Plasma Physics

TU Dresden

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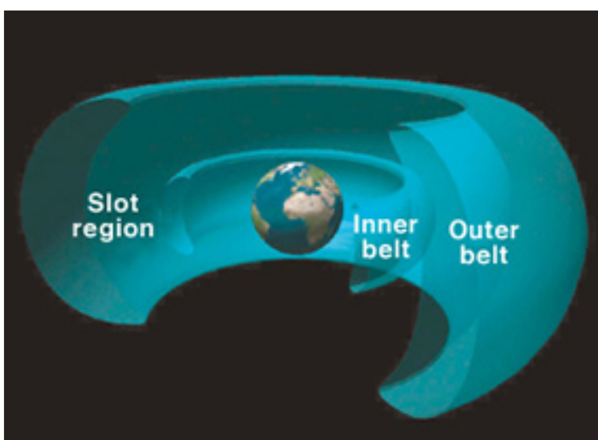
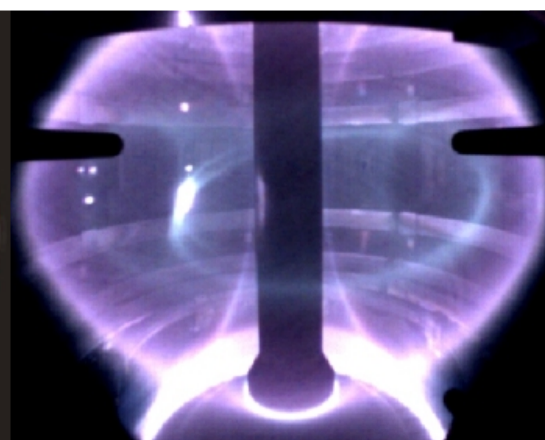
Lecture 10: Plasma instabilities



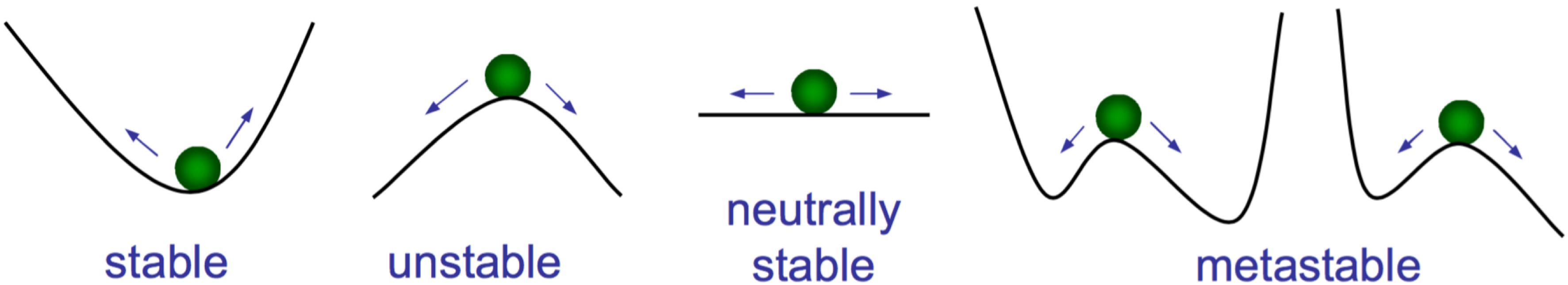



Plasma Physics: lecture 10

- Definition of instability
- Types of instabilities in plasma
- Magnetic field instabilities revisited
- The Rayleigh-Taylor instability
- Parametric instabilities
- Some examples of common instabilities

Plasma instabilities overview

- Plasmas are susceptible to unstable situations:



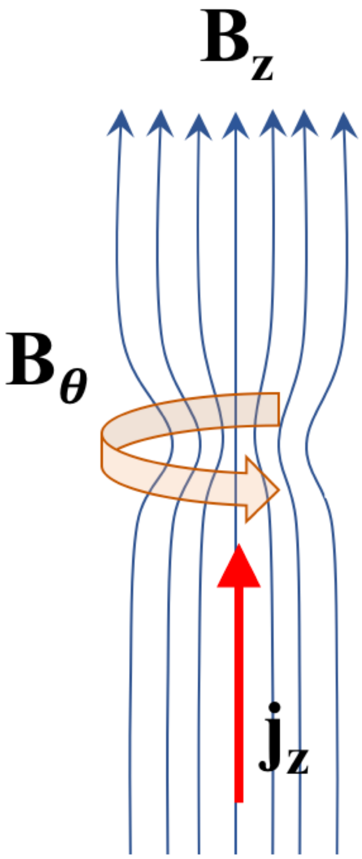
- Definition of stability: deflection from equilibrium position leads to a restoring force, which drives the system/object back to the equilibrium position.
- Also, a stable point is defined as potential energy minimum.
- Normal mode analysis: decompose initial perturbation into Fourier components $\propto \exp(-i\omega t)$

Types of plasma instabilities

- *Microinstabilities*: in homogenous plasmas, where the distribution function deviates substantially from a Maxwellian
  - Beam instabilities
  - Buneman instability
  - Weibel instability
  - Two plasmon decay
  - Parametric instability
  - Stimulated Brillouin and Raman scattering
- *Macroinstabilities*: characterized by inhomogeinity in real space
  - Magnetic pinch instabilities (sausage and kink)
  - Rayleigh-Taylor instability
  - etc.

The sausage instability

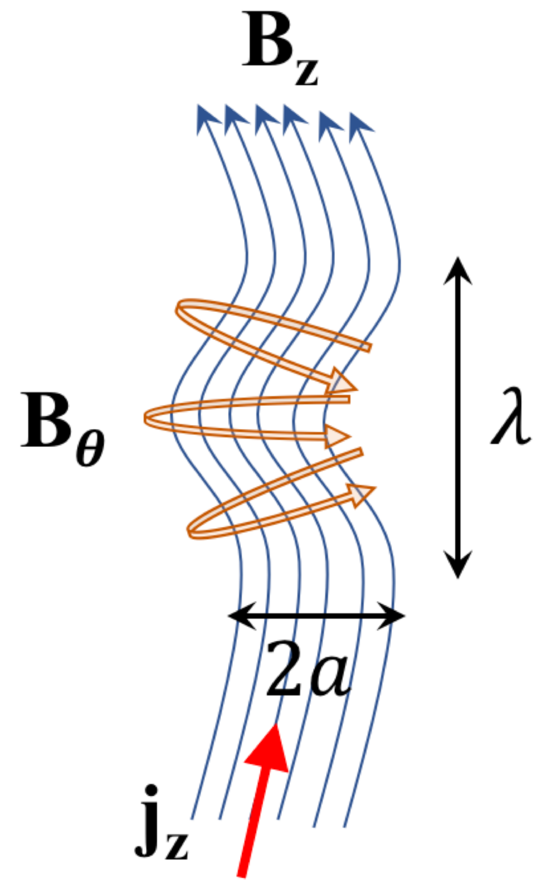
- Mode $m = 0$ MHD pinch instability, i.e. no instability structure in θ direction
- Neck with smaller radius forms by random fluctuation in plasma bulk
- As neck contracts the bulge expands
  - Neck region: $j \times B > \nabla P$ → **compression**
    - → Larger magnetic pressure $\frac{B^2}{2\mu_0}$
    - → Larger magnetic tension force $\frac{B^2}{\mu_0 r}$
  - Bulge region: $j \times B < \nabla P$ → **expansion**
- Positive feedback enhances instability



