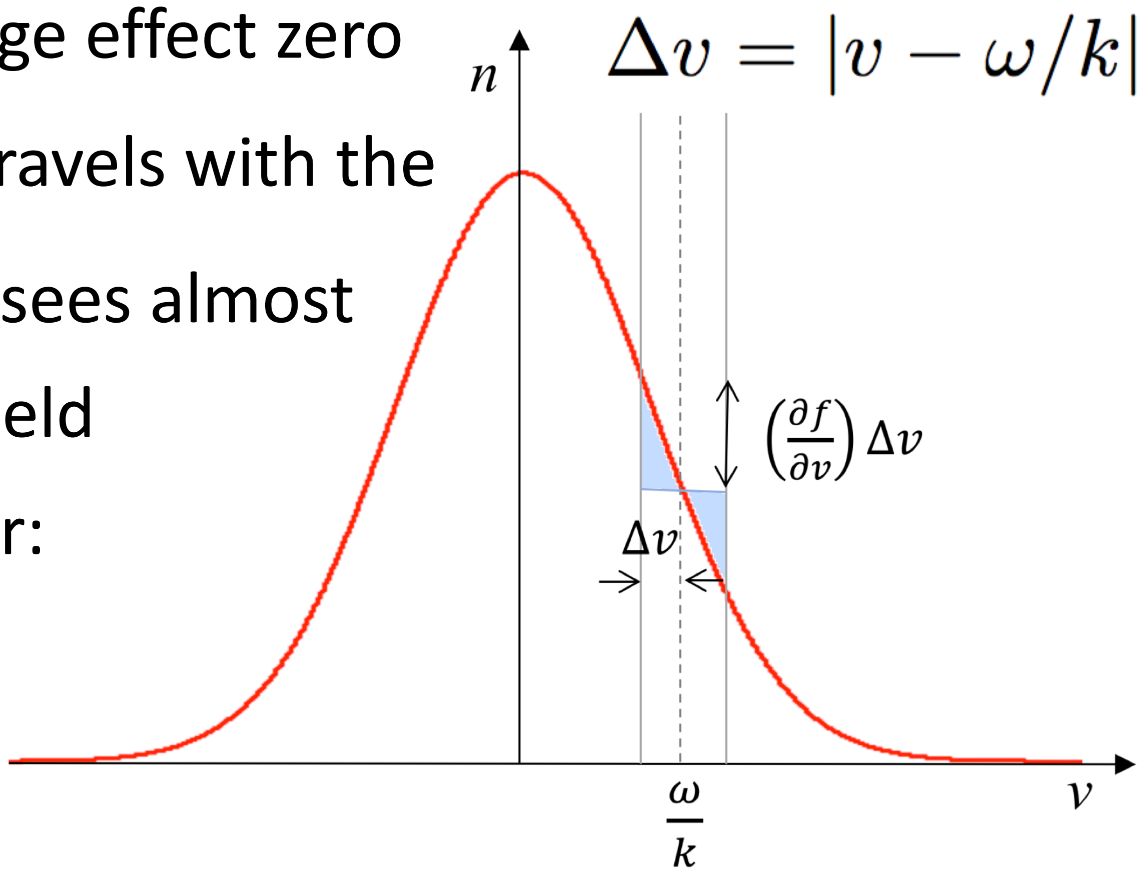
- Trapped particle oscillates within the field of the wave



