Plasma Physics

TU Dresden

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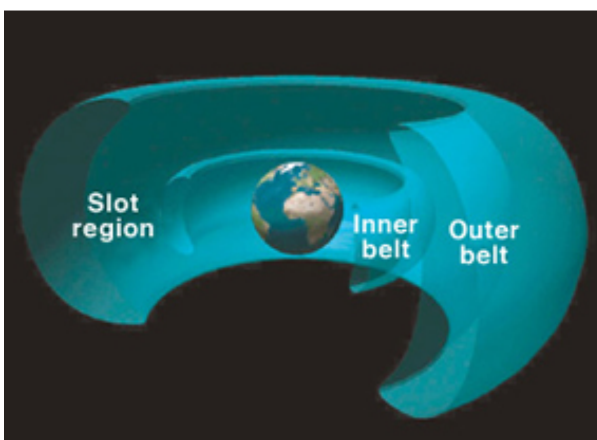
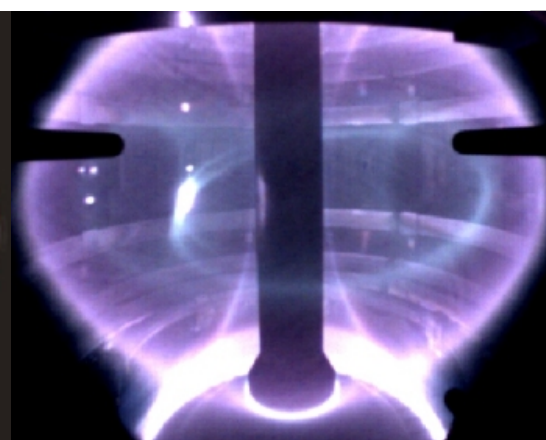
Lectures 6 – 7: Waves in plasmas





Plasma Physics: lecture 6

- Electrostatic plasma waves revisited
- Sound waves recap
- Ion acoustic waves
- Alfvén waves
- Conductivity and dielectric tensors
- Waves in cold magnetized plasma

Plasma Physics: lecture 6

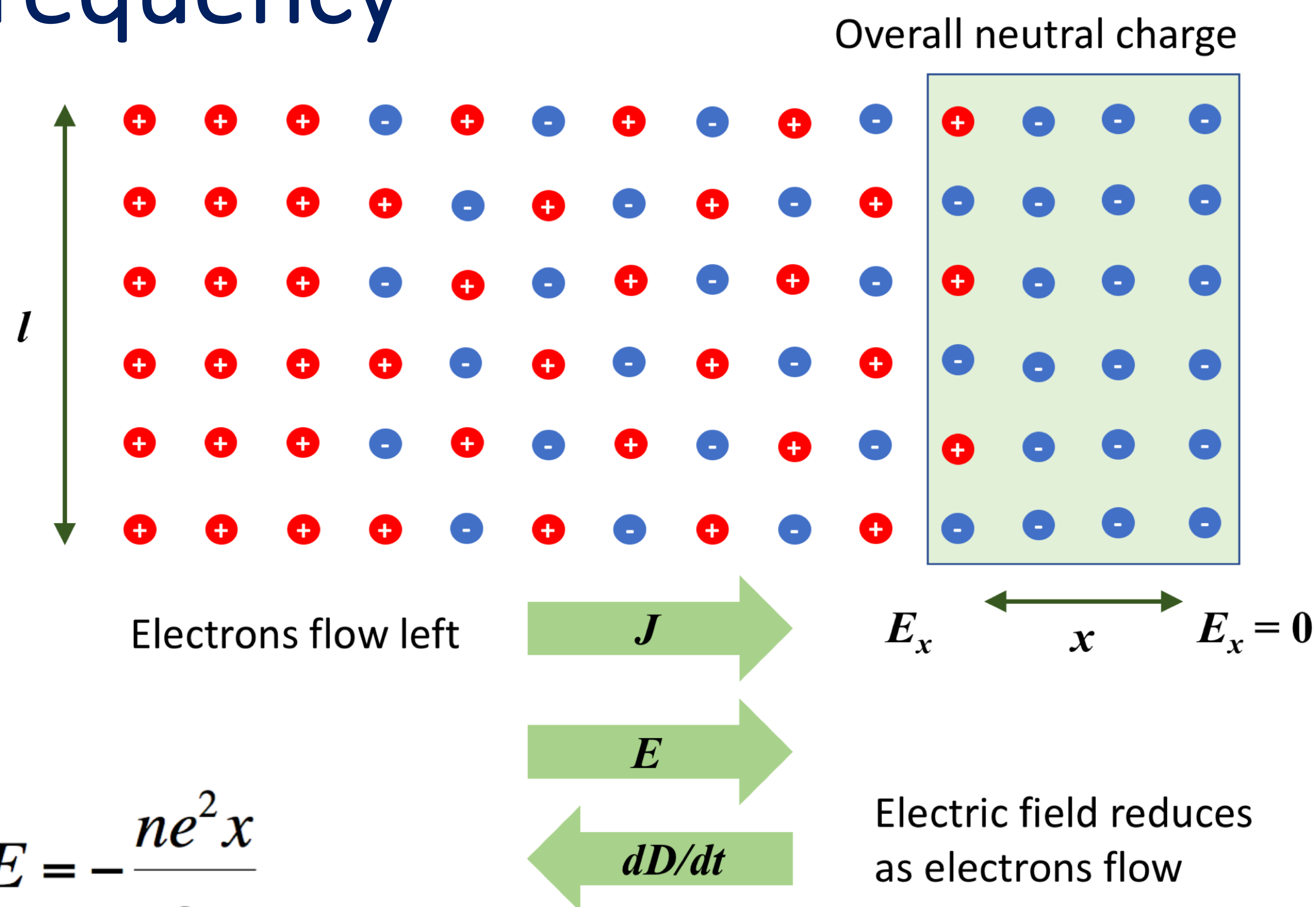
Part 1

Plasma frequency

Apply Gauss' law:

$$El^2 = \frac{nel^2x}{\varepsilon_0}$$

$$\rightarrow \quad E = \frac{nex}{\varepsilon_0}$$



$$F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -eE = -\frac{ne^2x}{\varepsilon_0}$$

Oscillation plasma frequency:

$$\boxed{\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}}$$

Electrostatic waves in plasma

- This plasma oscillation (or a Langmuir wave) is also called the **electrostatic wave**
- Since only the electric field oscillates
- There is **no oscillatory magnetic field** (as with light)
- From Faraday's law, we assume wave-like solution:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \mathbf{E} = \mathbf{E}_0 \exp(i[\omega t - \mathbf{k} \cdot \mathbf{r}])$$

- Thus: $-i\mathbf{k} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Electrostatic waves in plasma

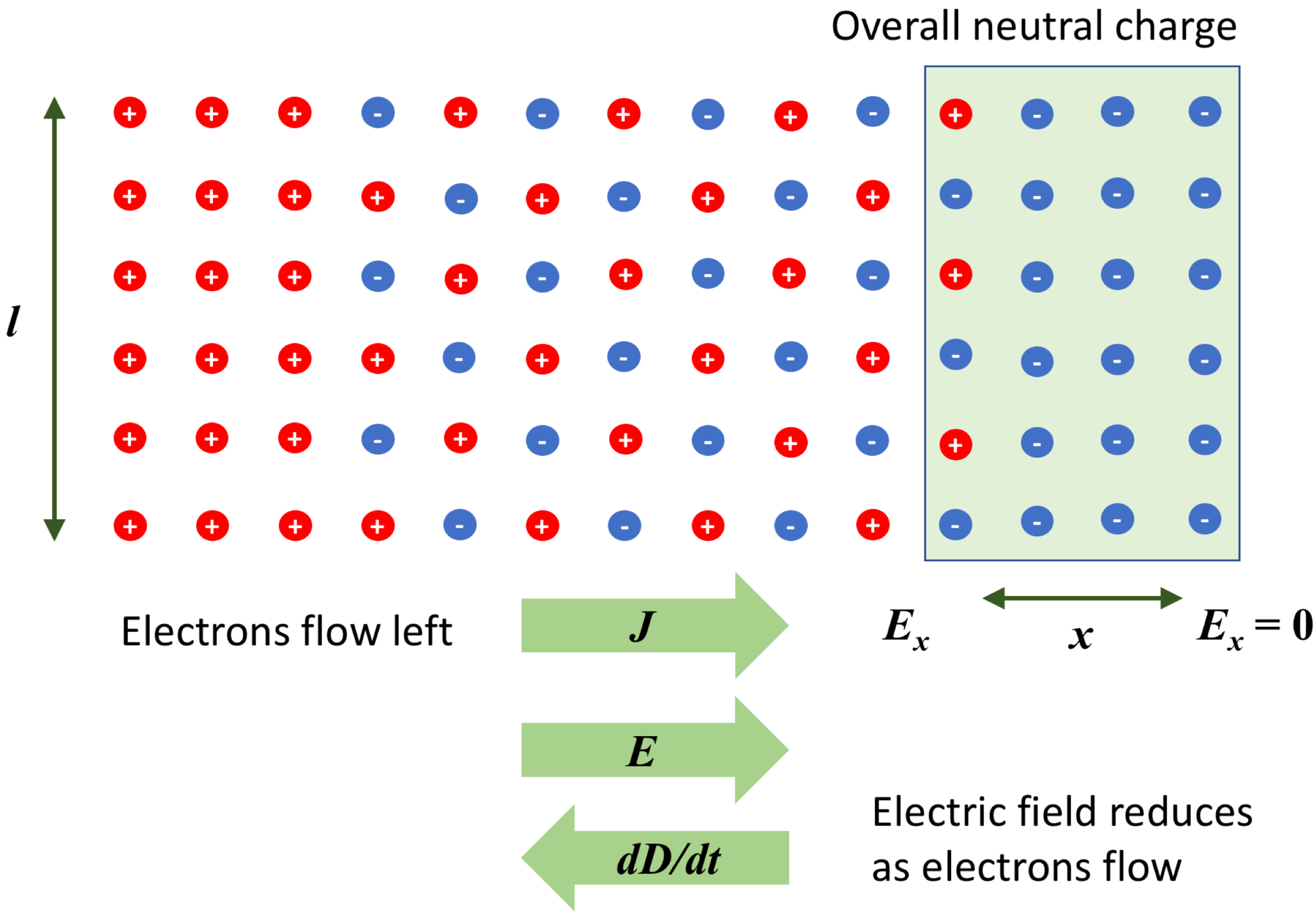
- For there to be no oscillating magnetic field, we need, thus for electrostatic waves we get:

$$\mathbf{k} \times \mathbf{E} = 0$$

- I.e. **the k-vector is parallel to the electric field**
- Now we look at Ampere's law: $$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$
- For there to be no magnetic field, we **require the conduction current and the displacement currents to be equal and opposite**:

$$\mathbf{J} = -\frac{\partial \mathbf{D}}{\partial t}$$

Electrostatic waves in plasma



Electrostatic waves in plasma

- At any instant the current is: $\mathbf{J} = -ne\mathbf{v} = -ne\dfrac{\partial x}{\partial t}$
- Our electric field: $E = \dfrac{nex}{\varepsilon_0}$
- Compute displacement current (proportional to the rate of change in the electric field):