$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

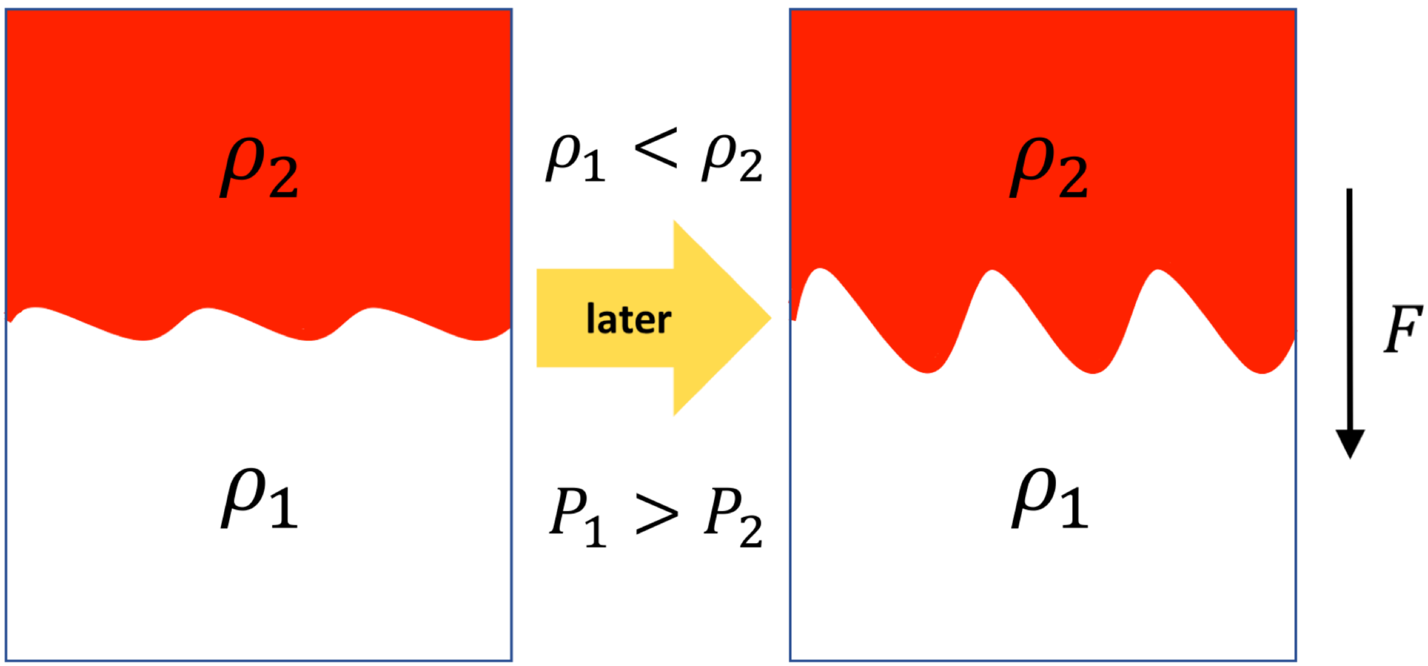
The kink instability

- Mode $m = 1$ MHD pinch instability, i.e. one period in θ direction for each period in z
- Random fluctuation in plasma flow "bends" the pinch slightly
  - Contracted region: B-lines concentrated → higher $\mathbf{j}\times\mathbf{B}$
  - Stretched region: B-lines rarefied → lower $\mathbf{j}\times\mathbf{B}$
  - → **net displacement**
- Positive feedback enhances instability as force imbalance increases



Rayleigh-Taylor instability

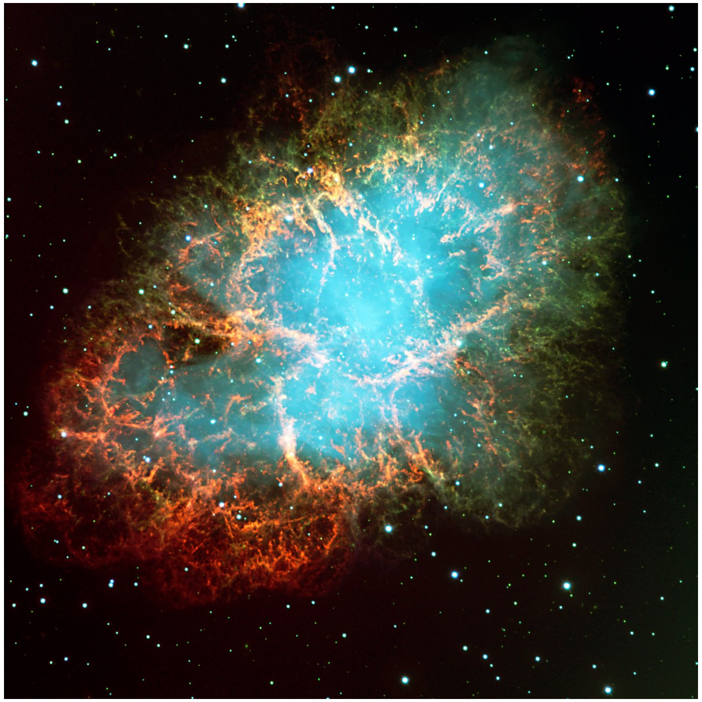
- The R-T instability is an instability that occurs whenever the pressure and density gradients are opposed.

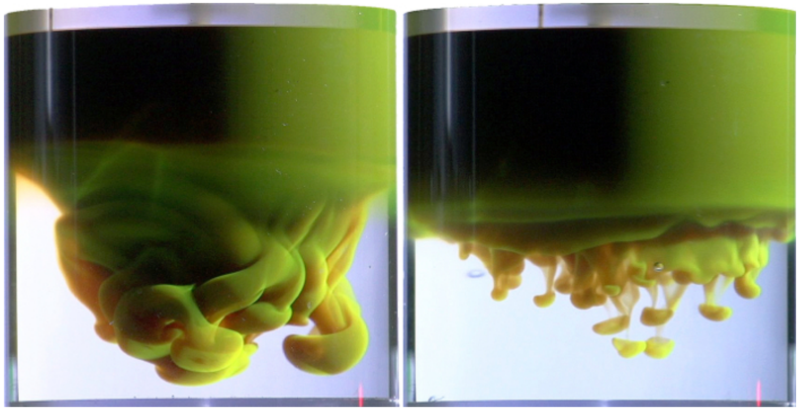


- Instability starts as uneven surface seed and grows with time → **positive feedback enhances the instability**.

Rayleigh-Taylor instability in nature

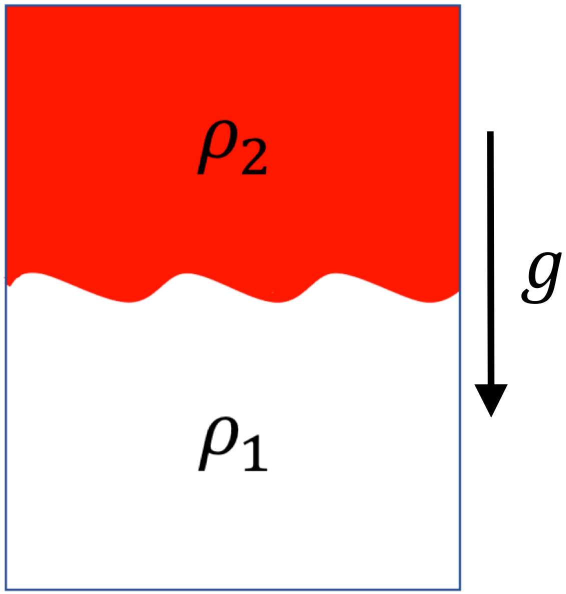
- The R-T instability is quite common in nature.
- It is seen in the universe, i.e. astrophysical shocks (supernova)
- R-T instability causes problems in inertial confinement fusion (ICF) causing shell break-up.





Rayleigh-Taylor growth rate

- Consider heavy fluid of density ρ_2 on top of light fluid of density ρ_2 in a gravitational field (g)
- Use fluid equations:



$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

Continuity equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \rho g \hat{\mathbf{z}}$$

Momentum equation

- In equilibrium the fluid has velocity $\mathbf{u} = 0$, but the pressure and density depend on z (vertical direction), i.e. $P = P(z)$ and $\rho = \rho(z)$

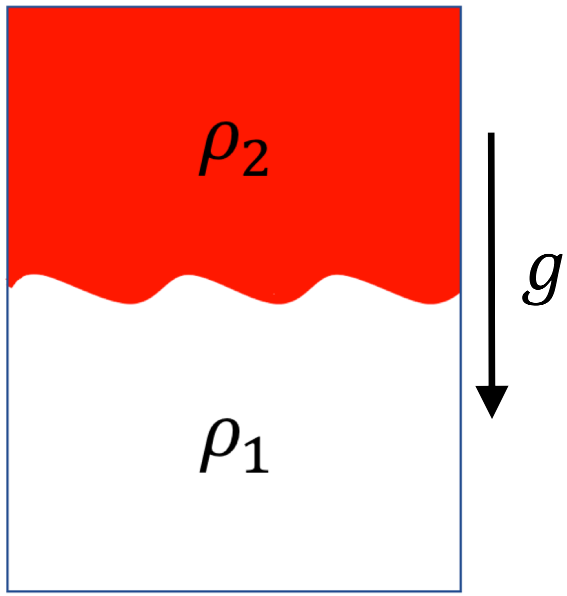
Rayleigh-Taylor growth rate

- Introduce a small perturbation, i.e. the ripples on the fluid interface (also in x and y):

$$P \rightarrow P + \delta P$$

$$\rho \rightarrow \rho + \delta\rho$$

$$\mathbf{u} \neq 0$$



- Insert in the continuity equation:

$$\frac{\partial(\rho + \delta\rho)}{\partial t} + \mathbf{u} \cdot \nabla(\rho + \delta\rho) = 0$$