Landau damping

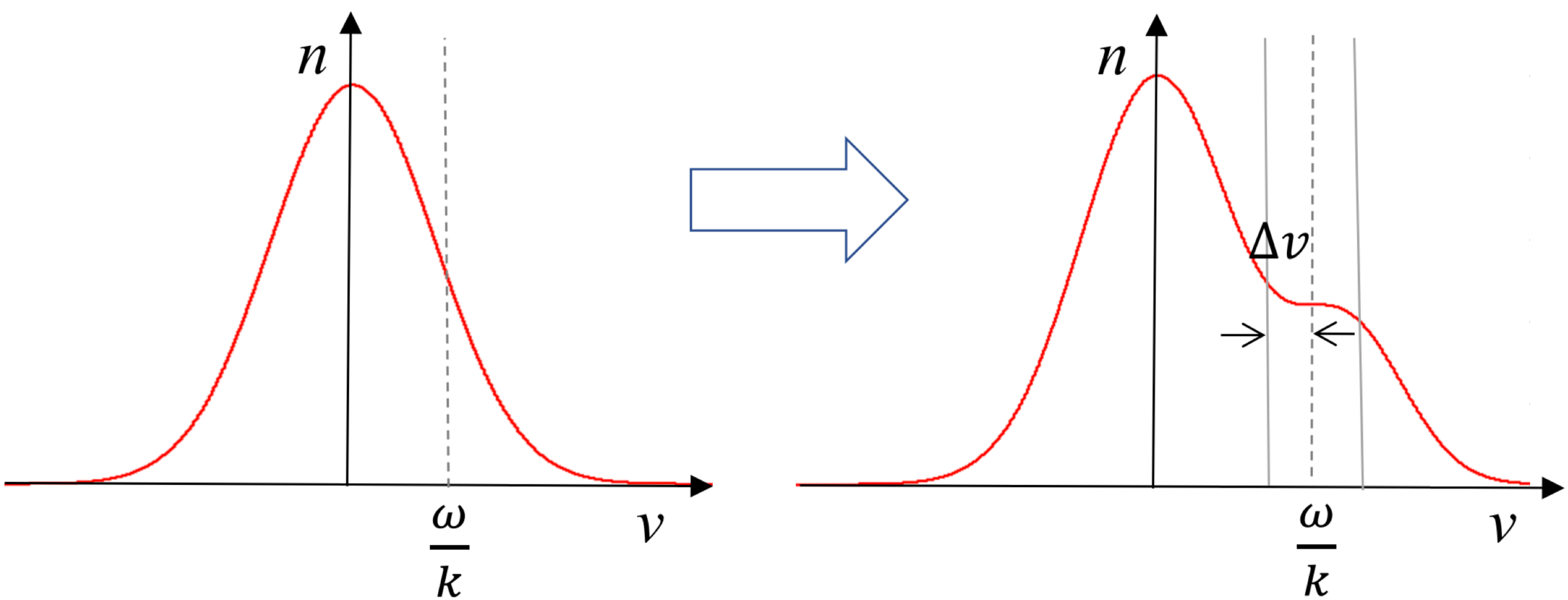
- $v_z \gg \frac{\omega}{k}$ → electron passes through the wave unchanged
- $v_z \ll \frac{\omega}{k}$ → electron oscillates with the wave, but average effect zero
- $v_z \approx \frac{\omega}{k}$ → electron travels with the wave and sees almost a steady field
- Electrons trapped for:

$$\frac{\omega}{k} - \Delta v < v < \frac{\omega}{k} + \Delta v$$



Landau damping

- For a Maxwellian distribution function, there are always more particles travelling more slowly than the wave, than faster than it.
- The wave accelerates particles and thus loses energy to them → the wave is damped (Landau damping).

