

## **A discrete population balance equation for binary breakage**

Liao, Y.; Oertel, R.; Kriebitzsch, S.; Schlegel, F.; Lucas, D.;

Originally published:

February 2018

**International Journal for Numerical Methods in Fluids 87(2018)4, 202-215**

DOI: <https://doi.org/10.1002/flid.4491>

Perma-Link to Publication Repository of HZDR:

<https://www.hzdr.de/publications/Publ-25920>

Release of the secondary publication  
on the basis of the German Copyright Law § 38 Section 4.

# A Discrete Population Balance Equation for Binary Breakage

Y. Lao<sup>a,\*</sup>, R. Oertel<sup>a</sup>, S. Kriebitzsch<sup>b</sup>, F. Schlegel<sup>a</sup>, D. Lucas<sup>a</sup>

<sup>a</sup>Helmholtz-Zentrum Dresden - Rossendorf, Institute of Fluid Dynamics, Bautzner Landstrasse 400, 01328 Dresden, Germany

<sup>b</sup>TU Bergakademie Freiberg, Institute of Energy Process Engineering and Chemical Engineering, Fuchsmühlenweg 9, 09599 Freiberg, Germany

---

## Abstract

The numerical solution of the population balance equation is frequently achieved by means of discretization, i.e., by the method of classes. An important concern of discrete formulations is the preservation of a chosen set of moments of the distribution, e.g. numbers and mass, while remaining flexible on the grid applied. As for the physical modeling of the breakup rate, two approaches exist. One type states the breakup rate of a mother particle and requires a function that describes the distribution of daughter particles. The other type gives the breakup rate between a mother and a daughter particle directly, usually under the assumption of binary breakage. The lack of an explicitly stated daughter size distribution function has implications on the formulation of the discrete equations, because existing formulations contain integrals over the daughter size distribution function. To the knowledge of the authors, no efficient formulations for this type of models exist. In the present work, a discrete formulation of the breakup terms due to binary breakage is proposed, which allows a direct implementation of both kinds of models and an efficient solution of the population balance equation, making it favorable for the coupling to computational fluid dynamics codes.

*Keywords:* Binary Breakup, Computational Fluid Dynamics, Incorporated Daughter Size Distribution, Method of Classes, Population Balance Equation

---

## 1. Introduction

2 The concept of population balance was first introduced by Hulburt & Katz (1964) and Randolph (1964)  
3 for the description of the dynamic behavior of particulate systems. Over the past five decades, population  
4 balance modeling found ever-increasing application in the field of chemical and pharmaceutical engineering. An  
5 extensive review was recently published by Ramkrishna & Singh (2014).

6 Meanwhile, integrating the solution of the population balance equation (PBE) into computational fluid  
7 dynamics (CFD) software is becoming a popular and promising way of taking into account polydispersity  
8 without spatially resolving all elements of the dispersed phase and their interactions. Thereby, local size changes  
9 may be considered, which affect the interfacial area density as well as the flow structure. Among other reasons,  
10 this information is required to predict heat- and mass transfer rates or the transition from the homogeneous  
11 to the heterogeneous regime in bubble columns. Numerous examples of such coupled methods can be found in  
12 the literature. For example, Cheung et al. (2013) and Liao et al. (2015) predicted the bubble size distribution

---

\*Corresponding author

Email address: y.liao@hzdr.de (Y. Lao)

13 in pipe flows. Chen et al. (2005) and Bannari et al. (2008) simulated gas-liquid bubble columns. Li et al.  
 14 (2013) and Sen et al. (2014) did numerical investigations on a fluidized-bed spray granulation process. Bellot  
 15 et al. (2014) dealt with the complex behavior in gas-stirred ladles. Metzger (2016) studied the fast precipitation  
 16 crystallization process.

