Plasma Physics

TU Dresden

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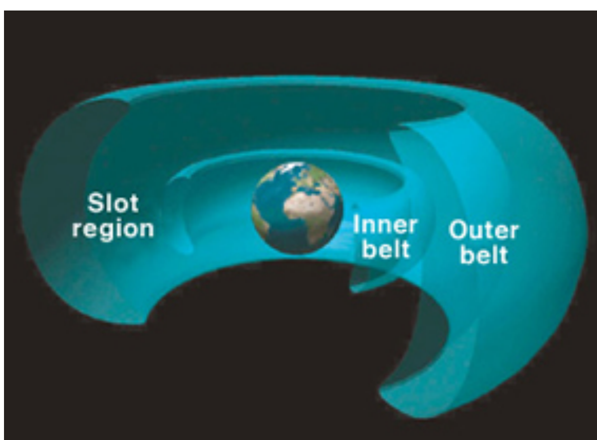
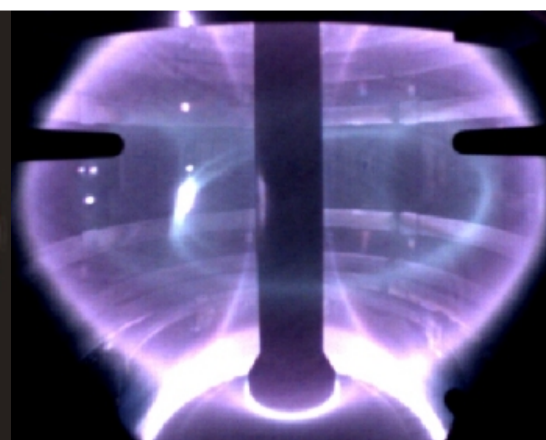
Lecture 5: Magneto-hydrodynamics






Plasma Physics: lecture 5

- Principles of MHD macroscopic models for plasma
- Continuity, momentum and energy equations
- Moments of the distribution functions
- Ohm’s law and the induction equation
- Magnetic flux freezing
- Magnetic pressure
- Plasma β parameter

Magneto-hydrodynamics

- For a large number of particles it is difficult to solve individual equations of motion.
- The primary weakness of kinetic models is that the modification of E and B fields by particle motion is neglected.
- The fluid model overcomes these issues.
- MHD solves the motion of fluid elements instead of tracing the individual particles.
- Two fluid model: electrons and ions two separate fluids that penetrate each other.
- **NB: in fluid dynamics particles strongly coupled, not the case in ideal plasmas!**

MHD basic assumptions

- Quasi-neutrality assumed → cannot model $\tau < \frac{1}{\omega_p}$ and $L < \lambda_{Debye}$ → **macroscopic model of plasma**
- Pressure scalar
- Velocity much smaller than speed of light
- Typical length scales much larger than kinetic length scales, e.g. gyro radii, skin depth, etc.
- Typical time scales much slower than kinetic time scales, e.g. gyro frequencies

MHD basic assumptions

- Does not track individual particles
- Only valid if particles localized by:
  - Collisions: $\lambda_{mfp} \ll L$
  - Magnetic field: $r_{Larmor} \ll L$ → **MHD cannot be used without B-fields!**
- Momentum exchange by collisions between species or by heavier ions dragging electrons along due to charge imbalance (if collisionless).
- MHD is a theory describes large-scale and slow phenomena compared to kinetic theory.

Components of MHD

- Mass density – ρ (kg/m^3)
- Fluid velocity – u (m/s)
- Internal energy – U (J/m^3)
- Pressure – P (Pa)
- Magnetic field – B (Tesla)
- Current density – j (A/m^2)

Maxwell's equations

- Gauss' law: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$
- Gauss' law for B-fields: $$\nabla \cdot \mathbf{B} = 0$$

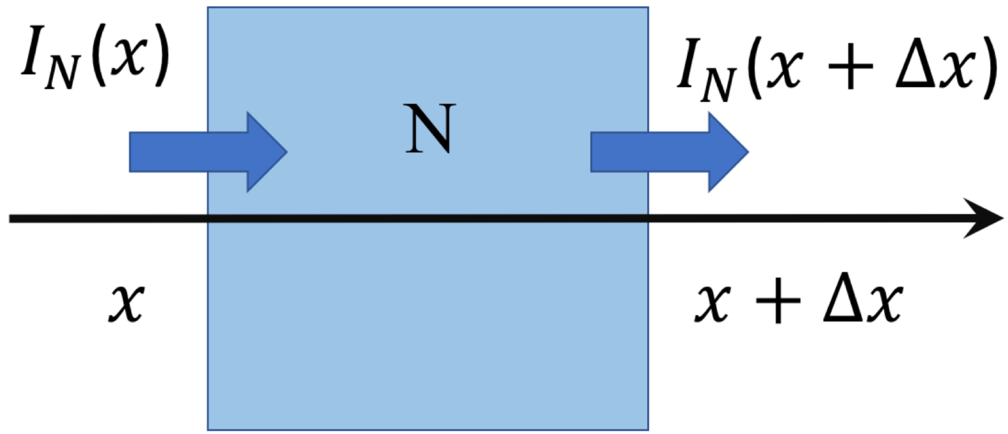
No magnetic monopoles

- Faraday's law: $$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Electromagnetic induction