- Linearize and subtract from equilibrium equation:

$$\frac{\partial(\delta\rho)}{\partial t} + u_z \frac{\partial \rho}{\partial z} = 0$$

Considering the z velocity component

Rayleigh-Taylor growth rate

- Repeat the same procedure with inserting perturbation into the momentum equation. Linearize and subtract equilibrium state. We then find:

$$\rho \frac{\partial u_x}{\partial t} = -\frac{\partial(\delta P)}{\partial x}$$
$$\rho \frac{\partial u_z}{\partial t} = -\frac{\partial(\delta P)}{\partial z} - g(\delta\rho)$$

- Assume wavelike solution for perturbation:

$$\mathbf{u} = \mathbf{u_0} \exp(ikx) \exp(\gamma t)$$
$$\delta\rho = \delta\rho_0 \exp(ikx) \exp(\gamma t)$$
$$\delta P = \delta P_0 \exp(ikx) \exp(\gamma t)$$

Growth rate γ

Rayleigh-Taylor growth rate

- Substitute the wavelike solution to the perturbed continuity equation:

$$\gamma \delta \rho = -u_z \frac{\partial \rho}{\partial z}$$

- And into the momentum equation result:

$$\gamma \rho u_x = -ik\delta P$$
$$\gamma \rho u_z = -\frac{\partial(\delta P)}{\partial z} - g\delta\rho$$

- Note that pressure and density depend on z, but the velocity has a transverse component in x.
- Solve for instability growth rate γ.

Rayleigh-Taylor growth rate

- Assume the fluid is incompressible (though the density profile can vary thanks to mixing of the fluids):

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0$$

$$iku_x + \frac{\partial u_z}{\partial z} = 0$$

- And substitute for u_x:

$$\delta P = -\frac{\gamma\rho}{k^2}\frac{\partial u_z}{\partial z}$$

- Now, substitute for δP and $\delta\rho$:

This equation describes the evolution of the fluid after the perturbation.

$$\boxed{\frac{\partial}{\partial z}\left(\rho\frac{\partial u_z}{\partial z}\right) = k^2\rho u_z - \frac{gk^2}{\gamma^2}u_z\frac{\partial \rho}{\partial z}}$$

Rayleigh-Taylor growth rate

- Apply boundary conditions (constraints):
  - Initial boundary between fluids at $z = 0$
  - Velocity u_z is continuous across boundary (no gap)
  - Spatial derivative $\partial u_z / \partial z$ continuous across the interface
  - Other quantities not conserved, i.e. density not continuous
- Thus, the jump in quantities across the interface:

$$\Delta \left(\rho \frac{\partial u_z}{\partial z} \right) = -\frac{gk^2}{\gamma^2} u_z \Delta(\rho)$$

$$\Delta f \equiv f(0)_+ - f(0)_-$$

For positive and negative sides

- For fluids 1 and 2:

$$\rho_2 \left(\frac{\partial u_z}{\partial z} \right)_2 - \rho_1 \left(\frac{\partial u_z}{\partial z} \right)_1 = -\frac{gk^2}{\gamma^2} u_z (\rho_2 - \rho_1)$$

Rayleigh-Taylor growth rate

- Assume that u_z grows exponentially with k: $u_z \propto \exp(kz)$
- u_z must go to zero at $\pm\infty$:

$$u_z = A\exp(kz) \qquad (z<0)$$
$$u_z = A\exp(-kz) \qquad (z>0)$$

- Substitute:

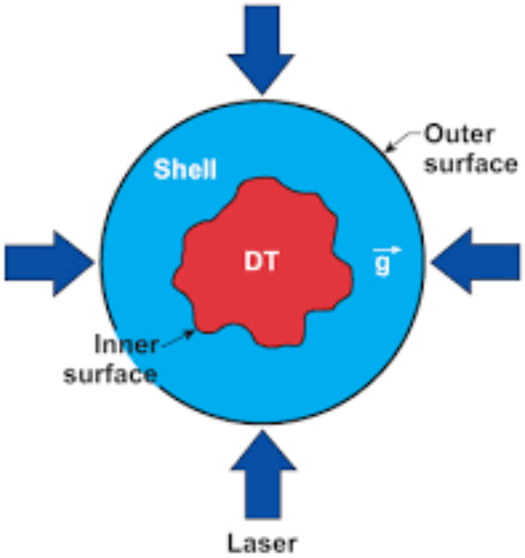
$$-(\rho_2+\rho_1)k = -\frac{gk^2}{\gamma^2}(\rho_2-\rho_1)$$

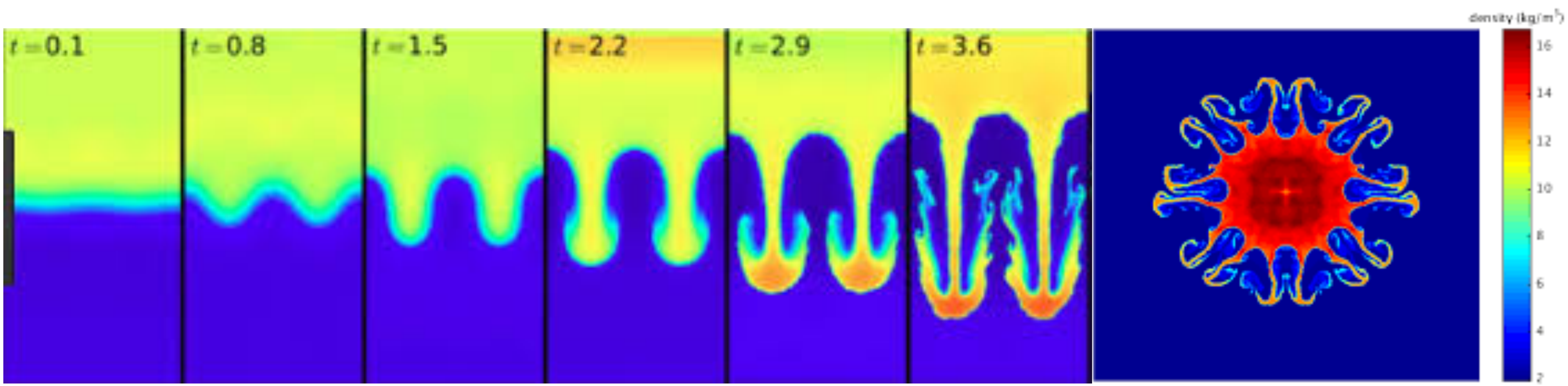
- Thus R-T instability growth rate:

$$\boxed{\gamma = \sqrt{\left(\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\right)gk}}$$