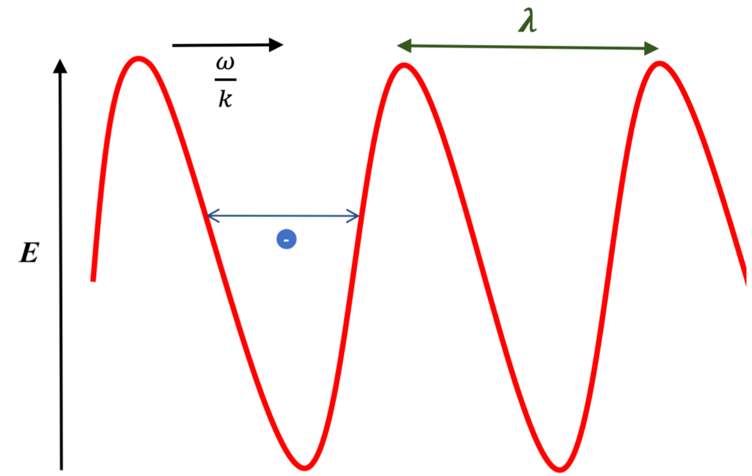


Landau damping

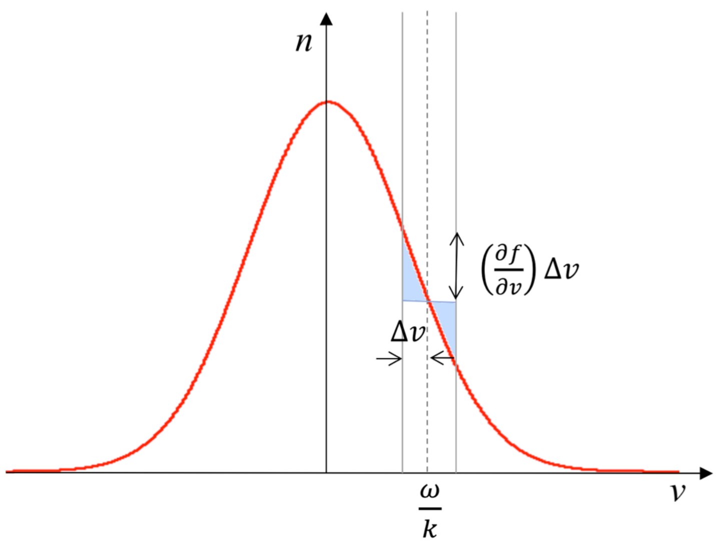
- Electron oscillates in a potential well:

$$\frac{1}{2}m(\Delta v)^2 \leq e\phi_0$$

$$\Delta v \leq \left(\frac{2e\phi_0}{m}\right)^{1/2}$$



- More particles accelerated than decelerated (Maxwellian) → **wave gives up energy to the electrons.**
- Self-limited, once $\frac{\partial f}{dv} = 0$, damping turns off.



Landau damping

- We explore the physical picture to get an approximate expression for the Landau damping rate.
- The potential associated with the wave ϕ_0 can be estimated as half the field amplitude times its wavelength: $E\lambda/2$
- We plug in to obtain the velocity range of trapped particles:

$$\Delta v = |v - \omega/k|$$

$$\Delta v \approx \left(\frac{eE}{mk}\right)^{1/2}$$

Landau damping

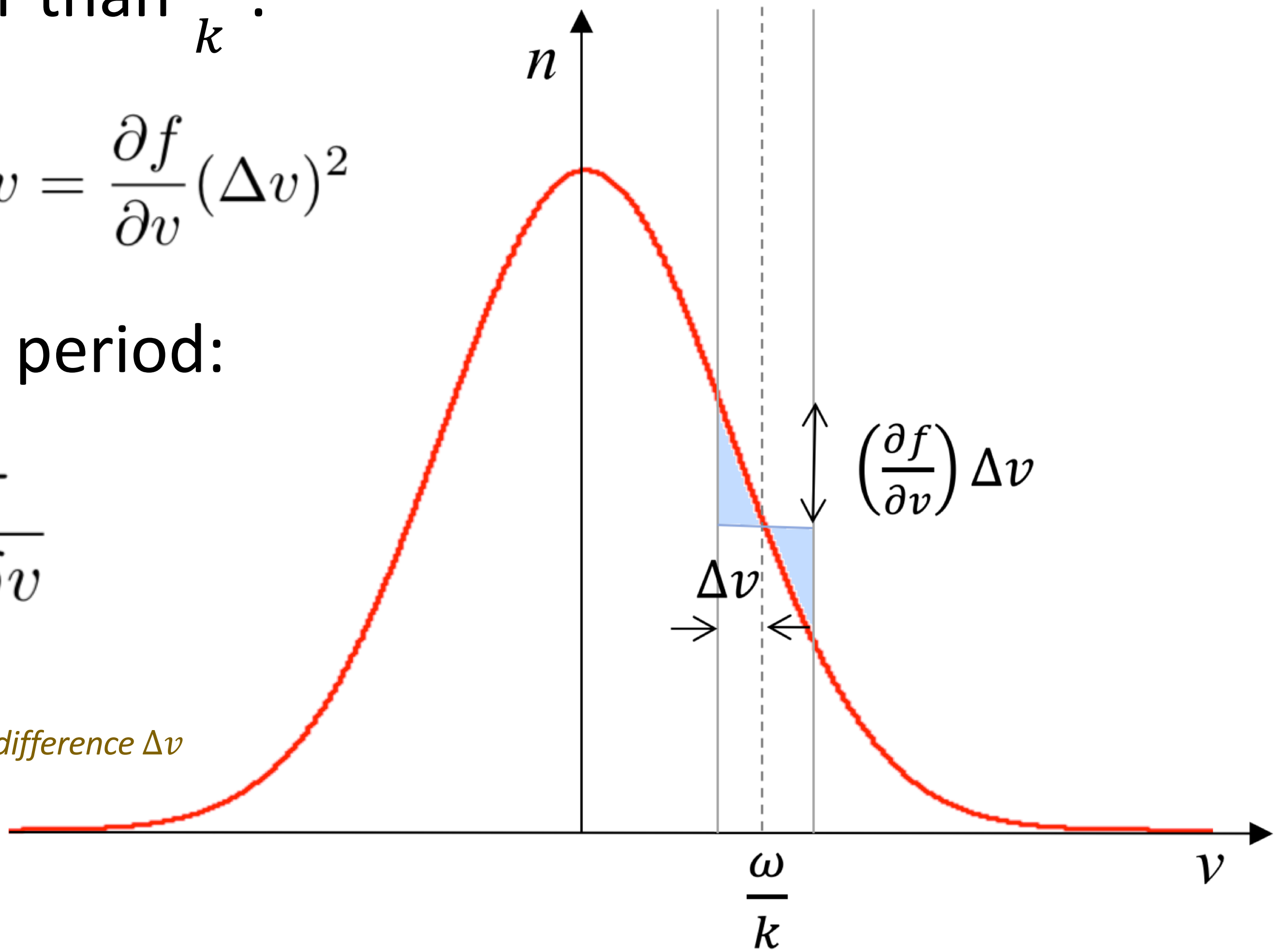
- Estimate the excess of trapped particles with initial velocities lower than $\frac{\omega}{k}$:

$$n_{v+} \approx \Delta v \frac{\partial f}{\partial v} \Delta v = \frac{\partial f}{\partial v} (\Delta v)^2$$

- One oscillation period:

$$\tau \approx \frac{\lambda}{2\delta v} = \frac{\pi}{k\delta v}$$

Set to the total velocity difference Δv



Landau damping

- Estimate the energy density of the electrostatic field as:

$$U_E = \frac{1}{2}\varepsilon_0 E^2$$

- Thus, the power loss of the wave: $P = \dfrac{dU_E}{dt}$

$$P = \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_0 \frac{E^2}{2}\right) = \varepsilon_0 E \frac{\mathrm{d}E}{\mathrm{d}t}$$