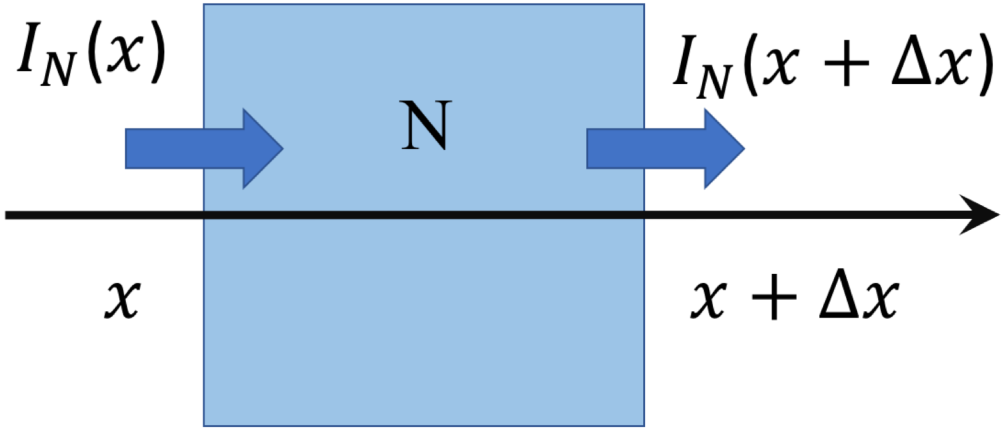
- Ampère's law: $$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

Continuity equation



- Conservation of particles → balance for the number of particles in a fixed cell of size $\Delta V = \Delta x \Delta y \Delta z$.
- The number of particles inside the interval $[x, x + \Delta x]$ is $\mathrm{N} = \mathrm{nA} \cdot \Delta x$ as $\mathrm{A} = \Delta y \Delta z$.
- The incident flux is: $I_N = n \cdot \Delta y \Delta z \cdot u_x$

Continuity equation



- When this flux is accelerated/decelerated by external forces, the flux on the exit side is larger/smaller and thus the number of particles inside the cell is diminished/increased:

$$-\frac{\partial N}{\partial t} = I_N(x+\Delta x) - I_N(x) \approx \frac{\partial I_N}{\partial x}\Delta x$$

Continuity equation (mass conservation)

- Before, we obtained:

$$\boxed{\frac{\partial n}{\partial t} + \nabla(n\mathbf{u}) = 0}$$

- Which can be rewritten in terms of current density:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0}$$

- For charge density: $\rho = \sum_\alpha n_\alpha q_\alpha$

- And current density: $\mathbf{j} = \sum_\alpha n_\alpha q_\alpha \mathbf{u}_\alpha$

Momentum equation

- MHD = hydrodynamics + Lorentz force:

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = q(E + \mathbf{v}\times\mathbf{B})$$

- For inhomogeneous flow:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial\mathbf{u}}{\partial t} + \frac{\partial\mathbf{u}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial\mathbf{u}}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial\mathbf{u}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}$$

- Vector $(\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, \mathrm{d}z/\mathrm{d}t)$ is just the velocity in the cell, thus:

Convective derivative

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}$$

Momentum equation

- Substitute into the expression for the Lorentz force and multiply by particle density n to obtain the balance of internal forces for many-particle system:

Mass density ρ

$$nm\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(E + \mathbf{v}\times\mathbf{B})$$

- Particle flux: $\Delta I_N(v_x) = v_x \Delta n(v_x)\,\Delta y \Delta z$

- By analogy, momentum flux:

$$\Delta I_p = m v_x \Delta n(v_x)|v_x|\Delta y \Delta z$$

Momentum equation

- The momentum balance in the interval $[\mathrm{x}, \mathrm{x} + \Delta\mathrm{x}]$:
  - Gain at x :
  $$\Delta I_p^+(x) = \sum_{v_x>0} [\Delta n(v_x)(mv_x)|v_x|]_x \, \Delta\mathrm{y}\Delta\mathrm{z}$$
  - Loss at x :
  $$\Delta I_p^-(x) = \sum_{v_x<0} [\Delta n(v_x)(mv_x)|v_x|]_x \, \Delta\mathrm{y}\Delta\mathrm{z}$$
  - Gain at $\mathrm{x} + \Delta\mathrm{x}$:
  $$\Delta I_p^+(x+\Delta x) = \sum_{v_x>0} [\Delta n(v_x)(mv_x)|v_x|]_{x+\Delta x} \, \Delta\mathrm{y}\Delta\mathrm{z}$$
  - Loss at $\mathrm{x} + \Delta\mathrm{x}$:
  $$\Delta I_p^-(x+\Delta x) = \sum_{v_x<0} [\Delta n(v_x)(mv_x)|v_x|]_{x+\Delta x} \, \Delta\mathrm{y}\Delta\mathrm{z}$$

Momentum equation

- The net gain/loss in momentum:

$$\frac{\partial p_x}{\partial t} = \Delta I_p^+(x) - \Delta I_p^+(x + \Delta x) + \Delta I_p^-(x) - \Delta I_p^-(x + \Delta x)$$

- Taylor expand and set the negative velocity intervals $|v_x|$ to $-v_x$:

$$\frac{\partial p_x}{\partial t} = -m \sum_{v_x=-\infty}^{\infty} \left([\Delta n(v_x) v_x^2]_{x+\Delta x} - [\Delta n(v_x) v_x^2]_x \right)$$

$$\rightarrow \quad \frac{\partial}{\partial t}(nmu_x)\Delta x \Delta y \Delta z = -m \frac{\partial}{\partial x}(n\langle v_x^2 \rangle)\Delta x \Delta y \Delta z$$