Rayleigh-Taylor instability in ICF

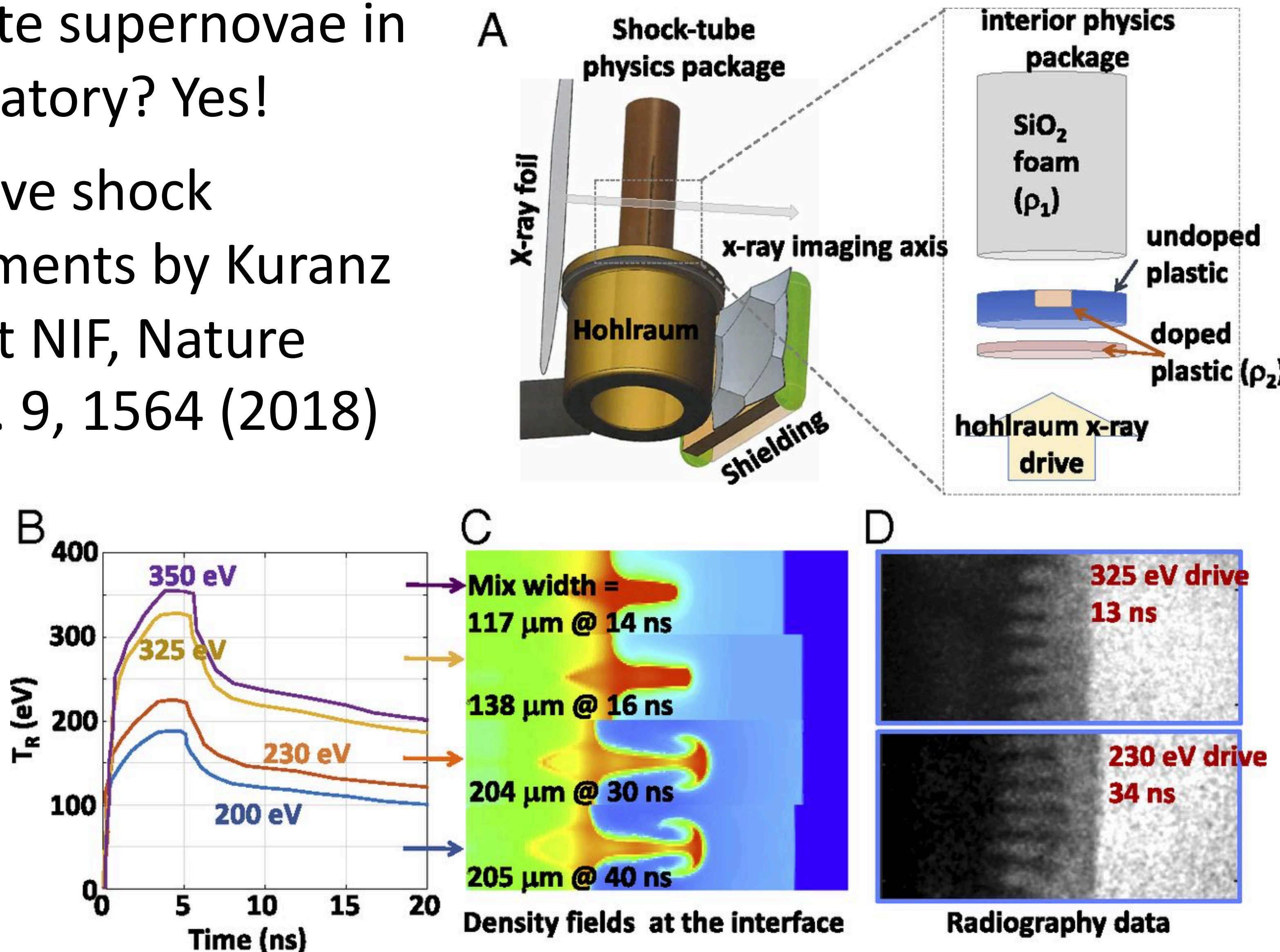
- The instability can be seeded by the fact that the laser/x-ray illumination is not perfectly spherically symmetric.
- It can also be seeded by a surface roughness in the original target.
- R-T limits the aspect ratio in ICF pellets.





Laboratory astrophysics

- Simulate supernovae in a laboratory? Yes!
- Radiative shock experiments by Kuranz *et al*. at NIF, Nature Comm. 9, 1564 (2018)



Parametric instabilities

- Waves exist within plasmas simply from thermal noise (plasma waves, ion acoustic waves).
- An incident wave, of frequency ω_0, can interact with one of these waves, of frequency ω_1, to produce a new wave of frequencies $\omega_2 = \omega_0 - \omega_1, \omega_3 = \omega_0 + \omega_1$.
- The wave of frequency ω_2 can beat with the laser to produce a frequency of ω_1 once more, that can enhance the wave within the plasma.
- If the rate at which laser energy is put into the mode with frequency ω_1 exceeds damping rates, then the mode can grow exponentially - this is known as a parametric instability.

Types of parametric instabilities

Name	Mode 1	Mode 2
Stimulated Brillouin Scattering	Scattered Light	Ion Acoustic Wave
Stimulated Raman Scattering	Scattered Light	Plasma Wave
Parametric Instability	Ion Acoustic Wave	Plasma Wave
Two Plasmon Decay	Plasma Wave	Plasma Wave

Parametric instabilities

- Let the incident laser have amplitude A_0, and the two other waves (which will grow) have amplitudes A_1 and A_2 and frequencies ω_1 and ω_2. In absence of a driver:

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \omega_1^2 A_1 = 0$$
$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}t^2} + \Gamma_2 \frac{\mathrm{d}A_2}{\mathrm{d}t} + \omega_2^2 A_2 = 0$$

Damping rates Γ_1 and Γ_2

- Now, turn on the laser:

Coupling constants c_1 and c_2

$$\frac{\mathrm{d}^2 A_1}{\mathrm{d}t^2} + \Gamma_1 \frac{\mathrm{d}A_1}{\mathrm{d}t} + \omega_1^2 A_1 = c_1 A_2 A_0$$
$$\frac{\mathrm{d}^2 A_2}{\mathrm{d}t^2} + \Gamma_2 \frac{\mathrm{d}A_2}{\mathrm{d}t} + \omega_2^2 A_2 = c_2 A_1 A_0$$

Parametric instabilities

- As modes 1 and 2 can grow so quickly, they have potentially a wide range of frequencies. We therefore use the method of Fourier transforms:

$$A(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t) A(t) \mathrm{d}t$$

- Assume that A_0 is unaffected, laser is not depleted:

$$A_0 = E_0 \cos(\omega_0 t) = \frac{E_0}{2}[\exp(i\omega_0 t) + \exp(-i\omega_0 t)]$$

- Take a Fourier transform of mode 1:

$$[\omega^2 - \omega_1^2 + i\omega\Gamma_1]A_1(\omega) + \frac{c_{1E_0}}{2}[A_2(\omega + \omega_0) + A_2(\omega - \omega_0)] = 0$$

Parametric instabilities

- Rewrite as:

$$D_1(\omega)A_1(\omega) + \frac{c_1 E_0}{2}[A_2(\omega+\omega_0) + A_2(\omega-\omega_0)] = 0$$

- Where: $D_i(\omega) = \omega^2 - \omega_i^2 + i\omega\Gamma_i$
- Notice that mode 1, at some frequency ω, couples with mode 2 at frequencies $\omega + \omega_0$, and $\omega - \omega_0$, so we should take the transform of the equation for the second mode at these frequencies:

$$D_2(\omega-\omega_0)A_2(\omega-\omega_0) + \frac{c_2 E_0}{2}[A_1(\omega) + A_1(\omega-2\omega_0)] = 0$$

$$D_2(\omega+\omega_0)A_2(\omega+\omega_0) + \frac{c_2 E_0}{2}[A_1(\omega) + A_1(\omega+2\omega_0)] = 0$$

Parametric instabilities

- One of the modes (say 1), may have a frequency close to that of the laser, assume $A_1(\omega) \gg A_1(\omega \pm 2\omega_0)$
- Also, mode 2 has lower frequency than the incoming laser, so consider $A_2(\omega - \omega_0)$ and ignore $A_2(\omega + \omega_0)$.
- Obtain pair of coupled equations (matrix form):

$$\begin{pmatrix} D_1(\omega) & c_1 E_0/2 \\ c_2 E_0/2 & D_2(\omega - \omega_0) \end{pmatrix} \begin{pmatrix} A_1(\omega) \\ A_2(\omega - \omega_0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Non-trivial solution:

$$D_1(\omega) D_2(\omega - \omega_0) - \frac{c_1 c_2 E_0^2}{4} = 0$$

Parametric instabilities

- With the approximations $\omega_0 - \omega \approx \omega_2$ and $\omega + \omega_1 \approx 2\omega_1$ we find:

$$-4\omega_1\omega_2\left(\omega - \omega_1 + i\frac{\Gamma_1}{2}\right)\left(\omega - \omega_0 + \omega_2 + i\frac{\Gamma_2}{2}\right) - \frac{c_1c_2E_0^2}{4} = 0$$

- For mode 1 to grow, it must have a complex frequency - we look at its real and imaginary parts: $\omega = \omega_\mathrm{R} + i\Gamma$
- Equate the real parts to zero:

$$(\omega_\mathrm{R} - \omega_1)(\omega_\mathrm{R} - \omega_1 - \Delta) + \psi E_0^2 = \left(\Gamma + \frac{\Gamma_1}{2}\right)\left(\Gamma + \frac{\Gamma_2}{2}\right)$$