- Then, for one oscillation period τ:

$$P = \varepsilon_0 E \frac{\delta E}{\tau}$$

Landau damping

- The total power loss then:

$$\text{Power} \quad = \quad (\text{No. of particles}) \times (\text{EnergyLost}) \frac{1}{(\text{Time})}$$

$$P = \varepsilon_0 E \frac{\delta E}{\tau} = \left(\frac{\partial f}{\partial v} (\Delta v)^2 \right) \times (mv\Delta v) \times \left(\frac{k\Delta v}{\pi} \right)$$

$$\approx \left(\frac{\partial f}{\partial v} \right) mv\Delta v^4 k$$

- Substituting for $\Delta v \approx (eE/mk)^{1/2}$, we get:

$$P \approx \left(\frac{\partial f}{\partial v} \right) \frac{ve^2E^2}{mk}$$

Landau damping

- Assume that the wave is dumped with the rate γ:

$$E = E_0 \exp(-\gamma t)$$

- Thus the dumping rate is: $$\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma E$$

$$\gamma = -\frac{1}{E}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{\varepsilon_0}\frac{P}{E^2}$$

- Substitute for P: $$\gamma = \left(\frac{\partial f}{\partial v}\right)\frac{e^2 v}{\varepsilon_0 m k} = \left(\frac{\partial f}{\partial v}\right)\frac{\omega_{\mathrm{pe}}^2 v}{n_e k}$$

- Given that $v = \omega/k \approx \omega_{pe}/k$: $$\gamma = \left(\frac{\partial f}{\partial v}\right)\frac{\omega_{\mathrm{pe}}^3}{k^2 n_e}$$

Landau damping

- Plug in 1D Maxwellian (simple analysis):

$$f = \left(\frac{n_e}{\sqrt{\pi}}\right)\left(\frac{m}{2k_B T}\right)^{3/2} v_z \exp\left(-\frac{m v_z^2}{2k_B T}\right)$$

- Get dumping rate:

$$\gamma \approx \frac{1}{\sqrt{\pi}}\frac{\omega_{\mathrm{pe}}^3}{k^2}\left(\frac{m}{2k_{\mathrm{B}}T}\right)^{3/2} v_z \exp\left(-\frac{m v_z^2}{2k_{\mathrm{B}}T}\right)$$

- Thus for $v = \omega/k \approx \omega_{pe}/k$:

Thermal velocity $\sqrt{k_B T/m}$

$$\gamma \approx \frac{1}{\sqrt{8\pi}}\frac{\omega_{\mathrm{pe}}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{m\omega_{\mathrm{pe}}^2}{2k^2 k_{\mathrm{B}}T}\right)$$

Plasma frequency

$$\gamma \approx \frac{1}{\sqrt{8\pi}}\frac{\omega_{\mathrm{pe}}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{1}{2k^2\lambda_{\mathrm{D}}^2}\right)$$

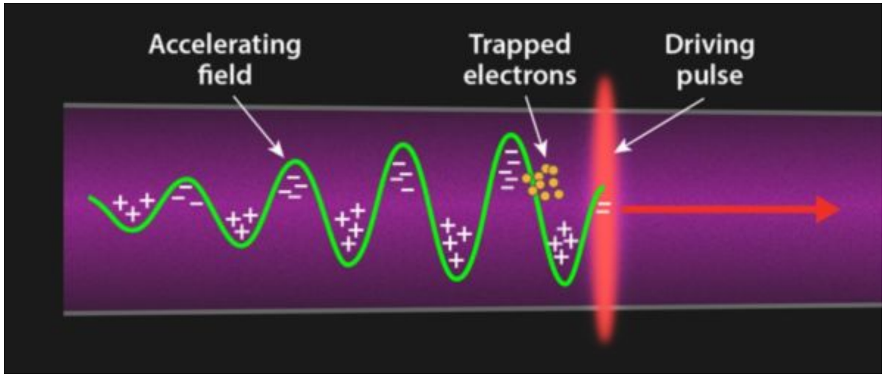
Debye length

Landau damping

- Waves are heavily damped (large γ) for wavelength close to or shorter than the Debye length (large $k\lambda_D$)
- Debye length is the distance a typical thermal electron travels in an oscillation period.
- Original assumption for light damping was that the phase velocity of the wave large compared with the thermal velocity, i.e. small kv_e/ω.
- The process is reversible → **can be used to drive plasma waves**
- As the wave damps, the electric field associated with it reduces and the faster particles eventually have enough energy to escape the trapping potential