Momentum equation

- Remembering the distribution function:

$$n\langle v_x^2\rangle = \int f(v_x)v_x^2 \cdot \mathrm{d}v_x$$

- The velocities break down into, the mean flow u_x and random thermal motion $\tilde{v}_x$: $v_x = u_x + \tilde{v}_x$

- Thus, the momentum conservation:

$$\frac{\partial}{\partial t}(nmu_x) = -m\frac{\partial}{\partial x}[n(\langle u_x^2\rangle + 2u_x\langle\tilde{v}_x\rangle + \langle\tilde{v}_x^2\rangle)]$$

Momentum equation

- From 1-D Maxwellian: $\frac{1}{2}m\,\langle\tilde{v}_x^2\rangle = \frac{1}{2}k_B T$

- By definition, average random motion is $\langle\tilde{v}_x\rangle = 0$

- Hence, the momentum balance becomes:

$$\frac{\partial}{\partial t}(nmu_x) = -\frac{\partial}{\partial x}\left[\underbrace{nmu_x^2}_{\text{Stagnation pressure}} + \underbrace{nk_B T}_{\text{Thermal pressure}}\right]$$

$$\rightarrow \quad \frac{\partial}{\partial t}(nmu_x) = -\frac{\partial P}{\partial x} \quad \rightarrow \quad \boxed{\rho\frac{\mathrm{d}u_x}{\mathrm{d}t} = -\frac{\partial P}{\partial x}}$$

Momentum equation

- Using continuity equation, we obtain:

$$nm\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x}$$

- And: $$nm\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(E + \mathbf{v}\times\mathbf{B})$$

- The full momentum transport equation:

$$\boxed{nm\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(E + \mathbf{v}\times\mathbf{B}) - \nabla P}$$

Momentum equation

- Generalize the MHD momentum equation by substituting the current density:

$$\boxed{\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = \mathbf{j}\times\mathbf{B} - \nabla P + F}$$

All external forces

- Current density due to the relative motion between electrons and ions:

$$\mathbf{j} = -n_e e\mathbf{u}_e + n_i e\mathbf{u}_i = ne(\mathbf{u}_i - \mathbf{u}_e)$$

Energy equation

- Can be written in different forms depending on thermodynamic variables used:

Specific internal energy

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon\right) + \nabla\left[\left(\frac{\rho u^2}{2} + \rho\epsilon + P\right)\cdot\mathbf{u}\right] = 0$$

- From intuitive analysis energy conservation follows:

$$\boxed{\overbrace{\frac{\partial \mathrm{U}}{\partial t} + (\mathbf{u}\cdot\nabla)U}^{\frac{\mathrm{d}U}{\mathrm{d}t}} = -P\nabla\cdot\mathbf{u} - \nabla\cdot(-\mathrm{K}\nabla\mathrm{T}) + \eta j^2 - \text{radiation}}$$

Fixed lab frame

Moving with fluid

Compressional heating/cooling

Heat flux (thermal conduction)

Ohmic heating

Energy equation (MHD)

- Consider the Poynting vector to include the effects of the magnetic field (EM field):

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{(\mathbf{u}\times\mathbf{B})\times\mathbf{B}}{\mu_0} = \frac{\mathbf{u}B^2}{\mu_0} - \frac{(\mathbf{u}\cdot\mathbf{B})\mathbf{B}}{\mu_0}$$

- The full MHD form thus follows:

Magnetic pressure *Magnetic tension force*

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon + \frac{B^2}{2\mu_0}\right) + \nabla\left[\left(\frac{\rho u^2}{2} + \rho\epsilon + P + \frac{B^2}{\mu_0}\right)\cdot\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{B})\mathbf{B}}{\mu_0}\right] = 0}$$

- See supplemental material for further notes on the energy equation.

Moments of the distribution function

- Relate the fluid description (MHD) to the kinetic theory.
- The Vlasov equation:

$$\frac{\partial f}{\partial t} + \mathbf{u}(\nabla_r f) + \frac{q}{m}\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot (\nabla_u f) = \left(\frac{\partial f}{\partial t}\right)_{collisions}$$

- n^{th} moment in velocity:

$$\int \mathbf{v}^n \frac{\partial f}{\partial t} \partial^3 \mathbf{v} + \int \mathbf{v}^n\ (\mathbf{u} \cdot \nabla_r) f \partial^3 \mathbf{v} + \frac{q}{m}\ \int \mathbf{v}^n\, (\mathbf{E} + \mathbf{v}\times\mathbf{B}) \cdot (\nabla_u f) \partial^3 \mathbf{v}$$

$$= \int \mathbf{v}^n \left(\frac{\partial f}{\partial t}\right)_{collisions}$$

- 0^{th} moment → **continuity equation**
- 1^{st} moment → **momentum equation**
- 2^{nd} moment → **energy equation**

Induction equation

- Ohm’s law in simplified form:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{v}\times\mathbf{B}) = \frac{1}{\eta}(\mathbf{E} + \mathbf{v}\times\mathbf{B})$$