- Where: $\Delta \equiv \omega_0 - \omega_1 - \omega_2$ and $\psi \equiv \dfrac{c_1c_2}{16\omega_1\omega_2}$

Parametric instabilities

- Equate the imaginary parts:

$$\omega_{\mathrm{R}} = \omega_1 + \frac{(\Gamma + \frac{1}{2}\Gamma_1)\Delta}{2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2}$$

- Eliminate ω_R by substitution:

$$E_0^2 = \frac{1}{\psi}\left\{(\Gamma + \frac{1}{2}\Gamma_1)(\Gamma + \frac{1}{2}\Gamma_2)\left[1 + \frac{\Delta^2}{(2\Gamma + \frac{1}{2}\Gamma_1 + \frac{1}{2}\Gamma_2)^2}\right]\right\}$$

- The instability will grow if Γ is positive, i.e. the threshold field is found for $\Gamma = 0$:

$$E_{0,\mathrm{th}}^2 = \frac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1 c_2}\left[1 + \frac{4\Delta^2}{(\Gamma_1 + \Gamma_2)^2}\right]$$

Parametric instabilities

- This threshold intensity will be a minimum when exactly on resonance – i.e. when $\Delta = 0$, so the minimum intensity is:

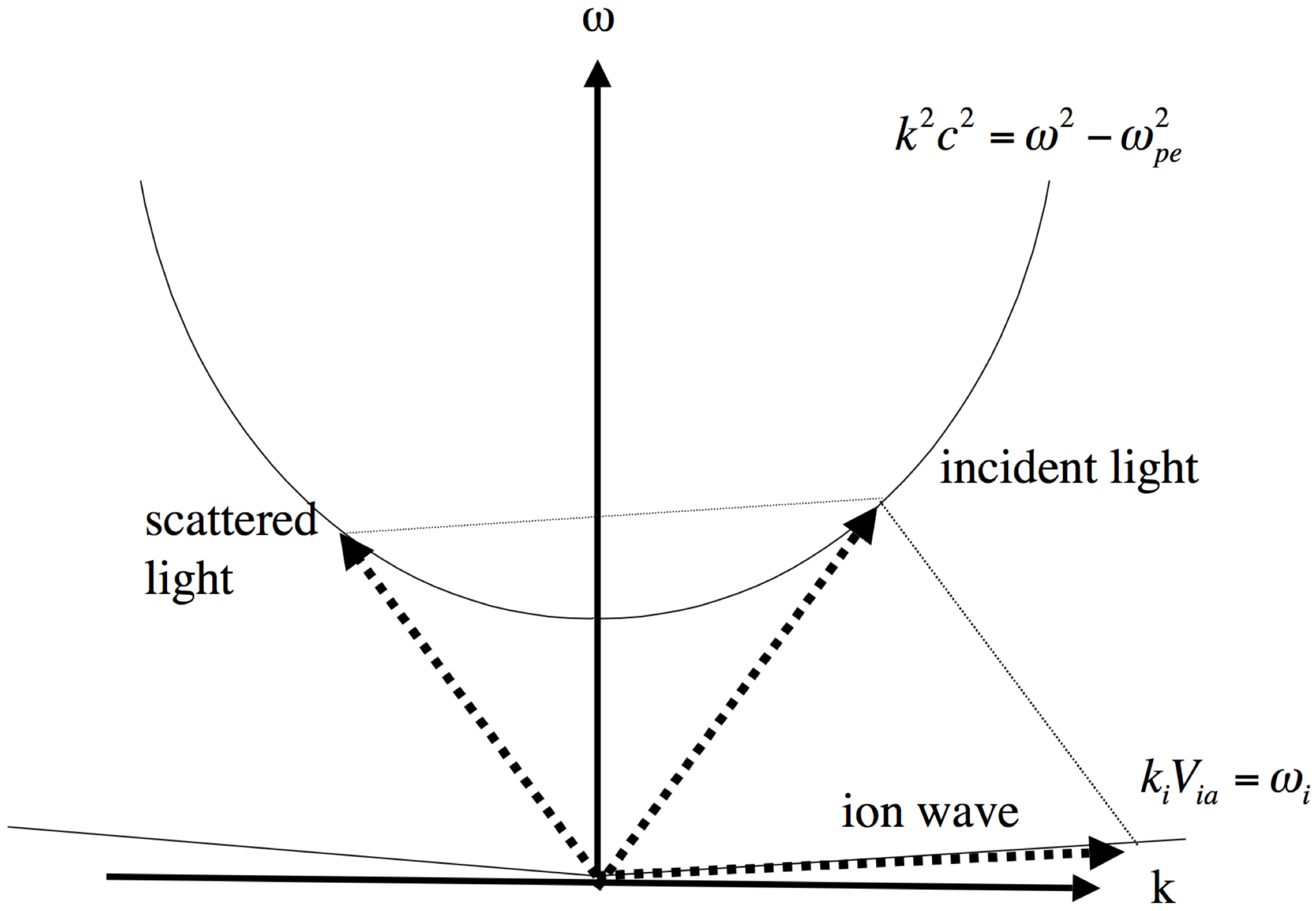
$$E_{0,\mathrm{min}}^2 = \frac{4\omega_1\omega_2\Gamma_1\Gamma_2}{c_1c_2}$$

- The growth rate will be a maximum if the damping of the two waves is zero, again on resonance: $\Gamma^2 = E_0^2\psi$

$$\boxed{\Gamma_{\mathrm{max}} = E_0\sqrt{\left(\frac{c_1c_2}{16\omega_1\omega_2}\right)}}$$

Stimulated Brillouin Scattering (SBS)

- Incoming laser generates ion-acoustic waves and scattered light at slightly lower frequency.



Stimulated Brillouin Scattering (SBS)

- Detailed calculation of the SBS growth rate is provided in the supplemental material to this lecture.
- SBS can take place all the way up to a density of:

$$n_e = n_c \left(1 + 4\frac{V_{\mathrm{ia}}^2}{c^2}\right)^{-1}$$

- SBS growth rate:

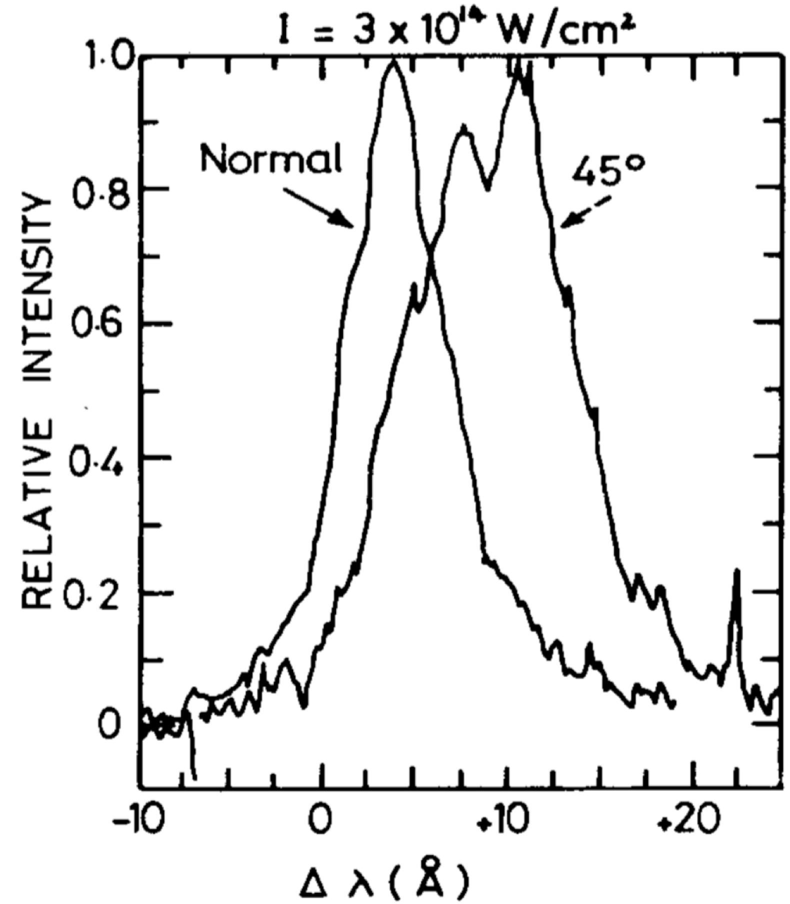
$$\Gamma_{\mathrm{brill}} = \frac{\omega_{\mathrm{pe}}^2 E_0}{4\omega_0 n_0}\left(\frac{c}{2\varepsilon_0 n_0}\sqrt{M k_{\mathrm{B}} T_e}\right)^{-1/2}$$

- For incident laser intensity:

$$\boxed{\Gamma_{\mathrm{brill}} = \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 V_{\mathrm{ia}} M}}}$$

Stimulated Brillouin Scattering (SBS)

- SBS is so damaging because it can reflect light out of the plasma, preventing it from being absorbed.
- For laser of $\lambda = 1\mu$m and intensity of 10^{14} Wcm^{-2} incident upon plasma of 1 keV and 10^{20} cm^{-3}, we get growth rate of 10^{12} s^{-1}.
- Significant scattering (and saturation) reached within picoseconds.
- Laser drive must be kept low in ICF and laser experiments.