17 A well accepted formulation of the PBE including aggregation and breakup as well as nucleation and growth  
 18 is presented in Ramkrishna (1985, 2000) and Jakobsen (2008). Since the focus of this work lies on the breakup  
 19 terms, we state the volume-based PBE as

$$\frac{\partial n(v)}{\partial t} = \underbrace{\int_0^\infty n(v')\Omega_t(v')\beta(v, v')dv'}_{B(v)} - \underbrace{n(v)\Omega_t(v)}_{D(v)}, \quad (1)$$

20 where  $n(v)$  represents the number density of particles with volume  $v$ . The source and sink terms  $B(v)$  and  $D(v)$   
 21 on the right hand side represent the birth rate of particles with volume  $v$  due to breakup of larger particles with  
 22 volume  $v'$  and their death rate due to breakup into smaller particles, respectively. The function  $\Omega_t(v')$  involved  
 23 in the sources terms is typically referred to as a breakup kernel. It describes the total breakup rate of a mother  
 24 particle of volume  $v'$ . The so-called daughter size distribution function  $\beta(v, v')$  represents the probability that  
 25 a daughter particle with volume  $v$  is generated by breakage of a mother particle with volume  $v'$ . Normal and  
 26 beta distribution functions are often adopted for this purpose (Coulaloglou & Tavlarides, 1977; Lee et al., 1987;  
 27 Martinez-Bazan et al., 1999; Laakkonen et al., 2006). Since no mass can be created or destroyed during the  
 28 breakup of a particle, it must satisfy

$$\int_0^{v'} v\beta(v, v')dv = v'. \quad (2)$$

29 Additionally, the following constraints must be fulfilled for binary breakage

$$\int_0^{v'} \beta(v, v')dv = 2, \quad (3a)$$

$$\beta(v, v') = \beta(v' - v, v'). \quad (3b)$$

31 These constraints express that only two daughter particles are generated per breakage event and that the  
 32  $\beta$  function is mirror-symmetrical with respect to  $v = 0.5v'$ . On the basis of Eq. 2 and Eq. 3a, the global  
 33 conservation of particle numbers and mass is satisfied during binary breakage processes

$$\int_0^\infty B(v)dv / \int_0^\infty D(v)dv = 2, \quad (4a)$$

$$\int_0^\infty vB(v)dv / \int_0^\infty vD(v)dv = 1. \quad (4b)$$

35 Analytical solutions of the PBE are limited to very few and simple forms of kernels (McCoy & Madras,  
 36 2003; Singh, 2014; Pinar, 2015). A detailed discussion about numerical approaches is given by Bayraktar  
 37 (2014). Stochastic methods (Meimaroglou & Kiparissides, 2007; Goodson & Kraft, 2004; Kruis et al., 2012)  
 38

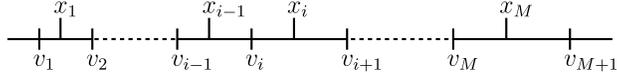


Figure 1: Discretization over the internal coordinate with representative sizes  $x_i$  and size group boundaries  $v_i$

39 like Monte-Carlo simulations are comparatively slow but very exact. They are often used for the purpose  
40 of validating other methods. The method of moments (Wick et al., 2017; Madadi-Kandjani & A., 2015) is  
41 efficient, especially its quadrature-based version. Based on the rationale that only averaged information about a  
42 distribution is relevant, a few low order moments need to be tracked. For example, if the PBE is written as a  
43 transport equation for the diameter-based number density function, then the Sauter mean diameter  $d_{32}$  can be  
44 determined by dividing the third by the second moment of the distribution. Since this is a relevant parameter  
45 for interfacial mass, energy and momentum exchange models in Euler-Euler simulations, it would be enough  
46 to solve for the zeroth to third moments. It is also possible to reconstruct the number density function from  
47 the moments as done by Yuan et al. (2012). The method of classes, also called discrete or sectional methods  
48 (Batterham et al., 1981; Hounslow et al., 1988; Kumar & Ramkrishna, 1996b) is an attractive and widely  
49 used alternative to the aforementioned approaches. It finds itself in between regarding the computational cost.  
50 Furthermore, its inherent information about the number density function is an appealing feature. The particle  
51 size is a fundamental property of any particulate system and it influences a variety of other properties, e.g., the  
52 settling and deposition of inhaled particles in the respiratory tract (Thomas, 2013). The method of classes is a  
53 popular method for coupling the PBE solution to a CFD simulation and also the focus of the current work.