Conductivity *Resistivity*

- Faraday’s law: $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
- Ampère’s law: $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$
- By substituting into the Ohm’s law, we get simplified version of the induction equation:

$$\boxed{\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} = -\frac{\eta}{\mu_0}\nabla\times(\nabla\times\mathbf{B}) + \nabla\times(\mathbf{v}\times\mathbf{B})}$$

Diffusive term *Convective term*

Resistive and ideal MHD

- Continuity equation: $$\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0$$
- Momentum equation: $$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} - \nabla P + F$$
- Energy equation (a version of):

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho \epsilon + \frac{B^2}{2\mu_0}\right) + \nabla\left[\left(\frac{\rho u^2}{2} + \rho \epsilon + P + \frac{B^2}{\mu_0}\right) \cdot \mathbf{u} - \frac{(\mathbf{u} \cdot \mathbf{B})\mathbf{B}}{\mu_0}\right] = 0$$

- Induction equation: $$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B})$$
- For ideal MHD, ignore the resistive terms η
- NB: These equations valid for $T_e = T_i = T$

Magnetic diffusion

- Consider plasma at rest **v** = 0, the induction equation reduces to time-dependent diffusion equation:

$$\frac{\partial \mathbf{B}}{\partial t} + \frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t} + \mathrm{D_B} \Delta \mathbf{B} = 0$$

- Where $\mathrm{D_B} = \frac{\eta}{\mu_0}$ is the diffusion coefficient.

- Estimate the diffusion time by setting:

$$\mathbf{B}(t) \propto \exp(-t/\tau_B)$$

Magnetic diffusion

- Thus, replace Laplacian by the square of a characteristic scale length:

$$\Delta \mathbf{B} = \mathbf{B}/l^2$$

- Obtain estimated diffusion time for the magnetic field:

$$\tau_B = \frac{\mu_0 l^2}{\eta}$$

- With decreasing resistivity the diffusion time increases.
- Inside Earth's core we obtain $\tau_B = 10^4$ years

Magnetic Reynolds number

- Starting from induction equation
- Diffusive term: $\frac{\eta}{\mu_0}\nabla\times(\nabla\times\mathbf{B}) \approx \frac{\eta}{\mu_0}\frac{B}{l^2}$
- Convective term: $\nabla\times(\mathbf{v}\times\mathbf{B}) \approx \frac{vB}{l}$
- Magnetic Reynolds number definition:

$$\boxed{R_m = \frac{|\nabla\times(\mathbf{v}\times\mathbf{B})|}{\left|\frac{\eta}{\mu_0}\nabla\times(\nabla\times\mathbf{B})\right|} = \frac{\mu_0 v l}{\eta}}$$

- R_m characterizes the ratio of mass flow to magnetic diffusion.

Frozen-in magnetic flux

- Hot plasmas have a very large conductivity compared to metals.
- Thus, hot ideal plasmas (astrophysical plasmas) can be assumed to have zero resistivity (ideal MHD).
- For $\eta \to 0$, $R_m \to \infty$, the induction equation reduces to:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

- And the continuity equation gives:

$$\nabla \cdot \mathbf{v} = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla)\rho\right) = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

Frozen-in magnetic flux

- Using a vector identity, we get:

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B} \cdot \nabla)\mathbf{v} + \frac{\mathbf{B}}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

- Use the quotient rule:

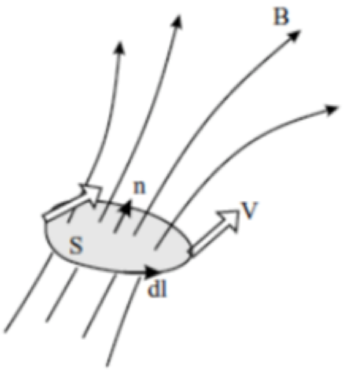
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \frac{1}{\rho}\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} - \frac{\mathbf{B}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t}$$

- Truesdell theorem:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right)\mathbf{v}}$$

Frozen-in magnetic flux for $\eta \to 0$

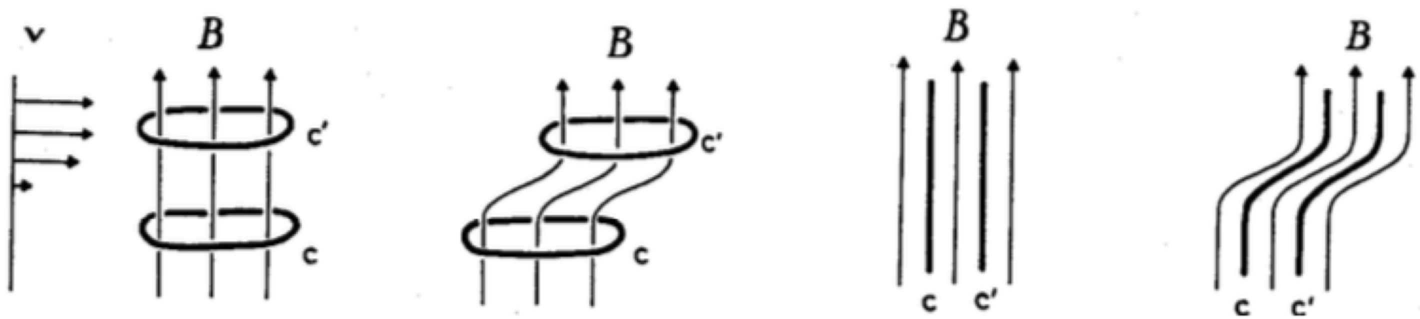
- Quantity $^{B}/_{\rho}$ is equivalent to the number of field lines per unit mass
- When mass flows perpendicular to the magnetic field the r.h.s. vanishes:
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = 0$$
- Mass motion can only occur together with the magnetic field