Stimulated Raman Scattering (SRS)

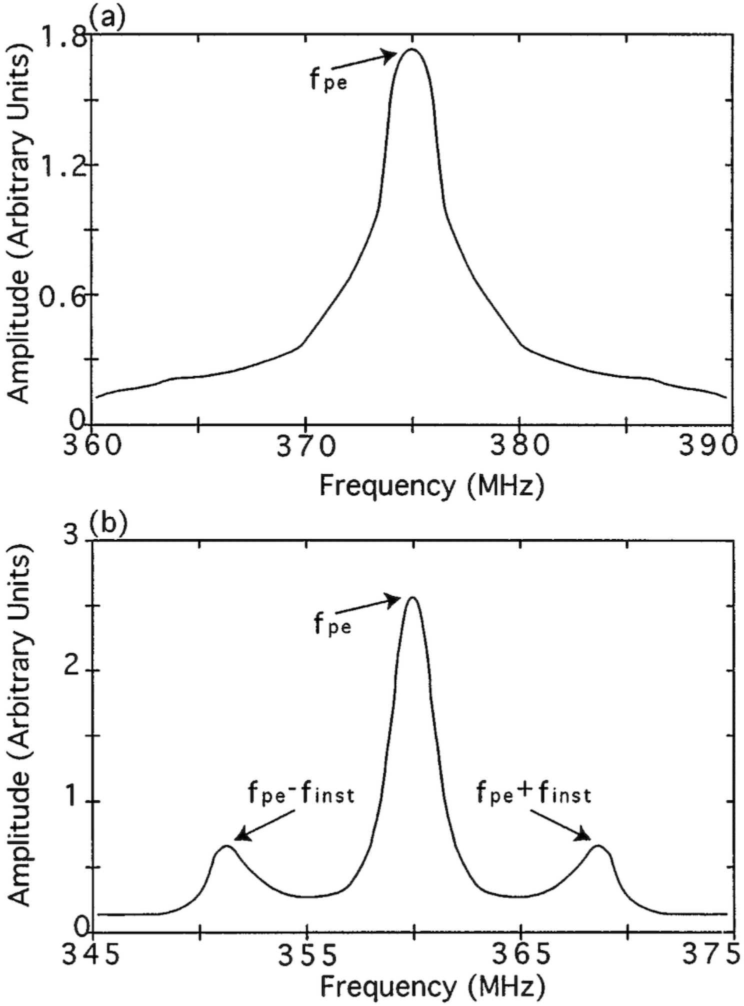
- Here the stimulated waves are the scattered wave and the electron plasma wave.
- The resultant high amplitude plasma waves result in hot electrons and are thus a problem for ICF (fuel preheat).
- As light cannot propagate above the local critical density, this instability cannot take place above the $\frac{1}{4}$ critical surface (if more than half the laser photon energy goes into the plasmon, the scattered wave cannot propagate.
- The growth rate is analogous to the SBS case:

$$\boxed{\Gamma_{\mathrm{Raman}} = \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 v_{\mathrm{ph}} m_e}}}$$

Replace with the plasma wave phase velocity v_{ph} and the electron mass m_e

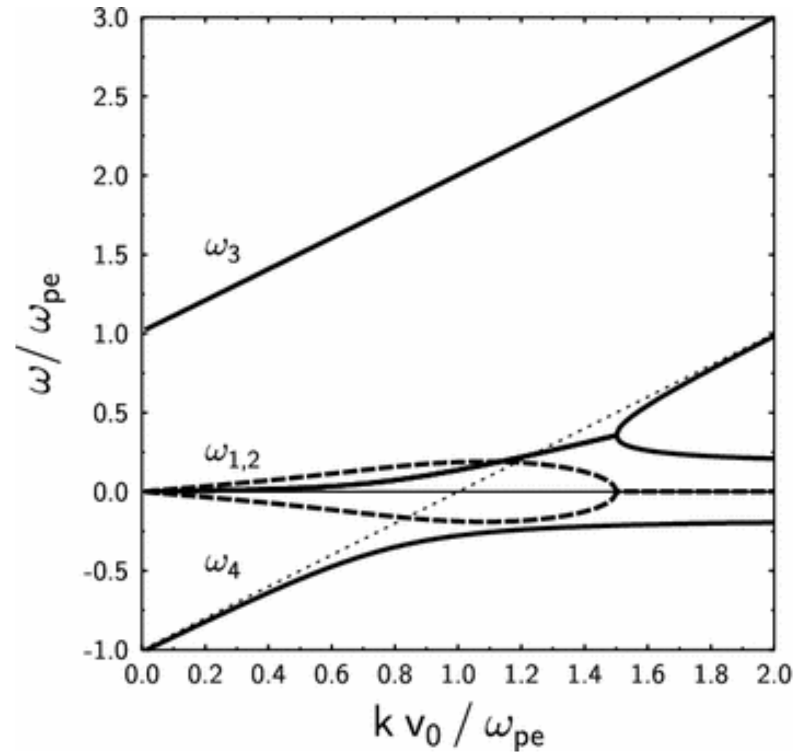
Beam plasma instability

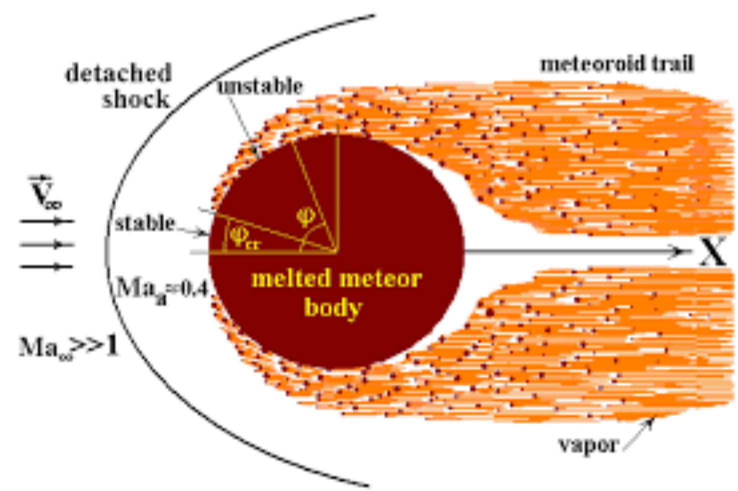
- Also called two-stream instability
- Langmuir oscillations can spontaneously grow in a non-equilibrium plasma, i.e. in presence of particle beams
- Two-stream instability can arise from the case of two cold beams, in which no particles are resonant with the wave, or from two hot beams, in which there exist particles from one or both beams which are resonant with the wave