54 Here, the class method with fixed pivots proposed by Kumar & Ramkrishna (1996a) is used. The continuous  
55 PBE (Eq. 1) is integrated over a size range  $[v_i, v_{i+1}]$ , which gives the total number of particles (or number  
56 concentration per control volume)  $N_i$  in this interval or class

$$N_i = \int_{v_i}^{v_{i+1}} n(v)dv . \quad (5)$$

57 The terms on the right-hand side of Eq. eq:PBE are closed by utilizing the mean value theorem on the breakup  
58 rate. The integrals are expressed as sums over sub-intervals as defined by the discretization. The population of  
59 particles is assumed to be concentrated at representative sizes  $x_i$ , giving a source-term coupled set of differential  
60 equations for the number of particles in each interval

$$\frac{\partial N_i}{\partial t} = \underbrace{\sum_{j=i}^M N_j \Omega_t(x_j)}_{B(x_i)} \int_{v_i}^{v_{i+1}} \beta(v, x_j)dv - \underbrace{N_i \Omega_t(x_i)}_{D(x_i)} . \quad (6)$$

61 A possible discretization over the internal coordinate is sketched in Fig. 1. Each class is given by its two  
62 boundary values  $v_i$  and  $v_{i+1}$ , which may be positioned in the middle of two adjacent representative values, i.e.,  
63  $v_i = (x_{i-1} + x_i)/2$ . The interval width of class  $i$  is  $\Delta v_i = v_{i+1} - v_i$ . For more details the reader is referred to  
64 Kumar & Ramkrishna (1996a).

65 As shown in Fig. 2, the daughter particles generated by breakup may have sizes different from the repre-  
66 sentative values of the classes. For example, if a particle with the representative volume  $x_j$  breaks up into one

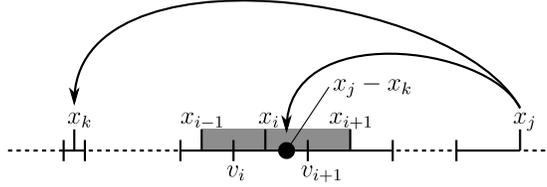


Figure 2: Breakup of a particle of volume  $x_j$  to a particle of volume  $x_k$  and its counterpart of volume  $x_j - x_k$  falling into the interval  $[x_i, x_{i+1}]$

67 daughter particle with the representative volume  $x_k$ , its counterpart with volume  $v = x_j - x_k$  often falls between  
 68 two classes, e.g. between classes  $i$  and  $i + 1$ . If the resultant particles with a volume unequal to any pivot are  
 69 assigned to the nearest class without adjustment, the mass and numbers would not be conserved in individual  
 70 events (Bove, 2005). To guarantee mass and number conservation, i.e., internal consistency of the equations  
 71 (Kumar & Ramkrishna, 1996a), both daughter particles have to be taken into account properly for each breakup  
 72 event and the daughter size distribution function  $\beta(v, x_j)$  in Eq. 6 should not simply be approximated by the  
 73 mean value theorem, i.e.,  $\beta(v, x_j) = \beta(x_i, x_j)$ . Therefore, the birth term requires further mathematical manip-  
 74 ulation. The focus of the present work is a discrete formulation which works well for all breakup kernel types.  
 75 The formulation is efficient for application in CFD and ensures both mass and number conservation. The paper  
 76 is structured as follows: in Sect. 2 we discuss some limitations in existing formulations. An alternative discrete  
 77 formulation is presented in Sect. 3 and its validation in Sect. 4. Finally, Sect. 5 concludes the paper.

## 78 2. Previous work

79 For the framework of breakup modeling, two different approaches are commonly used. Many breakup models  
 80 such as those of Coualoglou & Tavlarides (1977), Lee et al. (1987), Tsouris & Tavlarides (1994) and Martinez-  
 81 Bazan et al. (1999) use the classical framework given in Eq. 1, i.e., they employ a total breakup rate  $\Omega_t(v')$  and  
 82 a separate continuous function for the daughter size distribution  $\beta(v, v')$ . Other breakup models such as Luo &  
 83 Svendsen (1996), Lehr et al. (2002), Wang et al. (2003), Zhao & Ge (2007), Liao et al. (2011) and Xing et al.  
 84 (2015) introduce kernels with an incorporated binary daughter size distribution, thereby providing the partial  
 85 breakup rate  $\Omega_p(v, v')$  directly. It is mirror-symmetrical about  $v = v'/2$  and defined by the product of the total  
 86 breakup rate and the daughter size distribution