- This also means that the magnetic flux: $\frac{\mathrm{d}\Phi_B}{\mathrm{d}t} = 0$
- The magnetic flux is “frozen in” the plasma
- If plasma is compressed, mag. field increases

Magnetic diffusion & frozen-in flux

- Highly resistive plasmas: $R_m \ll 1$
  - Plasma can move through field
- Partially resistive plasmas: $R_m{\sim}1$
  - Convection stretching field lines, resistive diffusion straightening field lines
- Highly conductive plasmas: $R_m \gg 1$
  - Highly twisted field line result, Solar surface (R_m>10^4)



Magnetic pressure

- Start from Ampère's law: $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$

- Magnetic force:

$$\mathbf{j} \times \mathbf{B} = \nabla P = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B})$$

$$\mathbf{j} \times \mathbf{B} = -\frac{1}{2\mu_0}\nabla(B^2) + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}$$

- Momentum equation: $\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{j} \times \mathbf{B} - \nabla P$

Magnetic pressure

- Substitute for $\mathbf{j}\times\mathbf{B}$:

Magnetic pressure gradient

$$\boxed{\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla P - \frac{\nabla B^2}{2\mu_0} + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}}$$

Thermal pressure gradient

Magnetic tension force

- Pressure balance:

$$\nabla(P + P_B) = \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B}$$

Gas pressure

- Magnetic pressure in plasma: $\boxed{P_B = \frac{B^2}{2\mu_0}}$

Plasma β parameter

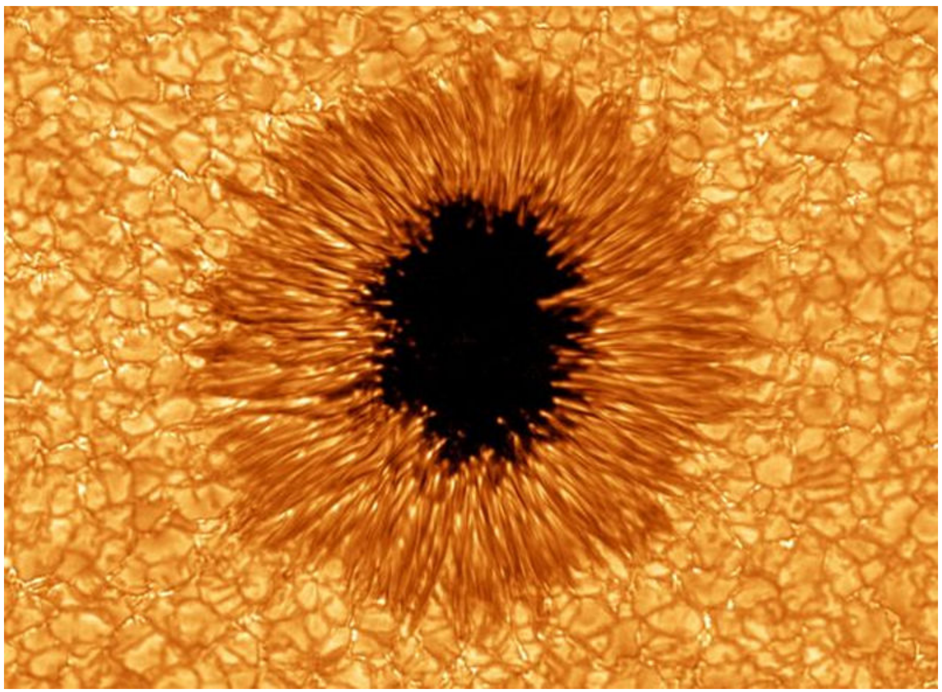
- The ratio of the plasma kinetic (thermal) pressure and the magnetic pressure:

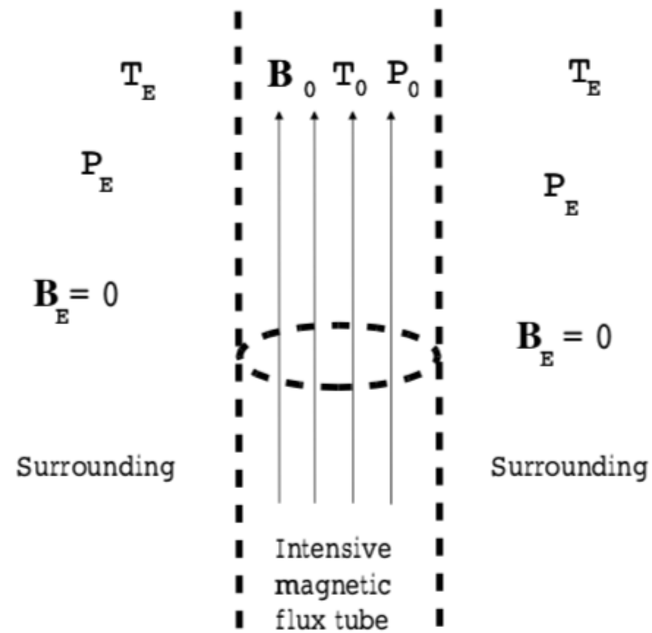
$$\beta = \frac{P}{B^2/_{2\mu_0}} = \frac{2\mu_0 P}{B^2} \approx \frac{2\mu_0 n k_B T}{B^2}$$