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = ne\frac{\partial x}{\partial t} = -\mathbf{J}$$

- The conduction and displacement currents in an electrostatic wave are opposite and equal in magnitude

Electrostatic waves in plasma

- Arise naturally from thermal fluctuations of charge density in plasma of finite temperature
- Can be driven externally, e.g. during scattering processes during an impact of a photon changing the density fluctuations
- Previously we obtained the Bohm-Gross dispersion relation for the electrostatic waves using the Vlasov equation with no damping term:

$$\boxed{\omega^2 = \omega_{pe}^2 + \frac{3k_B T}{m} \cdot k^2}$$

The dispersion relation for electrostatic waves in warm plasma

Sound waves

- Sound waves in neutral gases/fluids arise from oscillating density fluctuations.
- Starting from fluid equations, for ordinary fluids we have the (simplified) Navier-Stokes equation:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla P = -\frac{\gamma P}{\rho}\nabla\rho$$

- And the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0$$

- Assume oscillatory motion: $\mathbf{u} = \mathbf{v_1} exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$

Sound waves

- Linearize for stationary equilibrium with uniform P_0 and ρ_0 and get:

$$-i\omega\rho_0\mathbf{v}_1 = -\frac{\gamma P_0}{\rho_0} i\mathbf{k}\rho_1$$

$$-i\omega\rho_1 + \rho_0 i\mathbf{k}\cdot\mathbf{v}_1 = 0$$

- For a plane wave with $\mathbf{k} = k\hat{\mathbf{x}}$ and $\mathbf{v} = k\hat{\mathbf{v}}$, we eliminate ρ_1 and obtain:

$$-i\omega\rho_0 v_1 = -\frac{\gamma P_0}{\rho_0} ik\frac{\rho_0 ikv_1}{i\omega} \quad \rightarrow \quad \omega^2 v_1 = k^2\frac{\gamma P_0}{\rho_0} v_1$$

- Thus speed of sound waves:

$$\boxed{c_s = \frac{\omega}{k} = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k_B T}{M}\right)^{1/2}}$$

Phase velocity $\frac{\omega}{k}$

Ion acoustic waves

- Real plasmas have finite temperature resulting in density fluctuation → **equivalent of sound waves**
- Starting from fluid equations for waves in x-direction, continuity equation for ions:

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i u_i) = 0$$

- And the momentum equation for ions:

$$\frac{\partial}{\partial t}(n_i u_i) + \frac{\partial}{\partial x}(n_i u_i^2) = + \frac{n_i Z e E}{M} - \frac{1}{M}\frac{\partial P_i}{\partial x}$$

Ion mass

From: $$\boxed{nm\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = nq(E + \mathbf{v}\times\mathbf{B}) - \nabla P}$$ *For B=0*

Ion acoustic waves

- Take spatial and temporal derivatives and merge equations eliminating the $\partial^2(n_i u_i)/\partial t \partial x$ terms:

$$\frac{\partial^2 n_i}{\partial t^2} - \frac{\partial^2}{\partial x^2}(n_i u_i^2) + \frac{Ze}{M}\frac{\partial}{\partial x}(n_i E) - \frac{1}{M}\frac{\partial^2 P_i}{\partial x^2} = 0$$

- Linearize by neglecting products of perturbed quantities $n_i = n_{i0} + \tilde{n}$, $u_i = \tilde{u}_i$, $E = \tilde{E}$, and $P_i = P_{i0} + \tilde{P}_i$:

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_{i0} Ze}{M}\frac{\partial \tilde{E}}{\partial x} - \frac{1}{M}\frac{\partial^2 \tilde{P}_i}{\partial x^2} = 0$$

Ion acoustic waves

- Ideal gas adiabatic equation in 1-D: $\tilde{P}_i = 3k_B T_i \tilde{n}_i$
- Substituting for $\tilde{P}_i$:

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} + \frac{n_{i0} Ze}{M}\frac{\partial \tilde{E}}{\partial x} - \frac{3k_\mathrm{B}T_i}{M}\frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

- When the ions move, the electrons follow quickly (they are very light). Therefore the electric field is still due to the electron motion → restoring electric force and thermal force: $\frac{\partial}{\partial t}(n_e u_e) + \frac{\partial}{\partial x}(n_e u_e^2) = -\frac{n_e e E}{m} - \frac{1}{m}\frac{\partial P_e}{\partial x}$
- And the electrons are almost massless, thus :

$$n_e e E = -\frac{\partial P_e}{\partial x}$$

Ion acoustic waves

- And for isothermal equation of state for electrons, we have $\tilde{P} = \tilde{n} k_B T_e$. Hence:

$$n_e e\tilde{E} = -k_{\mathrm{B}} T_e \frac{\partial \tilde{n}_e}{\partial x}$$

- For $n_e = Z n_{i0}$ and $\tilde{n}_e = Z\tilde{n}_i$ we can differentiate with respect to x:

$$n_{i0} Z e \frac{\partial \tilde{E}}{\partial x} = -Z k_{\mathrm{B}} T_e \frac{\partial^2 \tilde{n}_i}{\partial x^2}$$

- And substitute:

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} - \frac{Z k_{\mathrm{B}} T_e}{M} \frac{\partial^2 \tilde{n}_i}{\partial x^2} - \frac{3 k_{\mathrm{B}} T_i}{M} \frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

i.e.

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} - \left(\frac{Z k_{\mathrm{B}} T_e + 3 k_{\mathrm{B}} T_i}{M} \right) \frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

Ion acoustic waves

- Wave equation form:

$$\frac{\partial^2 \tilde{n}_i}{\partial t^2} - \left(\frac{Zk_\mathrm{B}T_e + 3k_\mathrm{B}T_i}{M}\right)\frac{\partial^2 \tilde{n}_i}{\partial x^2} = 0$$

- Assuming wave-like solutions with the form of:

$$\tilde{n}_i \propto e^{[i\omega t - kx]} \quad \rightarrow \quad \omega = \pm k v_\mathrm{ia}$$

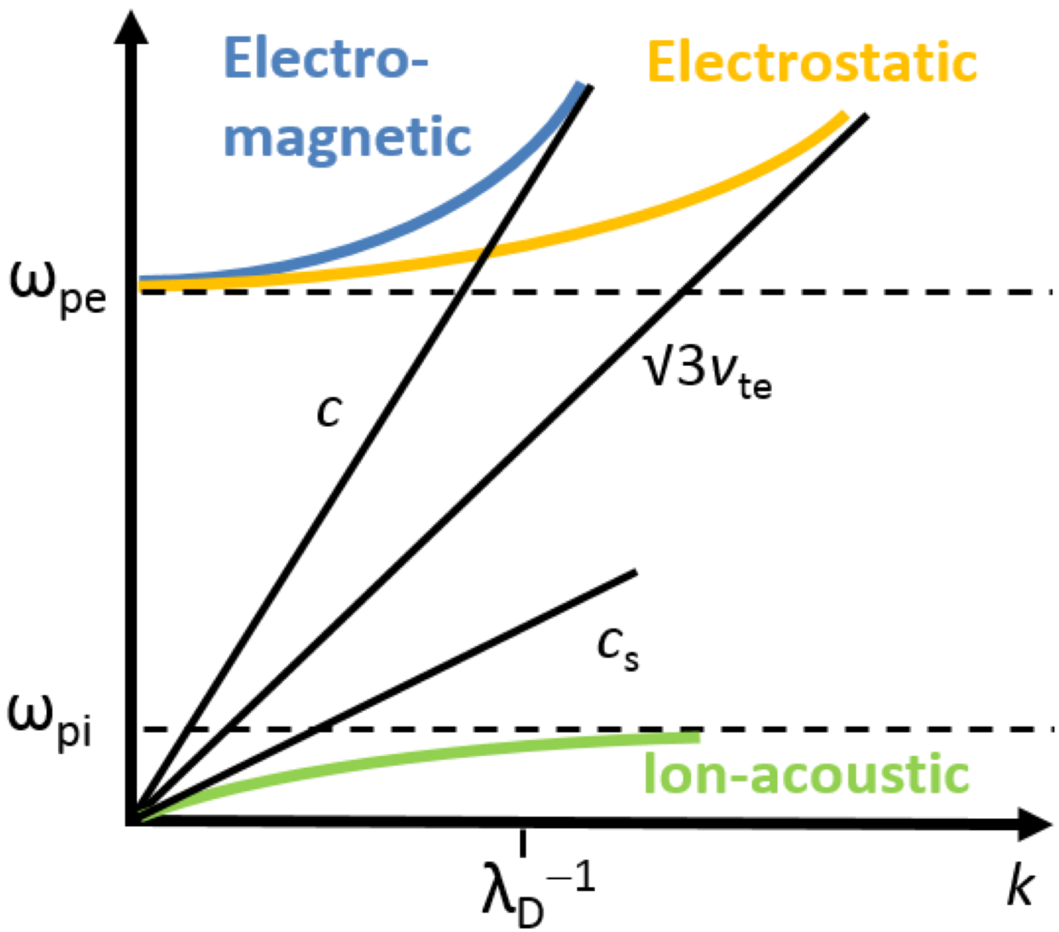
- Therefore the ion-acoustic frequency is:

$$\boxed{v_\mathrm{ia} = \sqrt{\left(\frac{Zk_\mathrm{B}T_e + 3k_\mathrm{B}T_i}{M}\right)}}$$

Ion acoustic waves

- These waves have low frequency compared to the electron electrostatic waves (plasmons, Bohm-Gross).
- Ion-acoustic waves are an analogue to sound waves in normal neutral gases/fluids (see supplemental notes to this lecture).
- The ions provide the inertia for the waves (density oscillations) and the plasma pressure provides the restoring force. The pressure is mitigated by the electrons. We treat the ions as adiabatic in their motion, but the electrons as isothermal (they are faster).
- Note that T_i and T_e are treated separately. The collisions between electrons and ions take a long time that the two species have significantly different temperatures.