$$\Omega_p(v, v') = \Omega_t(v')\beta(v, v') . \quad (7)$$

87 As discussed in Sect. 1 for  $\beta$ , an approximation of  $\Omega_p$  using the mean value theorem will also lead to incon-  
 88 sistency. When a partial breakup model is used in combination with a discrete formulation that is based on  
 89 a total breakup rate and a continuous daughter size distribution function, the total breakup rate needs to be  
 90 calculated for each representative value

$$\Omega_t(v') = \frac{1}{2} \int_0^{v'} \Omega_p(v, v') dv . \quad (8)$$

91 In a CFD simulation where the breakup rate depends on flow field parameters, the above integration would  
 92 need to be carried out for each control volume (Wang & Wang, 2007; Bannari et al., 2008), thereby increasing  
 93 the computational cost.

94 In the course of developing a discrete formulation that guarantees the conservation of mass and finally also  
 95 numbers, many techniques were proposed, frequently with a pure focus on coalescence. Considering particles  
 96 consisting exclusively of  $x_i = 2^{i-1}$  monomers, Vanni (2000) extended the work of Batterham et al. (1981) for  
 97 pure coalescence to the case of breakup. It was shown to be poor in capturing the shape of the number density  
 98 function due to the very coarse grid. In addition, the formulation fails in preserving the total particle number,  
 99 in spite of being mass conservative, as stated by Kumar & Ramkrishna (1996a).

100 An approximate method proposed by Marchal (1988) has no restriction on the choice of the representative  
 101 values and boundaries. However, it has a restriction on the distribution of daughter particles. Their formulation  
 102 is only applicable in cases where the size of the two daughters is precisely known (Vanni, 2000).

103 Hill et al. (1995) and Vanni (1999) tried to remove the inconsistency error by introducing correction factors  
 104 in the birth and death terms. Hill et al. (1995) determined these factors by evaluating the budget of the zeroth  
 105 and first moment, aiming at a correct prediction of numbers and total mass. The method was tested for a total  
 106 breakup rate with a power law form against theoretical and empirical daughter particle size distributions. It  
 107 was shown that in all cases mass is conserved, but the total number of particles is not correct in most cases.

108 The approximation method proposed by Vanni (1999) differs from that by Hill et al. (1995) only in the  
 109 choice of the second constraint for the determination of the correction factors. Instead of a correct prediction of  
 110 the total number of particles, they evaluated the net rate at which particles disappeared from each class from  
 111 the uncorrected PBE. The corrected death term was set equal to this rate.

112 For pure coalescence Gelbard & Tambour (1980) derived a general conservation equation for the quantity  $q$

$$q = \alpha v^\gamma n(v), \quad (9)$$

113 where  $\alpha$  and  $\gamma$  are constants, by applying the mean value theorem on the number density  $n(v)$  instead of the  
 114 breakup rate. In this way, the preservation of the  $\gamma$ th moment is satisfied automatically, but not those of the  
 115 moments with order different from  $\gamma$ . The approach was extended by Vanni (2000) to breakup events.

116 Keeping in mind, that the discrete formulation should exhibit number and mass conservation for any given  
 117 discretization over the internal coordinate, a very general and established framework was proposed by Kumar  
 118 & Ramkrishna (1996a). A benchmark study of several formulations for cases where coalescence and breakup  
 119 occur simultaneously was conducted by Vanni (2000). He concluded that the method of Kumar & Ramkrishna  
 120 (1996a) is favorable with respect to accuracy, efficiency and robustness. Therefore, it is used as a reference  
 121 in the present work. If a particle breaks into particles with sizes other than the representative ones given by  
 122 the discretization, this method allocates fractions between the two nearest neighbors. The fractions can be  
 123 calculated in a way that guarantees the preservation of two chosen moments of the distribution, e.g. numbers  
 124 and mass. For the latter, the discrete birth and death terms representing breakup are expressed as

$$B(x_i) = \sum_{j=i}^M \eta_{ji} N_j \Omega_t(x_j),$$

$$D(x_i) = N_i \Omega_t(x_i) .$$

126 The coefficient

$$\eta_{ji} = \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - v}{x_{i+1} - x_i} \beta(v, x_j) dv + \int_{x_{i-1}}^{x_i} \frac{v - x_{i-1}}{x_i - x_{i-1}} \beta(v, x_j) dv \quad (10)$$