Buneman instability

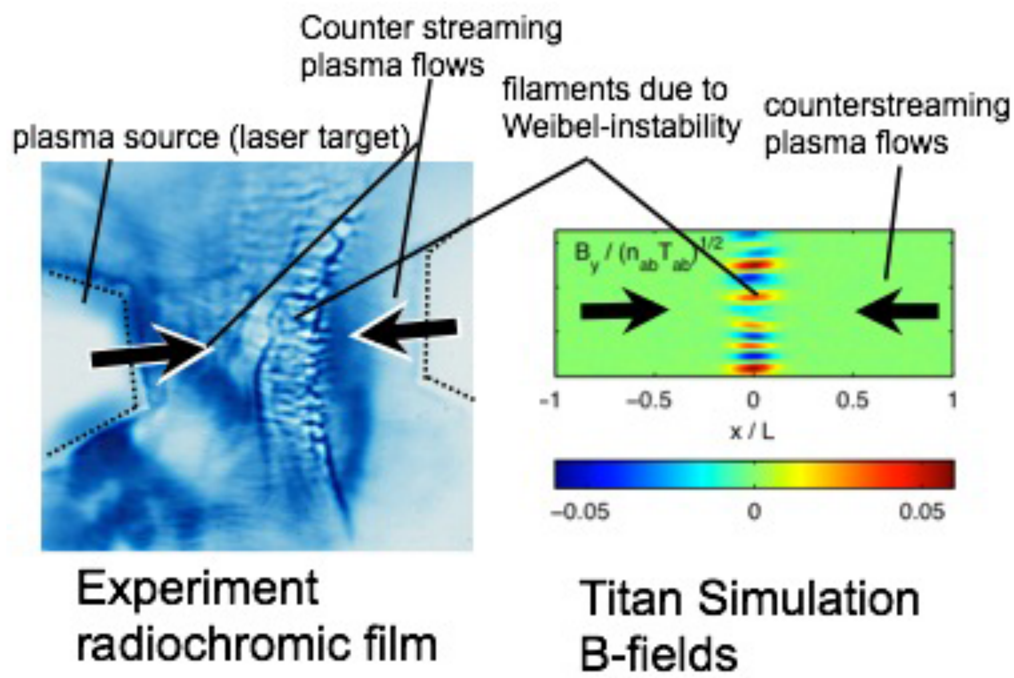
- Modified two-stream instability arising from the difference in drifts of electrons and ions exceeding the ion acoustic speed
- Present in the equatorial and polar ionospheric E-regions. In particular, it occurs in the equatorial electron jet due to the drift of electrons relative to ions and also in the trails behind ablating meteoroids.





Weibel instability

- Present in homogeneous or nearly homogeneous electromagnetic plasmas which possess an anisotropy in momentum (velocity) space as an extreme case of a beam instability.
- Perturbations are electromagnetic and result in filamentation as opposed to electrostatic perturbations (superposition of many counter-streaming beams) which would result in charge bunching.
- The Weibel instability causes exponential growth of electromagnetic fields.
- Common in astrophysical plasmas, such as collisionless shock formation in supernova remnants and gamma-ray bursts.



Summary of lecture 10

- Magnetic fusion (Z-pinch, torus configurations) are sensitive to sausage and kink instabilities that cause issues for plasma confinement
- The Rayleigh-Taylor Instability places a limit on the aspect ratio of the target that can be safely used without the shell breaking up due to hydrodynamic instabilities.
- SBS occurs above a certain threshold intensity, and causes light to be reflected away from the target and wasted.
- SRS again occurs above a certain threshold, and results in plasma waves that again cause hot electrons to be produced.
- Present research indicates ICF can just operate below the thresholds for SRS and SBS.

Supplemental material for lecture 10

Stimulated Brillouin Scattering (SBS)

- The dispersion relations for the incoming laser (k_0, ω_0), scattered light (k_s, ω_s), and ion-acoustic waves (k_i, ω_i), :

$$k_0^2 c^2 = \omega_0^2 - \omega_{\mathrm{pe}}^2$$
$$k_{\mathrm{s}}^2 c^2 = \omega_{\mathrm{s}}^2 - \omega_{\mathrm{pe}}^2$$
$$k_{\mathrm{i}}^2 V_{\mathrm{ia}}^2 = \omega_{\mathrm{i}}^2$$

- As the light cannot have a frequency lower than the plasma frequency, there is a minimum value of k_i which can backscatter the light, and this will produce backscattered light at k_s, just lower than zero, i.e. $k_s = 0$, $\omega_s \approx \omega_{pe}$, $k_0 \approx k_i$, and $\omega_s \approx \omega_{pe}$.
- The dispersion relation for the light is: $k_0^2 c^2 = \omega_0^2 - \omega_{\mathrm{pe}}^2$

Stimulated Brillouin Scattering (SBS)

- But because $k_i \approx k_0$:

$$k_{\mathrm{i,min}}^2 c^2 \approx \omega_0^2 - \omega_{\mathrm{pe}}^2$$
$$k_{\mathrm{i,min}}^2 c^2 \approx (\omega_{\mathrm{i}} + \omega_{\mathrm{pe}})^2 - \omega_{\mathrm{pe}}^2$$
$$k_{\mathrm{i,min}}^2 c^2 \approx \omega_{\mathrm{i}}^2 + \omega_{\mathrm{pe}}^2 + 2\omega_{\mathrm{i}}\omega_{\mathrm{pe}} - \omega_{\mathrm{pe}}^2$$

- And for $\omega_i \ll \omega_{pe}$: $k_{\mathrm{i,min}} \approx 2\frac{V_{\mathrm{ia}}}{c^2}\omega_{\mathrm{pe}}$
- Therefore: $\omega_{\mathrm{i,min}} \approx 2\frac{V_{\mathrm{ia}}^2}{c^2}\omega_{\mathrm{pe}}$
- Rewrite in terms of electron and critical densities:

$$4\frac{V_{\mathrm{ia}}^2}{c^2} = \frac{n_{\mathrm{c}}}{n_e} - 1$$

Stimulated Brillouin Scattering (SBS)

- Thus, SBS can take place all the way up to a density of:

$$n_e = n_c \left(1 + 4\frac{V_{\rm ia}^2}{c^2}\right)^{-1}$$

- Light scattered at high densities has frequency close to original laser frequency ω_0.
- At low density the scattered light has $k_s \sim k_i/2$ ($k_i \approx 2k_0$). Hence the dispersion relation for the scattered light is:

$$\frac{k_{\rm i}^2 c^2}{4} = \omega_{\rm s}^2 - \omega_{\rm pe}^2 \approx \omega_0^2 - \omega_{\rm pe}^2$$

$$\frac{\omega_{\rm i}^2 c^2}{4V_{\rm ia}^2} = \omega_0^2\left(1 - \frac{\omega_{\rm pe}^2}{\omega_0^2}\right) = \omega_0^2\left(1 - \frac{n_e}{n_c}\right)$$

$$\frac{\omega_{\rm i}}{\omega_0} = \frac{2V_{\rm ia}}{c}\sqrt{1 - \frac{n_e}{n_c}}$$

Stimulated Brillouin Scattering (SBS)

- The greatest fractional shift in the light frequency is of order $2V_{ia}/c$, which is a function of temperature and ion mass, but for a high Z ion, and assuming $A/Z = 2$, implies:

$$\left(\frac{\omega_0 - \omega_s}{\omega_0}\right)_{\max} \approx 5 \times 10^{-5} T_e^{1/2} \longleftarrow \text{Temperature in eV}$$

- Note: to examine the threshold intensities and dumping rate of the instability, we need to determine the coupling constants c_1, c_2 and damping rates Γ_1, Γ_2.
- Damping rate difficult due to combined effects (Landau damping with density and velocity gradients). Restrict analysis to obtaining c_1 and c_2.

Stimulated Brillouin Scattering (SBS)

- Denote the amplitude of the incident laser wave by E_0, that of the scattered wave by E.
- Assume that the ion acoustic wave gives rise to ion density fluctuations n_{ia} around the background density, n_0, such that: $n_{\mathrm{ia}} = n_{\mathrm{a}} \cos(k_{\mathrm{a}} x)$
- Oscillations in the ions are slow, the electrons are able to follow, keeping overall charge neutrality. Therefore there is also a periodic variation in the electron density, which is associated with the ion acoustic wave. We call this amplitude of the electrons n_{ea}, and $n_{ea} = n_{ia}$.
- The spatially averaged energies of the incident and reflected waves are $\varepsilon_0 E_0^2/2$ and $\varepsilon_0 E^2/2$.