- $\beta \gg 1$ → **Particles dominant** → dominate dynamics
- $\beta \sim 1$ → **particles ~ fields** (typical in astrophysics)
- $\beta \ll 1$ → **Field dominant** (ignore particles)

Magnetic diffusion & frozen-in flux

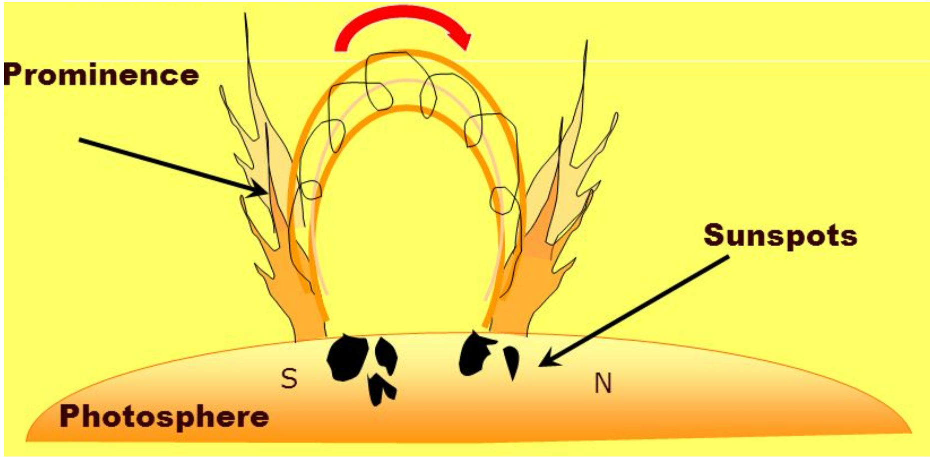
- Flux tubes:

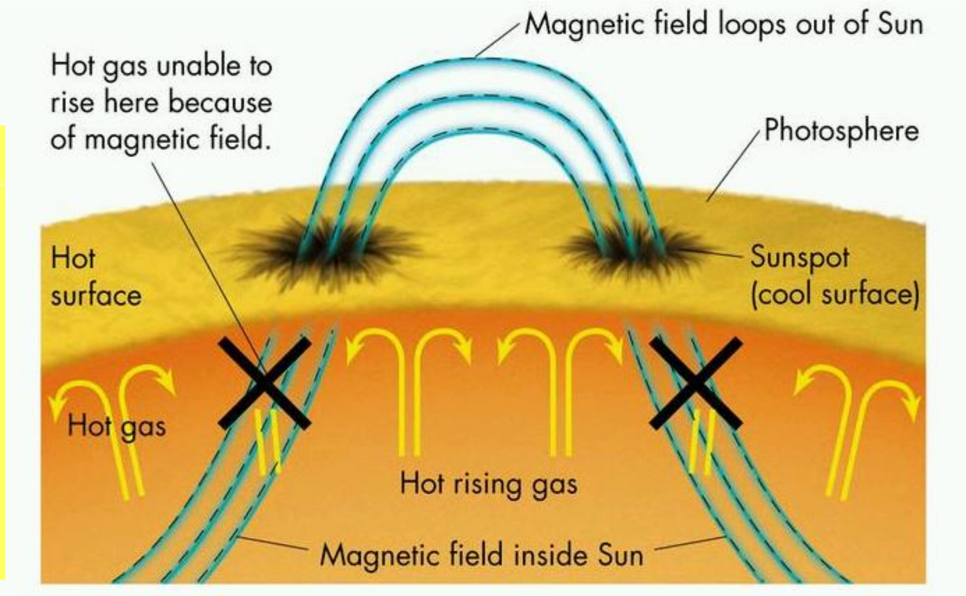


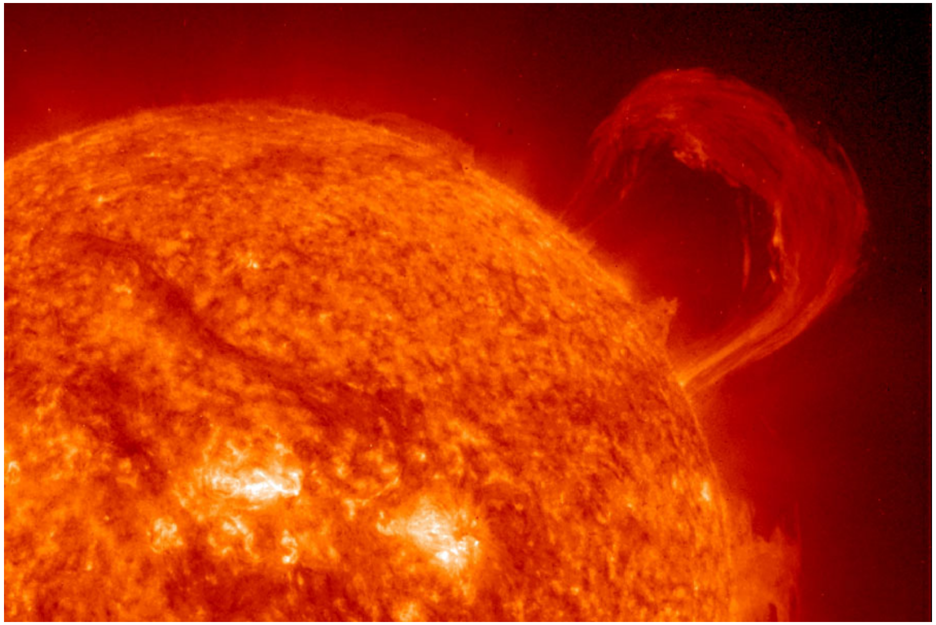


- Sunspots are an example of frozen-in flux (flux tubes) with strong magnetic fields