127 The first and second term reduce to zero for  $i = j$  and  $i = 1$ . Like all other methods discussed above it is based  
 128 on the classical framework and efficient if analytical expressions for the daughter size distribution are available,  
 129 or if its integration over  $v$  is manageable at low computational cost. If a partial breakup rate model is used, the  
 130 framework cannot be directly and readily applied anymore. As discussed at the beginning of this section,  $\Omega_t$   
 131 and  $\beta$  need to be obtained using Eq. 7 and 8, which affects the performance of a CFD simulation. Hagesaether  
 132 et al. (2002) introduced a formulation for partial breakup kernels stating

$$\begin{aligned} B(x_i) &= \sum_{j=i+1, i \neq M}^M N_j \Omega'_p(x_i, x_j) \\ &\quad + \sum_{j=1, i \neq M}^i \chi_{i+1, j} N_{i+1} \Omega'_p(x_j, x_{i+1}) \\ &\quad + \sum_{j=1, i \neq 1}^{i-1} (1 - \chi_{i+1, j}) N_i \Omega'_p(x_j, x_i) , \\ D(x_i) &= \sum_{j=1}^{i-1} N_i \Omega'_p(x_j, x_i) , \end{aligned}$$

133 where  $\chi_{i,j} = 2^{1+j-i}$ . However, this formulation is restricted to  $x_{i+1}/x_i = 2$ .

134 In the next section an efficient discrete formulation for binary breakage is given. It avoids the numerical  
 135 integration of Eq. 8 and allows a direct use of partial as well as total binary breakup rate models. Furthermore  
 136 it is flexible in terms of discretization over the internal coordinate while preserving both mass and numbers by  
 137 fulfilling the conditions given by the Eqs. 2, 3 and 4.

### 138 3. An alternative discrete formulation

139 As discussed in Sect. 1, a binary breakage event may lead to the formation of particles with volumes different  
 140 from the representative values if the mother and one daughter particle are positioned at a representative value.  
 141 In this case, the mean value theorem should not be applied on  $\beta$  or  $\Omega_p$  for these secondary particles to avoid  
 142 inconsistency. In the new discrete formulation, these particles are accounted for through their complementary  
 143 partners ( $x_k$  in Fig. 2), which may coincide with a representative volume. For that purpose, before the  
 144 discretization of Eq. 1 the birth by breakup term  $B(v)$  is rewritten to

$$\begin{aligned} B(v) &= \int_v^{\infty} n(v') \Omega_t(v') \beta(v, v') \theta(v \leq v'/2) dv' \\ &\quad + \int_v^{\infty} n(v') \int_0^{v'} \Omega_t(v') \beta(v'', v') \delta(v'' - (v' - v)) \theta(v'' \leq v'/2) dv'' dv' . \end{aligned} \quad (11)$$

145 The formulation in Eq. 11 exploits the symmetry relation for  $\beta$  given in Eq. 3b, by classifying the daughter  
 146 particles into two groups, i.e., one smaller than half of the respective mother particle size and the other one  
 147 larger than it. This is achieved through the step function

$$\theta(x) = \begin{cases} 0 & \text{if } x \text{ is false ,} \\ 1 & \text{if } x \text{ is true .} \end{cases}$$

148 The integration over all possible mother particles  $v'$  remains as given in Eq. 1. The first term represents the  
 149 breakage of  $v'$  into  $v$  for all daughter particles that are smaller than  $v'/2$ . The second term introduces an  
 150 additional inner integral over the range of complementary daughter particles  $v''$ . For a given mother particle  
 151 size  $v'$ , the secondary daughter particle with size  $v' - v''$  may contribute to the change of  $n(v)$ , as determined  
 152 by the delta function  $\delta(v'' - (v' - v))$ .  $v''$  is also set to be smaller than  $v'/2$  in order for the second term to  
 153 be active. Inversely this means, that the secondary daughter particle  $v' - v''$  has a size greater than  $v'/2$ . The  
 154 advantage of the reformulation becomes clear by deriving the discrete version. For that purpose, Eq. 11 is  
 155 integrated over a size range  $[v_i, v_{i+1}]$ , giving

$$\begin{aligned} \int_{v_i}^{v_{i+1}} B(v) &= \int_{v_i}^{v_{i+1}} \int_v^\infty n(v') \Omega_t(v') \beta(v, v') \theta(v \leq v'/2) dv' dv \\ &+ \int_{v_i}^{v_{i+1}} \int_v^\infty n(v') \int_0^{v'} \Omega_t(v') \beta(v'', v') \delta(v'' - (v' - v)) \theta(v'' \leq v'/2) dv'' dv' dv . \end{aligned}$$