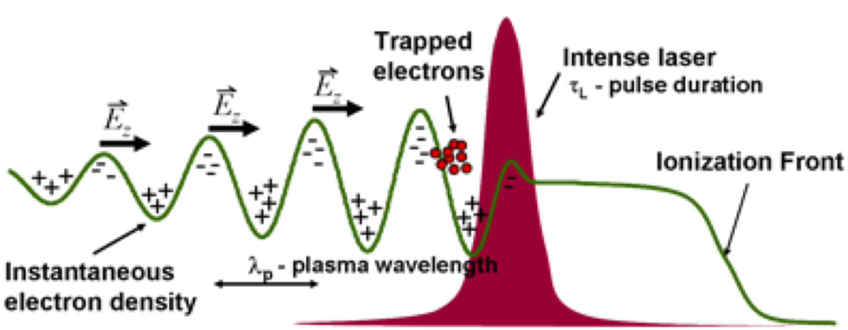
Plasma accelerators

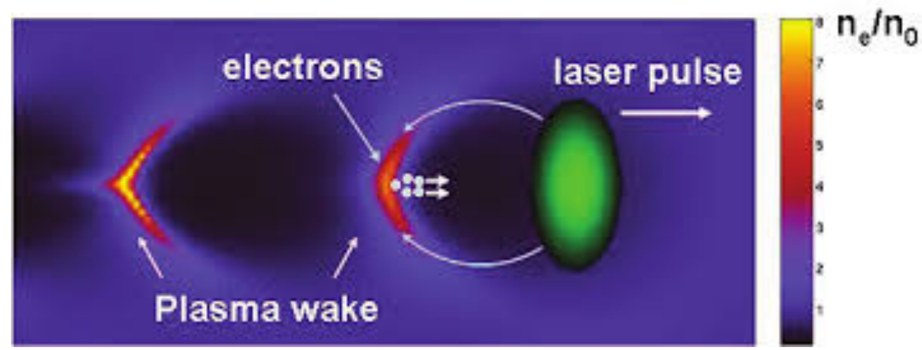
- New generation of particle accelerators also works by particles ‘surfing’ plasma waves.
- Inject a 30-fs very intense laser pulse into a plasma.
- Electrons oscillate in laser field, but due to gradient in field, get expelled.
- This leaves a ‘wake’ behind the pulse, almost devoid of electrons, with a huge E field.
- Any residual electrons trapped in this wake are accelerated to high energy



Plasma accelerators

- Length of wake region $\sim c\tau$
- The bubble is devoid of electrons, thus field $E \approx nec\tau/\varepsilon_0$
- Need max. 100 fs pulse to get laser field to eject electrons.
- Cannot have too high density or laser group velocity falls too far below c. Use $10^{23}\mathrm{m}^{-1}$.
- This leads to fields of $5\times10^{10}\mathrm{m}^{-1}$! i.e. a GeV in a couple of centimetres!
- Acceleration is ~1000 times greater than conventional accelerators.





Summary of lecture 4

- The continuity equation of the distribution function in phase space leads to the Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{u}(\nabla_r f) + \frac{q}{m}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot(\nabla_u f) = \left(\frac{\partial f}{\partial t}\right)_{collisions}$$

- Assuming periodic perturbations to the distribution function we recover our dispersion relation for plasma (Langmuir) waves:

$$\omega^2 = \omega_{\mathrm{pe}}^2 + 3k^2 v_e^2$$

- We find that the pole in the dispersion relation gives a rise to famous phenomenon of Landau damping at the rate of:

$$\gamma \approx \frac{1}{\sqrt{8\pi}}\frac{\omega_{pe}^4}{k^3}\left(\frac{1}{v_e^3}\right)\exp\left(-\frac{1}{2k^2\lambda_D^2}\right)$$