Plasma waves dispersions



MHD waves in magnetized plasma

- Low-frequency waves in magnetized plasma
- Remembering the momentum equation:

Magnetic pressure gradient

$$\boxed{\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla P - \frac{\nabla B^2}{2\mu_0} + \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}}$$

Sound wave

Shear Alfén waves (due to mag. tension)

Magnetosonic waves

- Longitudinal waves:
  - For B = 0 reduces to a simple sound wave
  - Get magnetic pressure waves for: $$\frac{\nabla B^2}{2\mu_0} \gg \nabla P$$

Transverse Alfvén waves

- Transverse waves:
  - From magnetic tension force: $\frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B}$
- Starting from equation for sound waves:

$$\left[\frac{\partial^2 n}{\partial t^2} - v_A^2 \frac{\partial^2 n}{\partial z^2}\right] = 0$$

- For period oscillations we have:

$$n(x,t) = \hat{n} \sin[k(x \pm v_A t)]$$

- Where $k = \frac{2\pi}{\lambda}$ is the wavenumber

Transverse Alfvén waves

- Starting from ideal MHD equations ($\eta = 0$), i.e. frozen-in flux, magnetic field cannot leave the plasma by diffusion
- Simplified momentum equation:

$$\rho_m \frac{\mathrm{d}\mathbf{v}_m}{\mathrm{d}t} = \mathbf{j}\times\mathbf{B}$$

- For $\eta \to 0$, $R_m \to \infty$, the induction equation:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}_m\times\mathbf{B})$$

- Apply vector identity as before (lecture 5):

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{B}\cdot\nabla)\mathbf{v}_m + (\mathbf{v}_m\cdot\nabla)\mathbf{B} - \mathbf{B}(\nabla\cdot\mathbf{v}_m)$$

Transverse Alfvén waves

- Ignore the contribution of soundwaves, i.e. no density fluctuation: ρ = const. and $\nabla \cdot \mathbf{v}_m = 0$
- For wavelike perturbation: $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$

$$\mathbf{v}_m = \mathbf{v}_0 + \mathbf{v}_1$$

- B-field along z-direction: $\mathbf{B}_0 = (0,0,B_0)$ and plasma assumed to be at rest: $\mathbf{v}_0 = 0$:

$$\rho_m \frac{\mathrm{d}\mathbf{v}_1}{\mathrm{d}t} = \mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}(\nabla\times\mathbf{B}_1)\times\mathbf{B}_0$$

$$\frac{\mathrm{d}\mathbf{B}_1}{\mathrm{d}t} = (\mathbf{B}_0 \cdot \nabla)\mathbf{v}_1$$

NB: We dropped second order term $(\mathbf{v}_1 \cdot \nabla)\mathbf{B}_1$

Transverse Alfvén waves

- Transverse displacement leading to magnetic field perturbations in the x-direction: $\mathbf{B}_1 = (B_{1x}, 0,0)$
- The double vector product:

$$(\nabla \times \mathbf{B}_1) \times \mathbf{B}_0 = (\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 - (\nabla \mathbf{B}_1) \cdot \mathbf{B}_0$$

= 0 as $\mathbf{B}_1$ *in x-direction and* $\mathbf{B}_0$ *in z-direction*

- And $(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 = B_0(\partial B_{1x}/\partial z)\hat{\mathbf{i}}_x$
- Simplify:

$$\rho_m \frac{\partial v_{1x}}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_{1x}}{\partial z}$$

$$\frac{\partial B_{1x}}{\partial t} = B_0 \frac{\partial v_{1x}}{\partial z}$$

Transverse Alfvén waves

- Combine above expressions to a set of wave equations for the transverse or shear Alfvén wave:

$$\left[\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right] v_{1x} = 0$$

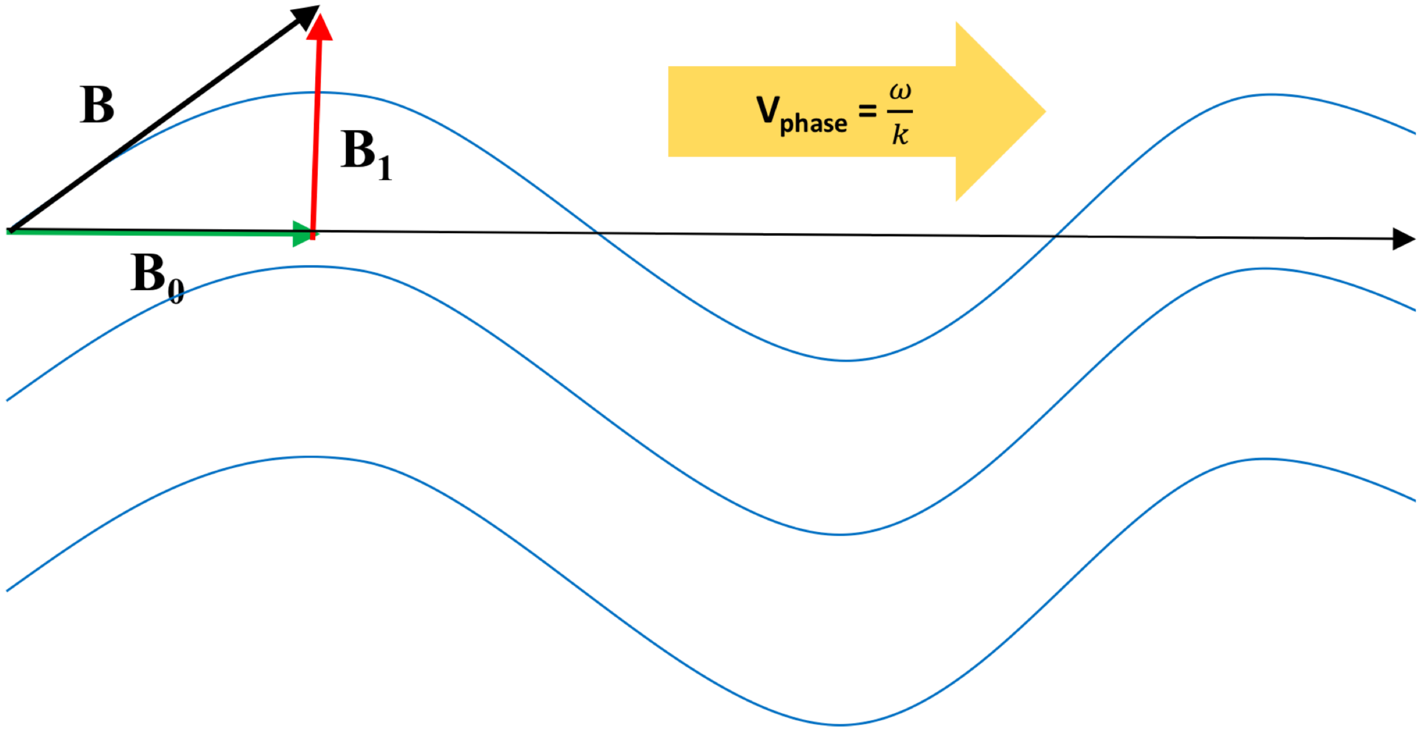
$$\left[\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right] B_{1x} = 0$$