Solar prominence







Summary of lecture 5

- Basic MHD equations:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0$$

Continuity equation: mass conservation

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{j}\times\mathbf{B} - \nabla P + F$$

Momentum equation

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon + \frac{B^2}{2\mu_0}\right) + \nabla\left[\left(\frac{\rho u^2}{2} + \rho\epsilon + P + \frac{B^2}{\mu_0}\right)\cdot\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{B})\mathbf{B}}{\mu_0}\right] = 0$$

Energy equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \frac{\eta}{\mu_0}\nabla\times(\nabla\times\mathbf{B})$$

Induction equation

Summary of lecture 5

- Magnetic Reynolds number: $$R_m = \frac{\mu_0 v l}{\eta}$$
- Magnetic pressure: $$P_B = \frac{B^2}{2\mu_0}$$
- Frozen-in flux: $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right) \mathbf{v}$$
- Plasma beta: $$\beta = \frac{P}{B^2/2\mu_0} = \frac{2\mu_0 P}{B^2}$$

Supplemental material for lecture 5

Energy equation derivation

- A nice overview of the energy equation derivation can be found here: https://www.youtube.com/watch?v=fjsxYmpth6A
- The derivation follows the same way as the momentum conservation equation, however the main difference is what components of the energy equation are being included.
- Here we start with the general fluid energy conservation equation that includes the kinetic, internal energy with rates of heat flux, gravitational potential energy flux, and the rate of viscous dissipation.
- Note that energy flux due to any other external force can be added. Many terms (gravity, viscosity, etc.) can be often ignored.
- We will also consider energy flux due to magnetic pressure and tension force.

Energy components

- Consider different forms of energy conservation:
- **Kinetic energy flux rate:** $E_K = \frac{1}{2}\rho u^2 . \mathbf{u}$
- **Internal energy:** $\epsilon = \frac{\alpha}{2} k_B T$ and for ideal gas: $PV = nk_B T$

Degrees of freedom

$$\rightarrow \frac{P}{\rho} = k_B T \rightarrow \epsilon = \frac{\alpha}{2} \cdot \frac{P}{\rho} = \frac{1}{\gamma - 1} \cdot \frac{P}{\rho} \quad \text{and} \quad \gamma = \frac{C_p}{C_v} = \frac{\alpha + 2}{\alpha} = \frac{5}{3}$$

For monatomic ideal gas

Assuming adiabatic (entropy preserving) process: $\frac{dP}{P} = \gamma \frac{d\rho}{\rho}$

- **Work done:** $W = PdV = \left[\frac{P}{\rho}\right]$
- **Enthalpy:** $H = \epsilon + PdV = \epsilon + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1} \cdot \frac{P}{\rho} \rightarrow \rho\epsilon = \frac{P}{\gamma - 1}$

Energy components

- **Gravity (energy flux rate):** $E_g = \rho \mathbf{u} \cdot \mathbf{g}$

Viscous stress tensor

- **Viscous energy dissipation rate:** $\Psi = \sum_{i,j} \pi_{ij} \frac{\partial u_i}{\partial x_j} = \pi \nabla \mathbf{u}$
- **For EM energy we use the Lorentz force:** $m \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = q(E + \mathbf{u} \times \mathbf{B})$
- **Poynting vector for EM energy flux:** $\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$
- **Conductive heat flow from Fourier's law:** $\mathbf{F}_q = -K_q \nabla T$

Coefficient of thermal conductivity

- **Ohmic heating:** ηj^2
- **Radiation losses:** e.g. Bremsstrahlung
- etc.

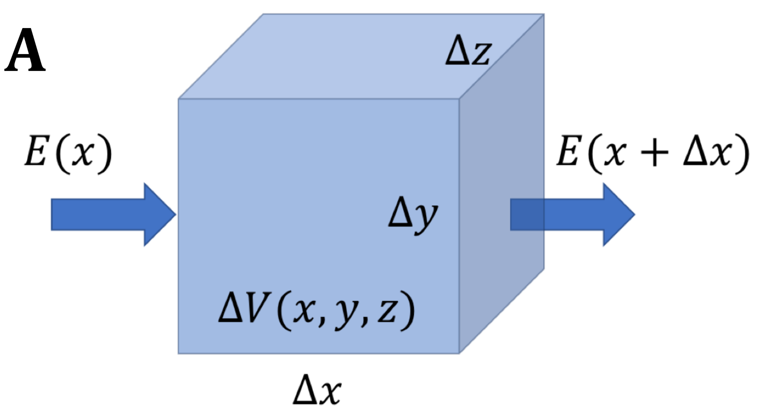
Conservation of energy

- Consider energy flux rate through a volume $\Delta V = \Delta\mathrm{x}\Delta\mathrm{y}\Delta\mathrm{z}$

Rate of energy accumulation = *Rate of energy flux in* − *Rate of energy flux out* + *Rate of added heat flux* + *Rate of work done*