Stimulated Brillouin Scattering (SBS)

- Determine the spatially averaged energy of the ion wave by the conservation of energy. Assuming that ion-waves are of low frequency and with isothermal equation of state.
- Define condensation s: $$s = \frac{n_{\mathrm{ia}}}{n_0} = -\frac{\int \mathrm{d}V}{V_0}$$
- The excess pressure is periodic, with the total pressure given by:

Ambient pressure

$$P = P_0 + p = P_0\left(1 + \frac{n_{\mathrm{ia}}}{n_0}\right)$$

Excess pressure $p = P_0 s$

- The compressive energy increase at any point associated with the wave is given by:

$$\Delta U = -\int p\mathrm{d}V = \int P_0 V_0 s \mathrm{d}s = \frac{1}{2} P_0 V_0 s^2$$

Stimulated Brillouin Scattering (SBS)

- By equipartition there will be an equal amount of kinetic energy associated with the movement of the ions in the wave, but spatially averaging the energy will bring in another factor of $1/2$, and thus the spatially averaged energy per unit volume of the sound wave is given by:

$$U_{\mathrm{a}} = \frac{1}{2} P_0 s^2 = \frac{1}{2} k_B T_e \frac{n_{\mathrm{a}}^2}{n_0}$$

- By conservation of energy thus:

$$\frac{1}{2}\varepsilon_0 E_0^2 = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2} k_B T_e \frac{n_{\mathrm{a}}^2}{n_0}$$

Note: We assumed that the laser heats up the electrons, although the electrons have time to equilibrate, they do not have enough time during the laser pulse to transfer much energy to the ions, so that the electron temperature is much higher than the ions.

Stimulated Brillouin Scattering (SBS)

- Now, consider the conservation of momentum density, remembering we are dealing with waves (quantized in units of $\hbar\omega$, each with momentum $\hbar\mathbf{k}$, thus multiply energy by $\mathbf{k}/\omega$):

$$\frac{1}{2}\varepsilon_0 E_0^2\left(\frac{\mathbf{k_0}}{\omega_0}\right) = \frac{1}{2}\varepsilon_0 E^2\left(\frac{\mathbf{k}}{\omega}\right) + \frac{1}{2}k_B T_e \frac{n_\mathrm{a}^2}{n_0}\left(\frac{\mathbf{k}_\mathrm{a}}{\omega_\mathrm{a}}\right)$$

- Substitute for E_0 from the energy conservation equation:

$$\varepsilon_0 E^2\left(\frac{\mathbf{k_0}}{\omega_0} - \frac{\mathbf{k}}{\omega}\right) = \frac{k_\mathrm{B} T_e n_\mathrm{a}^2}{n_0}\left(\frac{\mathbf{k}_\mathrm{a}}{\omega_\mathrm{a}} - \frac{\mathbf{k_0}}{\omega_0}\right)$$

Stimulated Brillouin Scattering (SBS)

- Consider densities low compared to n_c, thus $\mathbf{k} \approx -\mathbf{k_0}$ and the speed of light is c, i.e. $\omega/\mathrm{k} \approx \omega_0/k_0 \approx c$
- As the frequency of the ion acoustic wave is so low compared with that of the electromagnetic waves we can assume$\omega \approx \omega_0$ and $(1/V_{ia}) \gg (1/c)$:

$$\frac{2\varepsilon_0 E^2}{c} = \frac{k_\mathrm{B} T_e n_\mathrm{a}^2}{n_0}\left(\frac{1}{V_\mathrm{ia}}\right)$$

$$E = \left(\frac{c}{2\varepsilon_0 n_0}\sqrt{M k_\mathrm{B} T_e}\right)^{1/2} n_\mathrm{a}$$

Ion mass

We have obtained a relation between the amplitude of the scattered electromagnetic wave and the amplitude of the ion wave. What we really seek is how these waves grow as a function of time.

Stimulated Brillouin Scattering (SBS)

- Recall from electromagnetism, wave equation when current is present:

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

- For transverse waves $\mathbf{k} \cdot \mathbf{E} = 0$, thus:

$$-\nabla^2 \mathbf{E} = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial \mathbf{J}}{\partial t}$$

- This simply represents the propagation of the incident wave E_0 within the plasma. We recover the dielectric function of the plasma. However, we are interested in an ion wave present within the system. We thus conclude that the ion wave – the periodic variation in the ion (and hence electron) density is what gives rise to the scattered electromagnetic wave E. The extra periodic current is: $\mathbf{J} = -en_\mathrm{a}\mathbf{v_e}\cos(k_\mathrm{a}x)$

Stimulated Brillouin Scattering (SBS)

- The motion of the electrons is determined by the amplitude of the driving field (laser) according to:

$$m_e \frac{\partial v_e}{\partial t} = -eE_0 \cos(k_0 x - \omega_0 t)$$

$$v_e = -\frac{eE_0}{m_e \omega_0} \sin(k_0 x - \omega_0 t)$$

- Substitute for v_e:

Backscattered by the ion acoustic wave is represented by the term oscillating in space according to $(k_0 - k_a)$

$$J = \frac{n_a e^2 E_0}{2 m_e \omega_0} \{\sin[(k_0 + k_a)x - \omega_0 t] + \sin[(k_0 - k_a)x - \omega_0 t]\}$$

- The rate of change of current to use in the wave equation for the scattered radiation is:

$$\mathbf{k_0} - \mathbf{k_a} = \mathbf{k} \approx -\mathbf{k_0}$$

$$\frac{\partial J}{\partial t} = -\frac{n_a e^2 E_0}{2 m_e} \cos[(k_0 - k_a)x - \omega_0 t] \approx -\frac{n_a e^2 E_0}{2 m_e} \cos(k_0 x + \omega_0 t)$$

Stimulated Brillouin Scattering (SBS)

- Hence, the wave equation that describes the evolution of the amplitude of the scattered wave is:

Undamped simple harmonic oscillator

$$\frac{\partial^2 E}{\partial t^2} + k^2 c^2 E = \frac{\omega_{\mathrm{pe}}^2 n_{\mathrm{a}} E_0}{2 n_0} \cos(k_0 x + \omega_0 t)$$

- Notice that the first of our coupling constants, let us call it c_1, c_2 found after some algebra:

$$c_1 = \frac{\omega_{\mathrm{pe}}^2}{2n_0} \qquad c_2 = \frac{8\varepsilon_0}{M} \frac{\omega_{\mathrm{pe}}^2}{c^2}$$

- Note that for the resonant case when the frequency of E is the same as E_0, $\omega = \omega_0$, then the solution for E is:

$$E = \frac{\omega_{\mathrm{pe}}^2 E_0 t}{4\omega_0} \frac{n_{\mathrm{a}}}{n_0} \sin(k_0 x + \omega_0 t) = E(t) \sin(k_0 x + \omega_0 t)$$

Stimulated Brillouin Scattering (SBS)

- We have:

$$E(t) = \frac{\omega_{\mathrm{pe}}^2 E_0 t}{4\omega_0} \frac{n_{\mathrm{a}}}{n_0}$$

- Therefore:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \frac{\omega_{\mathrm{pe}}^2 E_0}{4\omega_0} \frac{n_{\mathrm{a}}}{n_0}$$

- But recall that the amplitude of the ion acoustic wave is proportional to the amplitude of the scattered electromagnetic wave (by conservation of energy and momentum of the waves). Thus by substituting:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \frac{\omega_{\mathrm{pe}}^2 E_0}{4\omega_0 n_0} \left(\frac{c}{2\varepsilon_0 n_0} \sqrt{M k_{\mathrm{B}} T_e} \right)^{-1/2} E$$

Stimulated Brillouin Scattering (SBS)

- Thus we obtain the SBS growth rate:

$$\Gamma_{\mathrm{brill}} = \frac{\omega_{\mathrm{pe}}^2 E_0}{4\omega_0 n_0}\left(\frac{c}{2\varepsilon_0 n_0}\sqrt{M k_{\mathrm{B}} T_e}\right)^{-1/2}$$

ion-acoustic wave grows exponentially with Γ_{brill}

Peak oscillatory velocity of the electron in the laser field

- Simplify:

$$\Gamma_{\mathrm{brill}} = \frac{1}{2\sqrt{2}}\frac{\omega_{\mathrm{pe}} v_{\mathrm{os}}\sqrt{m_e/M}}{\sqrt{c V_{\mathrm{ia}}}}$$

- For incident laser intensity:

$$\boxed{\Gamma_{\mathrm{brill}} = \frac{1}{2}\frac{\omega_{\mathrm{pe}}^2}{\omega_0}\frac{\sqrt{I_0}}{\sqrt{n_0 c^2 V_{\mathrm{ia}} M}}}$$