- Giving the Alfvén speed:

$$\boxed{v_A = \left(\frac{B_0^2}{\mu_0 \rho_m}\right)^{1/2}}$$

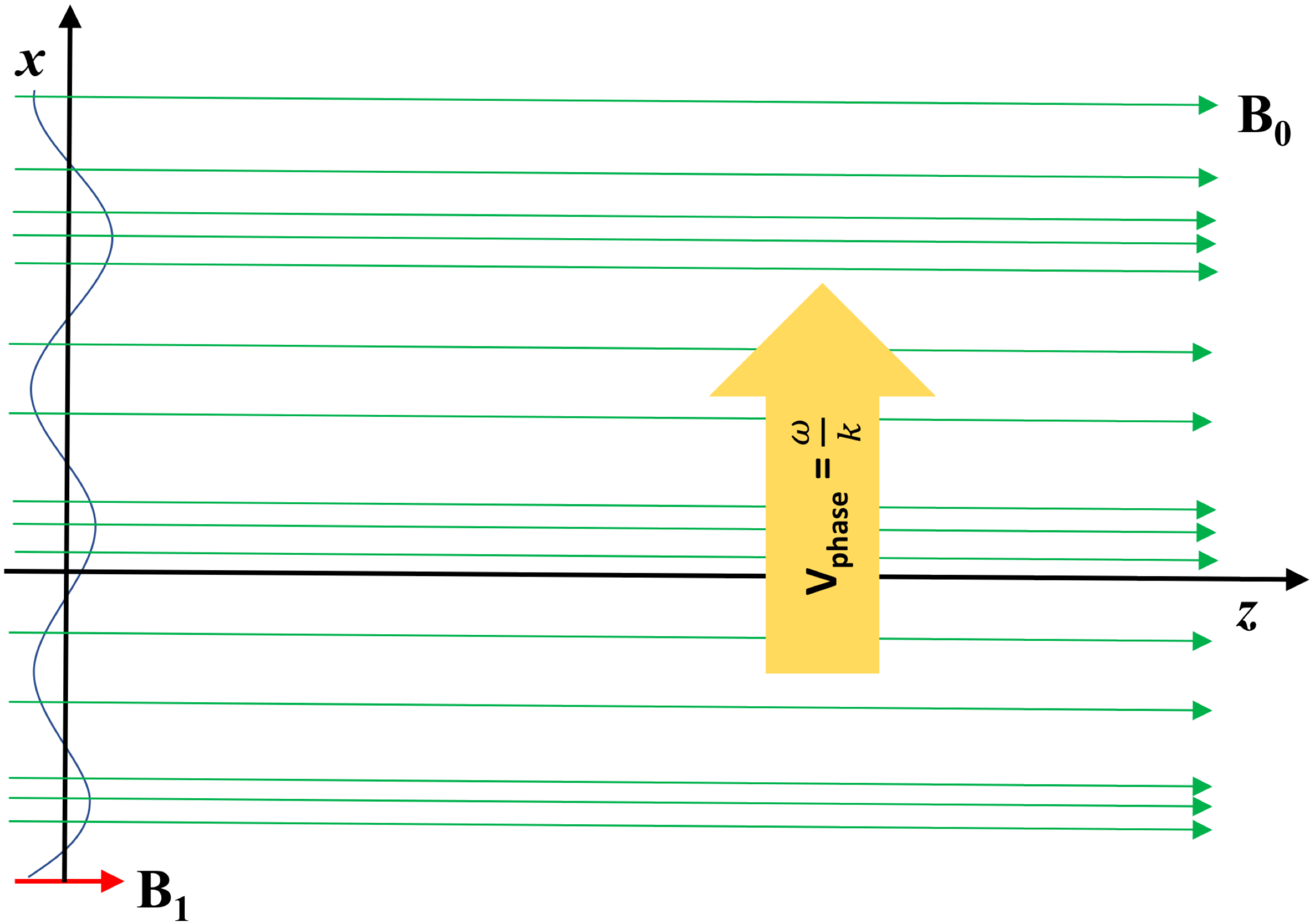
Transverse Alfvén waves

- The transverse Alfvén waves conserve the volume between the field lines (frozen-in flux).
- There is no compression of the plasma or field lines.
- Waves driven by mag. field tension:



Longitudinal Alfvén waves

- Compressional/longitudinal Alfvén waves propagate across the magnetic field, i.e. B-field is compressed.



Longitudinal Alfvén waves

- These are more similar to sound waves:

$$c_s = \frac{\omega}{k} = \left(\frac{\gamma P}{\rho}\right)^{1/2} \text{where} \quad P_B = \frac{B^2}{2\mu_0}$$

- For 2 degrees of freedom $\gamma = 2$
- Then the full magnetosonic wave propagates at speed:

$$v_\varphi = \frac{\omega}{k} = (v_A^2 + c_s^2)^{1/2} \text{ where } \boxed{v_A = \left(\frac{B_0^2}{\mu_0 \rho_m}\right)^{1/2}}$$

Accounting for kinetic pressure

Alfvén speed is the maximum speed of information transfer

- Alfvén Mach number $= \frac{u}{v_A}$ → shocks for $u > v_A$

→ MHD instabilities!!!

Nobel Prize for Physics

The Nobel Prize in Physics 1970 was given to Hannes O. G. Alfvén "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics".

Conductivity/dielectric tensor

- We have previously found the conductivity of unmagnetized plasma:

$$\sigma(\omega) = -i\frac{ne^2}{m\omega}$$

- And the dielectric (permittivity) function:

Plasma frequency

$$\varepsilon(\omega) = 1 - \frac{ne^2}{\varepsilon_0 m\omega^2} = 1 - \frac{i}{\varepsilon_0\omega}\sigma(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$

- In unmagnetized plasma, conductivity is a scalar:

$$\mathbf{J}(\omega) = \sigma(\omega)\,\mathbf{E}(\omega)$$

Conductivity/dielectric tensor

- In a magnetic field, the path of the electrons (and hence the current) is not necessarily parallel to the electric field. To take this into account we must let the conductivity be a 3 x 3 tensor:

$$\mathbf{J}(\omega) = \bar{\sigma}(\omega)\,\mathbf{E}(\omega)$$

- Therefore the dielectric permittivity must also be a tensor:

$$\bar{\varepsilon}(\omega) = \mathbf{I} - \frac{i}{\varepsilon_0 \omega}\bar{\sigma}(\omega)$$

Identity matrix

- Note that waves in plasma still satisfy dispersion equation (see lecture 3):

$$(\omega^2 \bar{\varepsilon}(\omega) - c^2 k^2)\mathbf{E} + c^2 \mathbf{k}(\mathbf{k}\cdot\mathbf{E}) = 0$$

Conductivity/dielectric tensor

- Assume the magnetic field to be along the z-axis.
- In the z-direction, we thus have no motion due to B-field and the oscillatory E-field gives (as before):

$$\sigma_{zz} = -i\frac{ne^2}{m\omega} = -i\frac{\varepsilon_0\omega_\mathrm{p}^2}{\omega}$$

- The electric field in the z direction never causes any motion in the x-y plane, and an electric field in the x-y plane never causes motion along the z-axis:

$$\boldsymbol{\sigma} = \begin{pmatrix} ? & ? & 0 \\ ? & ? & 0 \\ 0 & 0 & -i\frac{\varepsilon_0\omega_\mathrm{p}^2}{\omega} \end{pmatrix}$$

Conductivity/dielectric tensor

- Starting from the Lorentz force:

$$\mathbf{F} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = q(\mathbf{E} + [\mathbf{v} \times \mathbf{B}])$$

- And the cyclotron frequency:

$$\vec{\omega}_c = \frac{-q\mathbf{B}}{m}$$

- Thus, the motion of electrons in the plane perpendicular to the magnetic field:

$$\dot{\mathbf{v}}_\perp = -\frac{e}{m}\mathbf{E}_\perp + \boldsymbol{\omega}_{\mathrm{ce}} \times \mathbf{v}_\perp$$

$$\rightarrow \quad \ddot{\mathbf{v}}_\perp = -\frac{e}{m}\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\mathrm{ce}} \times \dot{\mathbf{v}}_\perp$$

Conductivity/dielectric tensor

- Substituting:

$$\begin{aligned}\ddot{\mathbf{v}}_\perp &= -\frac{e}{m}\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\mathrm{ce}} \times (-\frac{e}{m}\mathbf{E}_\perp + \boldsymbol{\omega}_{\mathrm{ce}} \times \mathbf{v}_\perp) \\ &= -\frac{e}{m}(\dot{\mathbf{E}}_\perp + \boldsymbol{\omega}_{\mathrm{ce}} \times \mathbf{E}_\perp) - \omega_{\mathrm{ce}}^2 \mathbf{v}_\perp\end{aligned}$$

- As the applied E-field and thus the velocity is oscillatory:

$$\mathbf{E}_\perp = \mathbf{E}_{\perp\mathbf{0}} \exp(i\omega t) \quad \rightarrow \quad \mathbf{v}_\perp = \mathbf{v}_{\perp\mathbf{0}} \exp(i\omega t)$$

- Rewrite the above equation of motion:

$$\mathbf{v}_{\perp\mathbf{0}}(\omega_{\mathrm{ce}}^2 - \omega^2) = -\frac{e}{m}(i\omega\mathbf{E}_{\perp\mathbf{0}} + \boldsymbol{\omega}_{\mathrm{ce}} \times \mathbf{E}_{\perp\mathbf{0}})$$

Conductivity/dielectric tensor

- The individual components are:

$$v_{x0}(\omega_{\mathrm{ce}}^2 - \omega^2) = -\frac{e}{m}(i\omega E_{x0} + \omega_{\mathrm{ce}} E_{y0})$$
$$v_{y0}(\omega_{\mathrm{ce}}^2 - \omega^2) = -\frac{e}{m}(i\omega E_{y0} - \omega_{\mathrm{ce}} E_{x0})$$

$$\boxed{\omega_p^2 = \frac{ne^2}{\varepsilon_0 m}}$$

- And since $\mathbf{J} = -ne\mathbf{v} = \sigma \cdot \mathbf{E}$:

$$\boldsymbol{\sigma} = \begin{pmatrix} i\frac{\varepsilon_0 \omega \omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & \frac{\varepsilon_0 \omega_{\mathrm{ce}} \omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & 0 \\ -\frac{\varepsilon_0 \omega_{\mathrm{ce}} \omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & i\frac{\varepsilon_0 \omega \omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & 0 \\ 0 & 0 & -i\frac{\varepsilon_0 \omega_{\mathrm{p}}^2}{\omega} \end{pmatrix}$$

Conductivity/dielectric tensor

- Since: $\bar{\varepsilon}(\omega) = \mathbf{I} - \dfrac{i}{\varepsilon_0 \omega}\bar{\sigma}(\omega)$

- Therefore, we get the full dielectric tensor:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} 1 + \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & \frac{-i\omega_{\mathrm{ce}}}{\omega}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & 0 \\ \frac{i\omega_{\mathrm{ce}}}{\omega}\frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & 1 + \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} & 0 \\ 0 & 0 & 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \end{pmatrix}$$

- Remember: $\boxed{\omega_p^2 = \dfrac{ne^2}{\varepsilon_0 m}}$

Conductivity/dielectric tensor

- Simpler form:

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

- Where:

$$\varepsilon_1 = 1 + \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2}$$

$$\varepsilon_2 = \frac{\omega_{\mathrm{ce}}}{\omega} \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2}$$

$$\varepsilon_3 = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}$$

Plasma Physics: lecture 7

Part 2

Waves in magnetized plasma

- Using the dielectric (conductivity) tensor and plugging it into the dispersion relation, we can derive waves in magnetized plasmas.
- Rewrite the dispersion relation:

$$(\omega^2 \overline{\varepsilon}(\omega) - c^2 k^2)\mathbf{E} + c^2 \mathbf{k}(\mathbf{k} \cdot \mathbf{E}) = 0$$

- As: $\mathbf{M} \cdot \mathbf{E} = 0$

$$\mathbf{M} = \frac{\omega^2}{c^2}\boldsymbol{\varepsilon} + \mathbf{kk} - k^2\mathbf{I}$$

Waves in magnetized plasma

- Solutions depend on the polarization of the wave (direction of the k-vector), which we track with: $\mathbf{N} = \frac{c\mathbf{k}}{\omega}$
- Thus using the dielectric tensor we find:

$$\mathbf{M} = \frac{\omega^2}{c^2}\boldsymbol{\varepsilon} + \mathbf{kk} - k^2\mathbf{I} \qquad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_y^2 - N_z^2 & -i\varepsilon_2 + N_x N_y & N_x N_z \\ i\varepsilon_2 + N_x N_y & \varepsilon_1 - N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & \varepsilon_3 - N_x^2 - N_y^2 \end{pmatrix}$$

Waves propagating ∥ to B-field

- For waves parallel to the B-field: $N_x = N_y = 0$
- The tensor becomes:

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_z^2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N_z^2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

- Non-trivial solution for $\mathbf{M}\cdot\mathbf{E} = 0$ occur for roots of $\det(\mathbf{M}) = 0$:

$$\varepsilon_3[(\varepsilon_1 - N_z^2)^2 - \varepsilon_2^2] = 0$$

- Which has three solutions:

$$\varepsilon_3 = 0 \quad \text{and} \quad \varepsilon_1 - N_z^2 = \pm\varepsilon_2$$

Waves propagating ∥ to B-field

- The first solution of $\varepsilon_3 = 0$ simply reproduces plasmon waves at the plasma frequency: $\omega^2 = \omega_p^2$
- In this case we have:

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{bmatrix} (\varepsilon_1 - N_z^2) & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & (\varepsilon_1 - N_z^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- I.e. $\mathbf{M} \cdot \mathbf{E} = 0$ satisfied for E field $(0,0,E_z)$, thus we get:

$$\mathbf{k} \cdot \mathbf{E} \neq 0 \quad , \quad \mathbf{k} \times \mathbf{E} = 0$$

- Consistent with an electrostatic wave introduced before.

Waves propagating ∥ to B-field

- Now we focus on the other two solutions (EM waves?):

$$\varepsilon_1 - N_z^2 = \pm\varepsilon_2$$

- Starting with: $\varepsilon_1 - \varepsilon_2 = N_z^2$
- We defined k to lie along z-direction: $N_z^2 = \dfrac{c^2k^2}{\omega^2}$
- Substitute:

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega + \omega_{\mathrm{ce}})}$$

- If there is no B-field, $\omega_{ce} = 0$ and we obtain known dispersion of electromagnetic waves:

$$\boxed{\omega^2 = \omega_{\mathrm{p}}^2 + c^2k^2}$$

Waves propagating ∥ to B-field

- For: $\varepsilon_1 + \varepsilon_2 = N_z^2$
- We then get:

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_\mathrm{p}^2}{\omega(\omega - \omega_\mathrm{ce})}$$

- For no B-field, $\omega_{ce} = 0$ the dispersion of EM waves is once again reproduced:

$$\omega^2 = \omega_p^2 + c^2 k^2$$

- Substituting for $\varepsilon_1 \pm \varepsilon_2 = N_z^2$, we get:

$$\mathbf{M} = \frac{\omega^2}{c^2} \begin{pmatrix} \pm\varepsilon_2 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \pm\varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}$$

Waves propagating ∥ to B-field

- The solution for $\mathbf{M} \cdot \mathbf{E} = 0$, we require the electric field of the form: $(E_x, \mp iE_y, 0)$, i.e. the k-vector is perpendicular to the electric field as expected for electromagnetic waves!
- Unlike in unmagnetized plasma, where the EM wave was linearly polarized, the EM waves in magnetized plasma are **circularly polarized for $\mathbf{k} \parallel \mathbf{B}$**.
- Consider the "left-handed" wave first:

$$n^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega + \omega_{\mathrm{ce}})}$$

Waves propagating ∥ to B-field

- Notice that for ω very large compared to ω_p and ω_{ce} we reproduce the vacuum dispersion. At high frequency there is no time in one period to interact with the plasma oscillations or the gyratory motion of electrons.
- At low ω we get to the cut-off point where $k = 0$ and the waves get infinitely long wavelength, i.e. below the cut-off frequency ω_{LC} waves can no longer propagate:

$$1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega + \omega_{\mathrm{ce}})} = 0$$

$$\omega^2 + \omega_{\mathrm{ce}}\omega - \omega_{\mathrm{p}}^2 = 0$$

$$\rightarrow \quad \boxed{\omega_{\mathrm{LC}} = \frac{-\omega_{\mathrm{ce}} + \sqrt{\omega_{\mathrm{ce}}^2 + 4\omega_{\mathrm{p}}^2}}{2}}$$

→ *Wave reflected*

Waves propagating ∥ to B-field

- Now, the "right-handed" wave:

$$c^2k^2 = \omega^2 - \frac{\omega^2\omega_{\rm p}^2}{\omega(\omega - \omega_{\rm ce})}$$

- For large ω we recover $\omega = ck$ again.
- For small ω compared to ω_p and ω_{ce} we see that:

$$c^2k^2 \approx \frac{\omega\omega_{\rm p}^2}{(\omega_{\rm ce} - \omega)}$$

- There is a pole at $\omega = \omega_{ce}$, i.e. resonance → *Wave absorbed*
- But we ignored the motion of ions here (and got $k \propto \sqrt{\omega}$), which is not correct at low ω → Alfvén waves

Waves propagating ∥ to B-field

- If ω is just above ω_{ce}, then k is imaginary → no propagation.
- But if ω is increased further, the first term on r.h.s exceeds the second term and waves can propagate again:

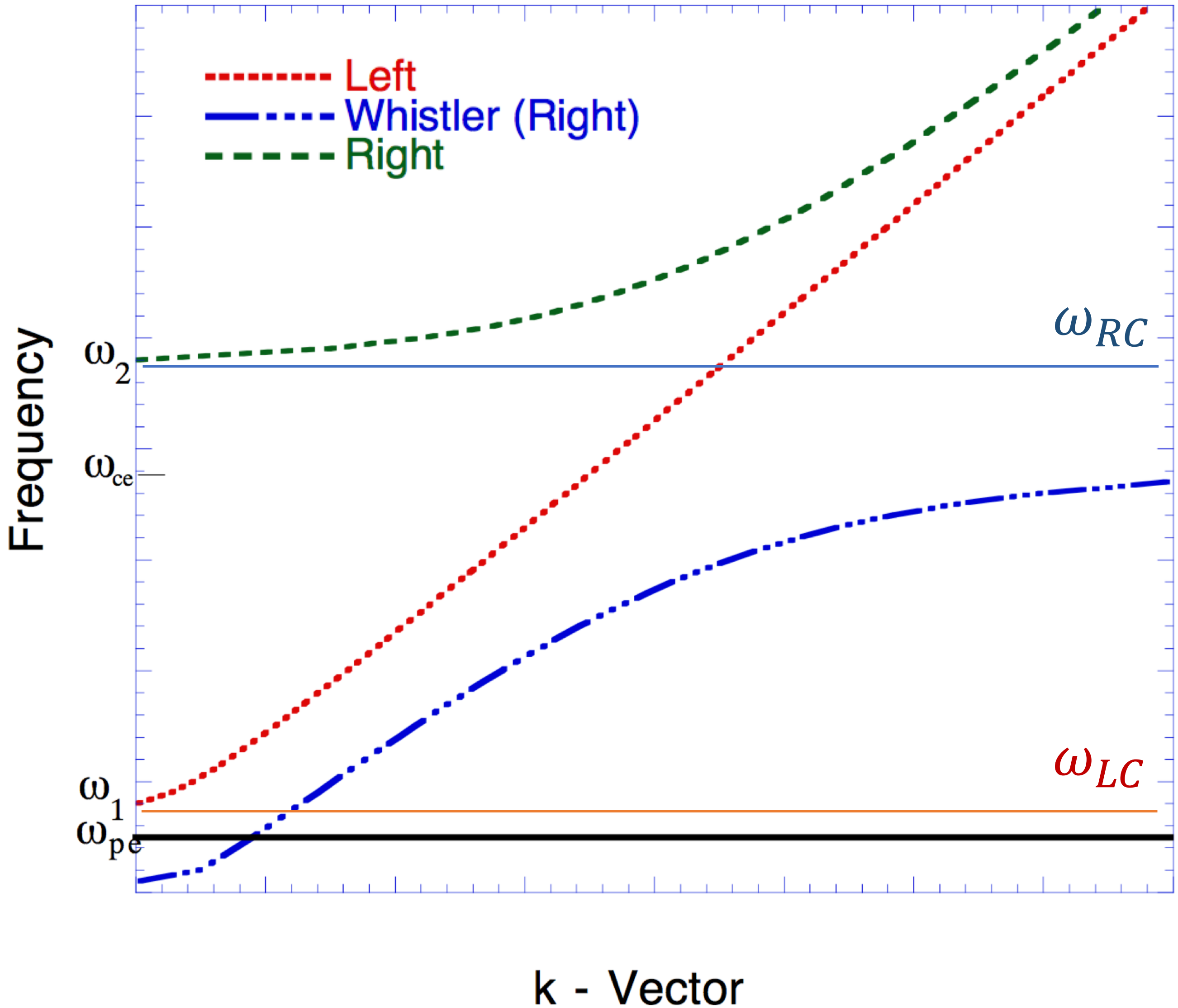
$$c^2k^2 = \omega^2 - \frac{\omega^2\omega_{\mathrm{p}}^2}{\omega(\omega - \omega_{\mathrm{ce}})}$$