+ *Other energy flux components (e.g. gravitational potential, viscosity, material stress, B-field, heat conduction, radiation, etc.)*

- Energy per unit mass: $E = \epsilon_i + E_K = \epsilon_i + \frac{1}{2}u^2$ *Also denoted as U*
- Rate of energy accumulation: $\frac{\partial}{\partial t}(\rho E \mathbf{u} \Delta x \Delta y \Delta z)$
- Energy flow rate through a surface: $\rho E \mathbf{u}.\mathbf{A}$



= *Rate of energy flux in* − *Rate of energy flux out*

$$= [\rho E u_x]_\mathrm{x}\Delta\mathrm{y}\Delta\mathrm{z} + [\rho E u_y]_\mathrm{y}\Delta\mathrm{z}\Delta\mathrm{x} + [\rho E u_z]_\mathrm{z}\Delta\mathrm{x}\Delta\mathrm{y}$$

$$-[\rho E u_x]_{x+\Delta x}\Delta\mathrm{y}\Delta\mathrm{z} + [\rho E u_y]_{y+\Delta y}\Delta\mathrm{z}\Delta\mathrm{x} + [\rho E u_z]_{z+\Delta z}\Delta\mathrm{x}\Delta\mathrm{y}$$

Conservation of energy

- Rate of added heat flux by conduction: q_x, q_y, q_z
- Net rate to the volume $\Delta V = \Delta\mathrm{x}\Delta\mathrm{y}\Delta\mathrm{z}$:

$$= [q_x]_{\mathrm{x}}\Delta\mathrm{y}\Delta\mathrm{z}+\left[q_y\right]_{\mathrm{y}}\Delta\mathrm{z}\Delta\mathrm{x}+[q_z]_{\mathrm{z}}\Delta\mathrm{x}\Delta\mathrm{y}$$

$$-[q_x]_{x+\Delta x}\Delta\mathrm{y}\Delta\mathrm{z}+\left[q_y\right]_{y+\Delta y}\Delta\mathrm{z}\Delta\mathrm{x}+[q_z]_{z+\Delta z}\Delta\mathrm{x}\Delta\mathrm{y}$$

- Work done by force $\mathbf{F}$ over distance dx: $\mathbf{F}.dx$ *and other directions if applicable*
- Rate of work done on volume ΔV: $\mathbf{F.u}$
- The flux rate of work done on ΔV: $\dfrac{\mathbf{F.u}}{A} = -P\mathbf{.u}$

$$= [Pu_x]_{\mathrm{x}}\Delta\mathrm{y}\Delta\mathrm{z}+\left[Pu_y\right]_{\mathrm{y}}\Delta\mathrm{z}\Delta\mathrm{x}+[Pu_z]_{\mathrm{z}}\Delta\mathrm{x}\Delta\mathrm{y}$$

$$-[Pu_x]_{x+\Delta x}\Delta\mathrm{y}\Delta\mathrm{z}+\left[Pu_y\right]_{y+\Delta y}\Delta\mathrm{z}\Delta\mathrm{x}+[Pu_z]_{z+\Delta z}\Delta\mathrm{x}\Delta\mathrm{y}$$

- Rate of work done by gravity: $(\rho\Delta\mathrm{x}\Delta\mathrm{y}\Delta\mathrm{z})\ \mathbf{u}\cdot\mathbf{g}$ can be added, etc.

Energy conservation with B-field

- In the most basic case we ignore gravity, viscosity, material stresses, EM fiends, etc.
- Considering the particle distribution function, dividing by $\Delta x \Delta y \Delta z$ and taking the limit of $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$ as before for the momentum equation, we get:

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon\right) = -\nabla\left[\rho\mathbf{u}\left(\epsilon + \frac{u^2}{2}\right) + P\mathbf{u}\right]}$$

- For monatomic ideal gas and adiabatic process:

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \frac{P}{\gamma-1}\right) = -\nabla\left[\left(\frac{\rho u^2}{2} + \frac{\gamma}{\gamma-1}P\right)\mathbf{u}\right]$$

Energy equation

- Consider the Poynting vector (units W/m^2):

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{(\mathbf{u}\times\mathbf{B})\times\mathbf{B}}{\mu_0} = \frac{\mathbf{u}B^2}{\mu_0} - \frac{(\mathbf{u}\cdot\mathbf{B})\mathbf{B}}{\mu_0}$$

Vector identity: $(\mathbf{A}\times\mathbf{B})\times\mathbf{B} = \mathbf{A}B^2 - (\mathbf{A}\cdot\mathbf{B})\mathbf{B}$

- The full MHD form follows:

Magnetic pressure

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \rho\epsilon + \frac{B^2}{2\mu_0}\right) + \nabla\left[\left(\frac{\rho u^2}{2} + \rho\epsilon + P + \frac{B^2}{\mu_0}\right)\cdot\mathbf{u} - \frac{(\mathbf{u}\cdot\mathbf{B})\mathbf{B}}{\mu_0}\right] = 0$$

- Thus, the complete expression for energy conservation for an ideal gas plasma shock used in lecture 10:

Steady state: $\frac{\partial}{\partial t} \to 0$

$$\left[\rho u_n\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\cdot\frac{P}{\rho}\right) + u_n\frac{B^2}{\mu_0} - \mathbf{u}\cdot\mathbf{B}\frac{B_n}{\mu_0}\right] = 0$$