- Cut-off frequency for the right-hand wave at $k = 0$:

$$\boxed{\omega_{\mathrm{RC}} = \frac{\omega_{\mathrm{ce}} + \sqrt{\omega_{\mathrm{ce}}^2 + 4\omega_{\mathrm{p}}^2}}{2}}$$

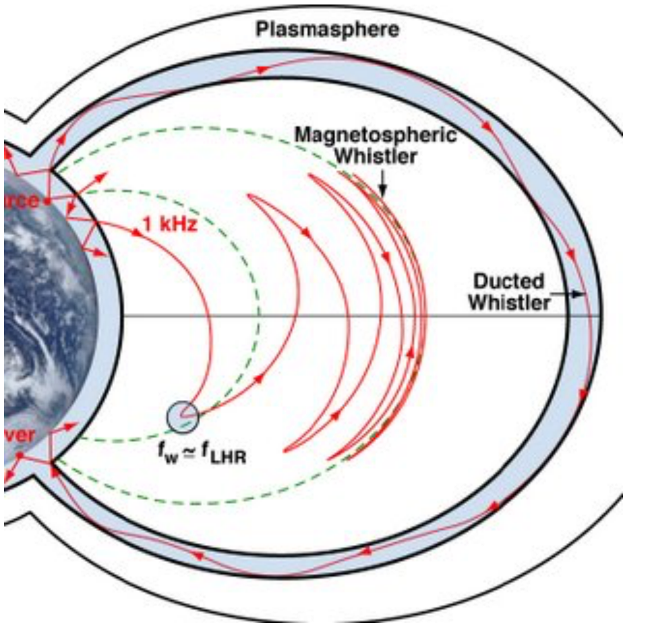
→ *Wave reflected*

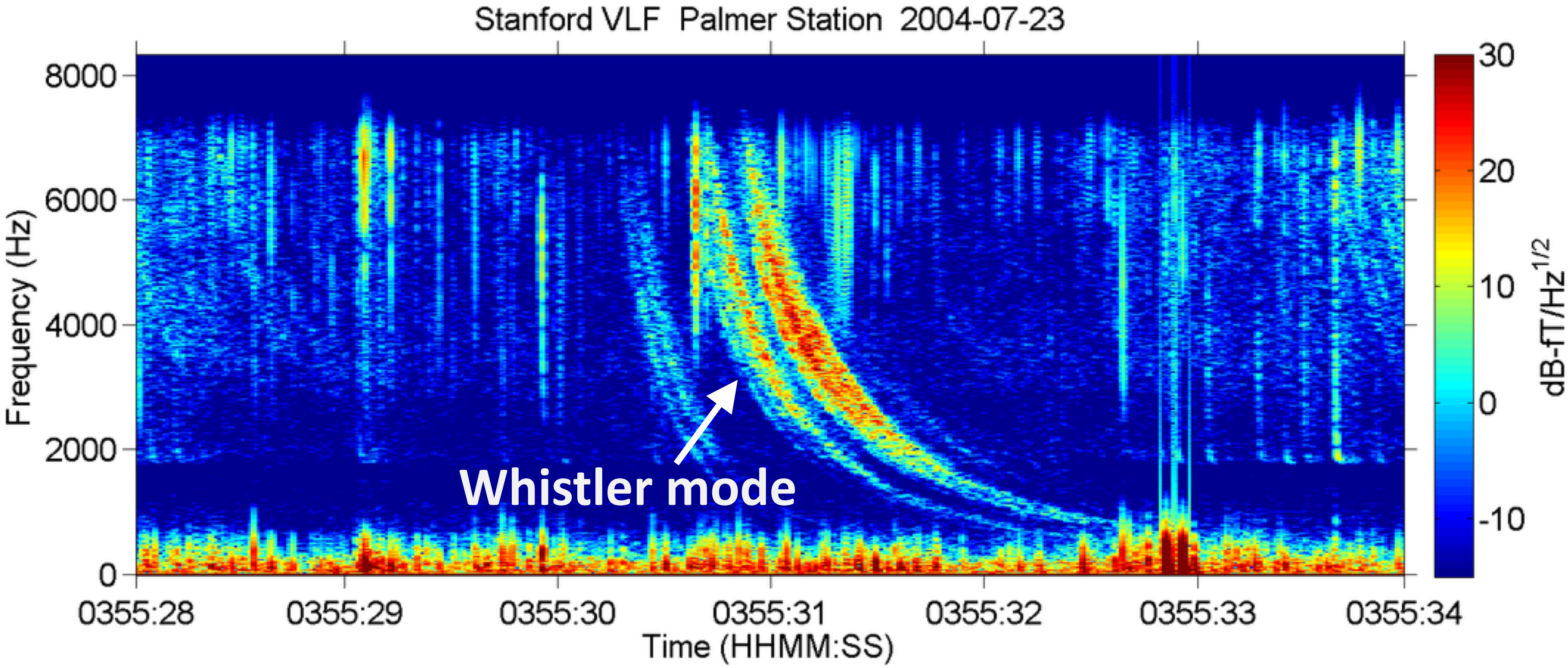
Waves propagating ∥ to B-field

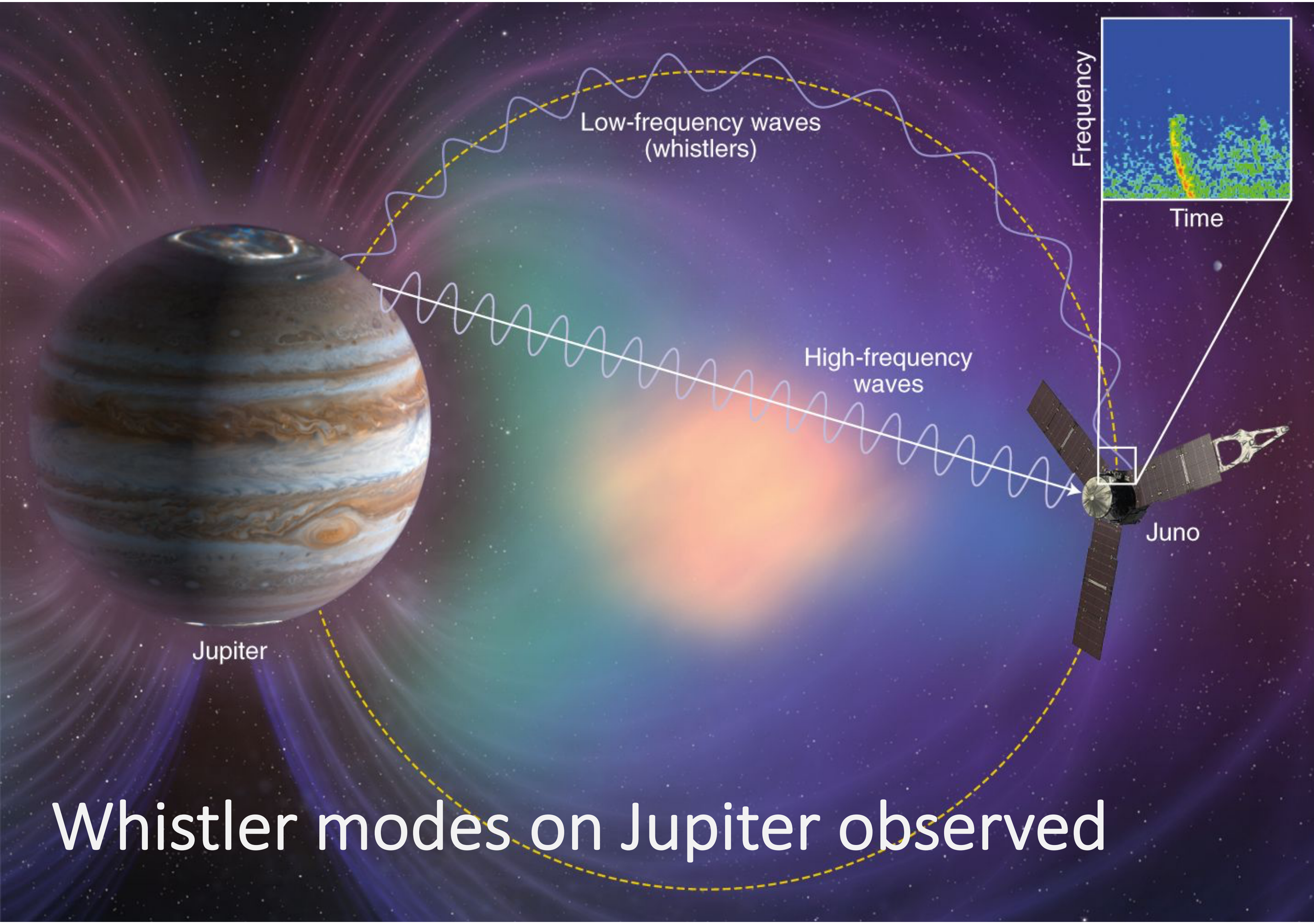


Whistler modes

- Whistler modes detected at low frequency radio (AM)
- Comes from a lightning strike on an opposite hemisphere







Whistler modes on Jupiter observed

Waves propagating ⊥ to B-field

- The analysis using the dielectric tensor can be followed the same way but with keeping either N_x or N_y non-zero too, with all other components set to zero.
- There is no physical differences between the waves with either N_x or N_y non-zero terms.
- Such analysis produces new electromagnetic modes called the O-mode (ordinary) and two extraordinary ones (X-mode).

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{pmatrix} \varepsilon_1 - N_y^2 - N_z^2 & -i\varepsilon_2 + N_x N_y & N_x N_z \\ i\varepsilon_2 + N_x N_y & \varepsilon_1 - N_x^2 - N_z^2 & N_y N_z \\ N_x N_z & N_y N_z & \varepsilon_3 - N_x^2 - N_y^2 \end{pmatrix}$$

Waves propagating ⊥ to B-field

- If the wave propagation is ⊥ to the B-field, the electron motion will be affected by the B-field
- We choose propagation along x-axis (k-vector ∥ to x-axis)
- N_x non-zero, $N_y = N_z = 0$
- Care must be taken as such waves tend to be elliptically polarized, i.e. the wave develops an E_x component becoming partly longitudinal and party transverse. Thus we must allow the E-field to have both x and y components.

Waves propagating ⊥ to B-field

- And we obtain a new form of the conductivity tensor:

$$\mathbf{M} = \frac{\omega^2}{c^2}\begin{bmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - N_x^2 & 0 \\ 0 & 0 & \varepsilon_3 - N_x^2 \end{bmatrix}$$

- Again, non-trivial solutions for $\mathbf{M} \cdot \mathbf{E} = 0$ occur for roots of $\det(\mathbf{M}) = 0$:

$$(\varepsilon_3 - N_x^2) \cdot [\varepsilon_1 \cdot (\varepsilon_1 - N_x^2) - \varepsilon_2^2] = 0$$

- Again, offers three solutions

Waves propagating ⊥ to B-field

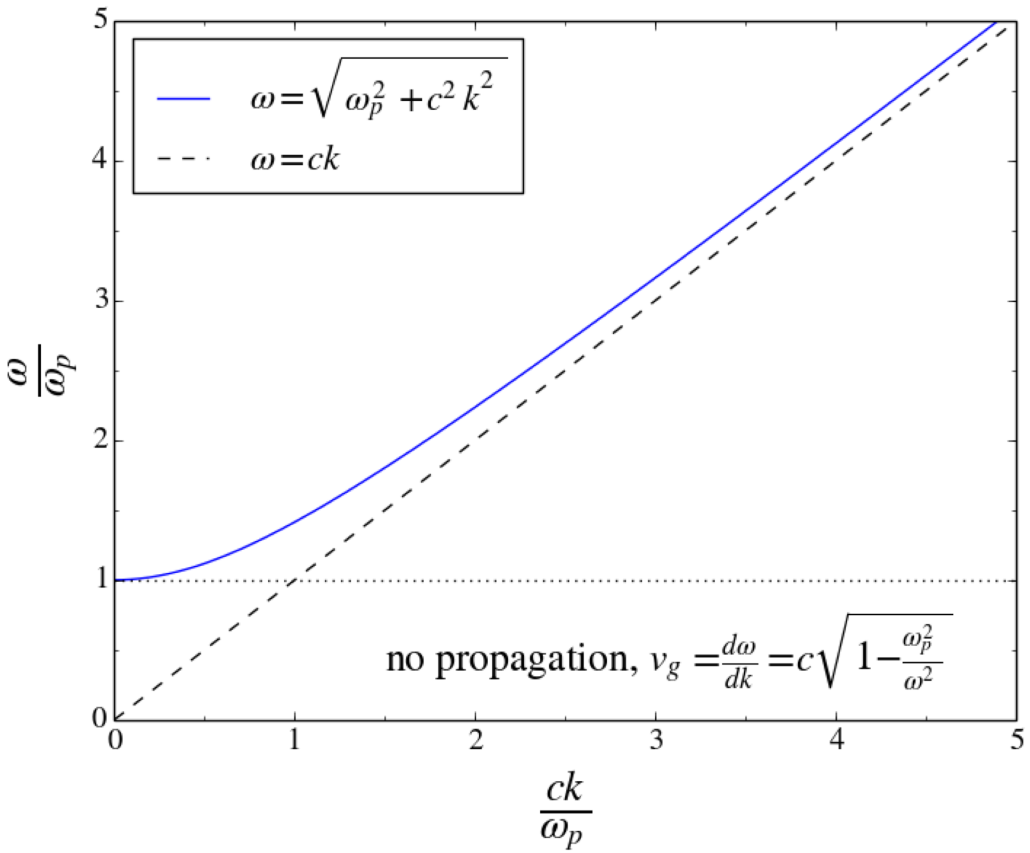
- The first solution gives: $(\varepsilon_3 - N_x^2) = 0$

$$\rightarrow \quad \varepsilon_3 = N_x^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}$$

- I.e. $\mathbf{M} \cdot \mathbf{E} = 0$ again satisfied for E field (0,0,E_z), thus we get the electric field aligned with the static B-field, and the refractive index is not affected by the B-field
- Resultant dispersion relation is of the electromagnetic wave seen in lecture 3: $$\omega^2 = \omega_\mathrm{p}^2 + c^2 k^2$$
- This is the "ordinary wave" or **O-mode, linearly polarized**

Waves propagating ⊥ to B-field

- The cut-off frequency of the "ordinary wave" is the plasma frequency as shown in lecture 3:



$$\omega^2 = \omega_\mathrm{p}^2 + c^2 k^2$$

$$\omega = \omega_\mathrm{p}$$

→ *Wave reflected*

Waves propagating ⊥ to B-field

- Now, solutions for:

$$\begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 \\ i\varepsilon_2 & \varepsilon_1 - N_x^2 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0$$

- Non-vanishing solutions for $\mathbf{E}$ when the matrix determinant is zero:

$$\rightarrow\ (\varepsilon_1 - N_x^2) \cdot \varepsilon_1 - \varepsilon_2^2 = 0 \ \rightarrow\ N_x^2 = \varepsilon_1 - \frac{\varepsilon_2^2}{\varepsilon_1}$$

- Get dispersion relation by substituting for N_x, ε_1 and ε_2:

$$N_x^2 = \frac{c^2 k^2}{\omega^2} \qquad \varepsilon_1 = 1 + \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} \qquad \varepsilon_2 = \frac{\omega_{\mathrm{ce}}}{\omega} \frac{\omega_{\mathrm{p}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2}$$

Waves propagating ⊥ to B-field

- Define the `upper hybrid frequency': $\boxed{\omega_{uh}^2 \equiv \omega_{\mathrm{p}}^2 + \omega_{ce}^2}$
- After much algebra: $$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} \cdot \frac{\omega^2 - \omega_{\mathrm{p}}^2}{\omega^2 - \omega_{uh}^2}$$
- Giving the "extraordinary wave" or **X-mode**
- Resonance occurs when $k \to \infty$ as $\omega \to \omega_{uh}$ → *Wave absorbed*
- Cut-off happens when $k = 0$: → *Wave reflected*
- After some more algebra: $$1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} = \pm \frac{\omega_{ce}}{\omega}$$

$$\rightarrow \quad \omega^2 \mp \omega\omega_{ce} - \omega_{\mathrm{p}}^2 = 0$$

Waves propagating ⊥ to B-field

- Solving the quadratic equation, we get two cut-off frequencies for the left-hand wave:

$$\omega_{LX} = \frac{-\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{\mathrm{p}}^2}}{2}$$

→ left-hand cut-off

- And the right-hand wave at $k = 0$:

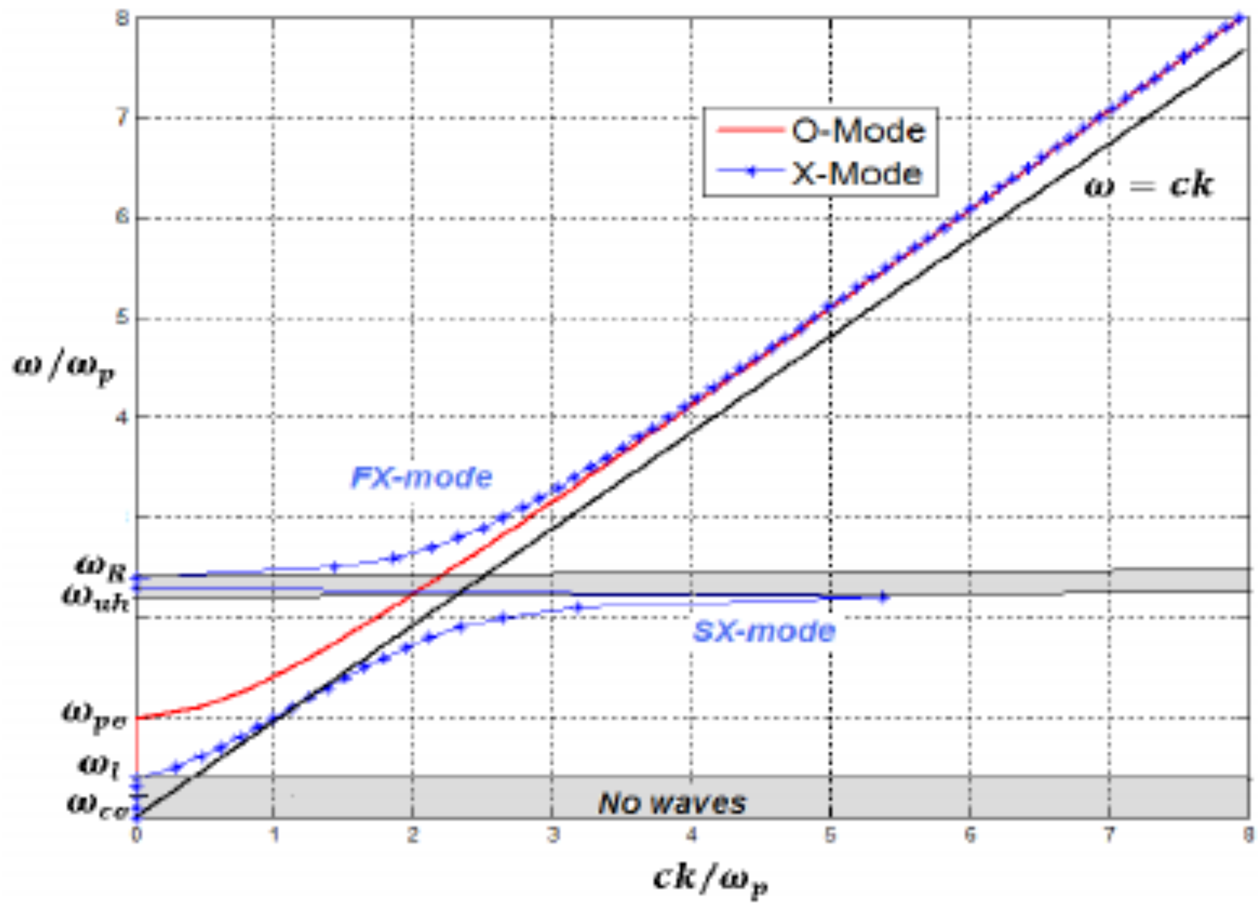
→ Wave reflected

$$\omega_{RX} = \frac{\omega_{ce} + \sqrt{\omega_{ce}^2 + 4\omega_{\mathrm{p}}^2}}{2}$$

→ right-hand cut-off

- The same solutions as for the waves with $\mathbf{k} \parallel \mathbf{B}$
- These are also EM waves, just dispersion relation affected by the B-field → **elliptically polarized for $\mathbf{k} \perp \mathbf{B}$**

Waves propagating ⊥ to B-field



Alfvén waves revisited

- The Alfvén waves can alternatively be derived from the conductivity/dielectric tensor for ions too.
- Let's go back and consider right circularly polarized waves travelling parallel to the B- field.
- At very low frequencies we cannot ignore the response of the heavy ions - they have time to move.
- The ion plasma frequency for $n_i = n/Z$ ions:

$$\omega_{\mathrm{pi}}^2 = \frac{(n/Z)(Ze)^2}{\varepsilon_0 M}$$
$$= \frac{nZe^2}{\varepsilon_0 M}$$

Alfvén waves revisited

- Redefine the dielectric tensor with new terms:

$$\varepsilon_1 = 1 + \frac{\omega_{\mathrm{pe}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} + \frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{ci}}^2 - \omega^2}$$

$$\varepsilon_2 = \frac{\omega_{\mathrm{ce}}}{\omega} \frac{\omega_{\mathrm{pe}}^2}{\omega_{\mathrm{ce}}^2 - \omega^2} - \frac{\omega_{\mathrm{ci}}}{\omega} \frac{\omega_{\mathrm{pi}}^2}{\omega_{\mathrm{ci}}^2 - \omega^2}$$

$$\varepsilon_3 = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} - \frac{\omega_{\mathrm{pi}}^2}{\omega^2}$$

- Note: ion gyro-frequency has an opposite sign: $-\omega_{ci}$
- The "right-handed" wave expression then becomes:

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega(\omega - \omega_{\mathrm{ce}})} - \frac{\omega_{\mathrm{pi}}^2}{\omega(\omega + \omega_{\mathrm{ci}})}$$

$$= 1 + \frac{\omega_{\mathrm{pe}}^2 + \omega_{\mathrm{pi}}^2}{(\omega_{\mathrm{ce}} - \omega)(\omega_{\mathrm{ci}} + \omega)}$$

Alfvén waves revisited

- In the low frequency limit $\omega \to 0$:

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{\omega_{\mathrm{pe}}^2}{\omega_{\mathrm{ce}}\omega_{\mathrm{ci}}}$$

- Substituting for ω_{pe}^2, ω_{ce} and ω_{ci}:

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{n_i\mu_0 M c^2}{B^2}$$

- Refractive index:

$$n^2 = \frac{c^2k^2}{\omega^2} \approx 1 + \frac{c^2}{v_A^2}$$

where

$$\boxed{v_A = \left(\frac{B_0^2}{\mu_0\rho_m}\right)^{1/2}}$$

→ *Alfvén speed*

Alfvén waves revisited

- Rearrange to obtain the phase velocity of the waves:

$$\frac{\omega}{k} = \frac{v_A}{\sqrt{1 + \frac{v_A^2}{c^2}}} \approx v_A$$

- As typically $v_A \ll c$, the low frequency waves travel at Alfvén speed as shown previously
- The expression for left-handed waves also reduce to v_A for very low frequencies
- These are the same waves due to magnetic tension force derived for ideal MHD (frozen-in flux)

Lower hybrid resonance

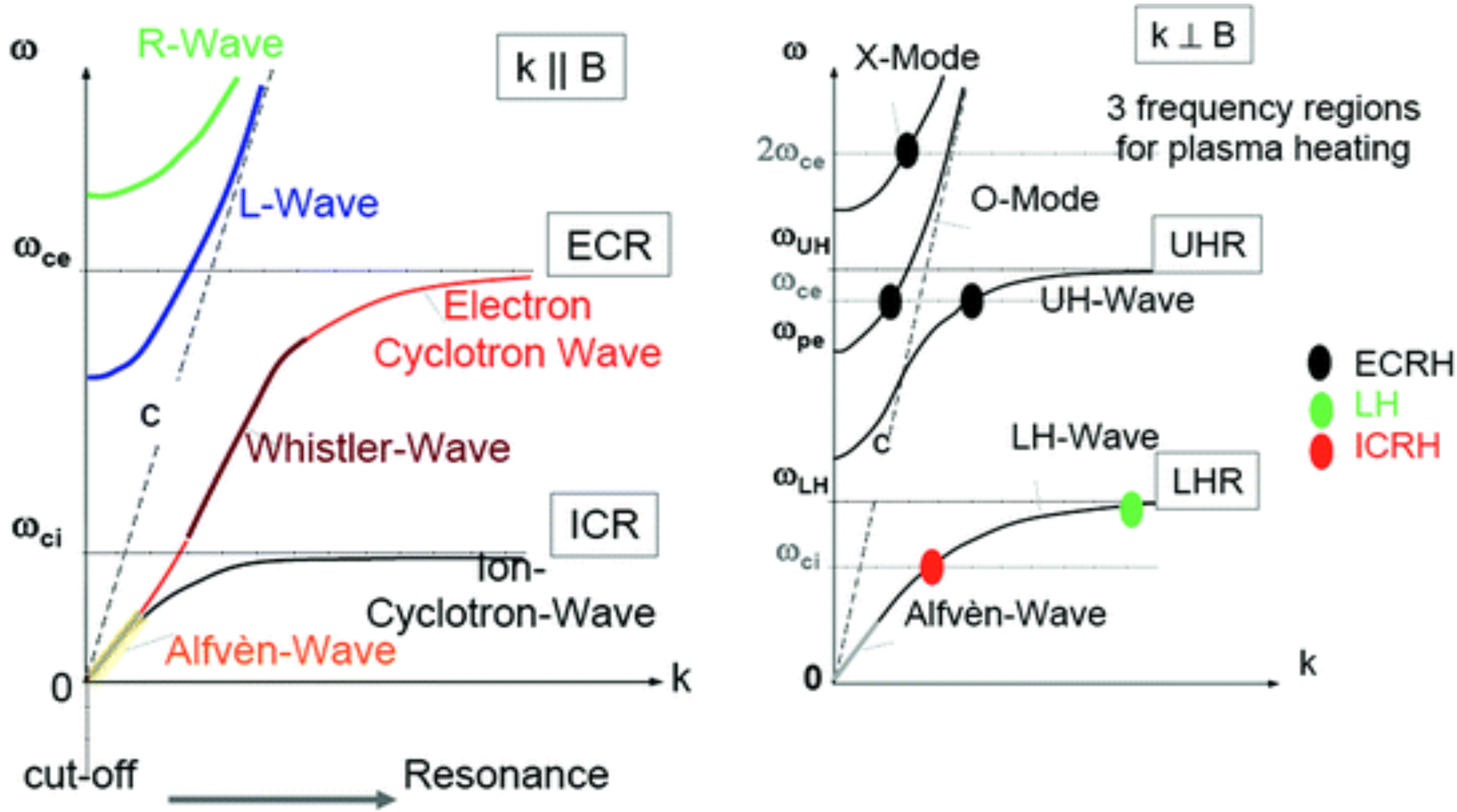
- Similarly, the ion motion needs to be added if we want investigate the low frequency behavior for X-waves, we obtain the `lower hybrid resonance frequency':

$$\omega_{lh} = \left(\omega_{ci}^2 + \frac{\omega_{\mathrm{pi}}^2 \omega_{ce}^2}{\omega_{\mathrm{pe}}^2 + \omega_{ce}^2}\right)^{1/2}$$

→ *Wave absorbed*

- The X-mode has a resonance at $\omega = \omega_{lh}$ since $k \to \infty$ as $\omega \to \omega_{lh}$
- In the limit of high electron density $\omega_{ce}^2 \ll \omega_{\mathrm{pe}}^2$, the lower hybrid frequency approaches ω_{ci}

Plasma waves ZOO



Summary of lectures 6 – 7

- Ion-acoustic waves equivalent to soundwaves in plasma.
- At very low frequencies ion dynamics are important, and we find Alfvén waves (frozen-in flux).
- The Alfvén speed:

$$v_A = \left(\frac{B_0^2}{\mu_0 \rho_m}\right)^{1/2}$$

- In a magnetized plasma the conductivity and dielectric permittivity are tensors.
- Waves exist when $\mathbf{M} \cdot \mathbf{E} = 0$, where $\mathbf{M} = \frac{\omega^2}{c^2}\boldsymbol{\varepsilon} + \mathbf{kk} - k^2\mathbf{I}$
- For waves parallel to the B field, the electrostatic wave exists as before, plus circularly polarized waves.
- Waves travelling perpendicular to the B-field give rise to O and X waves.