156 The integrals are expressed as sums over sub-intervals. The total breakup rate,  $\Omega_t(v')$  and the daughter size  
 157 distribution functions,  $\beta(v, v')$  and  $\beta(v'', v')$ , are now approximated by the mean value theorem. Furthermore,  
 158 the integral over the number density  $n(v)$  is substituted by  $N_i$  using Eq. 5, yielding

$$\begin{aligned} B(x_i) &= \sum_{j=i}^M N_j \Omega_t(x_j) \beta(x_i, x_j) \int_{v_i}^{v_{i+1}} \theta(v \leq x_j/2) dv \\ &+ \sum_{j=i}^M \sum_{k=1}^j N_j \Omega_t(x_j) \beta(x_k, x_j) \int_{v_i}^{v_{i+1}} \delta(x_k - (x_j - v)) dv \int_{v_k}^{v_{k+1}} \theta(v'' \leq x_j/2) dv'' . \end{aligned} \quad (12)$$

159 A further elimination of the remaining integrals gives

$$B(x_i) = \sum_{j=i}^M N_j \left[ \Omega_t(x_j) \beta(x_i, x_j) \Delta v_i(j) + \sum_{k=1}^j \Omega_t(x_j) \beta(x_k, x_j) Y_{ijk} \Delta v_k(j) \right] . \quad (13)$$

160 Possible inconsistencies introduced by the mean value theorem may be compensated by an appropriate dis-  
 161 cretization of the Dirac function (Engquist et al., 2005). The discrete  $\delta$  function for class  $i$ , referred to as  
 162 regularized weight function  $Y_{ijk}$ , determines the fraction that is assigned to class  $i$  if a secondary daughter  
 163 particle with size  $x_j - x_k$  falls into the neighborhood of  $x_i$ . For the exact conservation of mass and numbers, it  
 164 is given by

$$Y_{ijk} = \begin{cases} \frac{(x_j - x_k) - x_{i-1}}{x_i - x_{i-1}} & \text{if } x_{i-1} \leq x_j - x_k < x_i , \\ \frac{x_{i+1} - (x_j - x_k)}{x_{i+1} - x_i} & \text{if } x_i \leq x_j - x_k < x_{i+1} , \\ 0 & \text{else .} \end{cases} \quad (14)$$

165 Note that it is possible to achieve the preservation of any two chosen moments by adapting the function  $Y_{ijk}$ .

166 For more details the reader is referred to Kumar & Ramkrishna (1996a).

167 The integrals over the step functions in Eq. 12 result in

$$\Delta v_i(j) = \begin{cases} v_{i+1} - v_i & \text{if } v_{i+1} \leq x_j/2, \\ x_j/2 - v_i & \text{if } v_i < x_j/2 < v_{i+1}, \\ 0 & \text{if } v_i \geq x_j/2. \end{cases}$$

168 The  $\beta$  function in Eq. 13 is now stated in discrete form. Combining with Eq. 7 and Eq. 8, the discrete form of  
169 the partial breakup rate states

$$\Omega_p(x_i, x_j) = \Omega_t(x_j)\beta(x_i, x_j), \quad (15)$$

170 and that of the total breakup rate

$$\Omega_t(x_i) = \sum_{j=1}^i \Omega_p(x_j, x_i)\Delta v_i(j). \quad (16)$$

171 As a result, the discrete formulation of the birth term in Eq. 13 is transformed into

$$B(x_i) = \sum_{j=i}^M N_j \left[ \Omega_p(x_i, x_j)\Delta v_i(j) + \sum_{k=1}^M \Omega_p(x_k, x_j)Y_{ijk}\Delta v_k(j) \right], \quad (17)$$

172 and the death by breakup term in Eq. 6,  $D(x_i)$ , is rewritten as

$$D(x_i) = N_i \sum_{j=1}^i \Omega_p(x_j, x_i)\Delta v_j(i). \quad (18)$$

173 The formulation is therefore generally applicable for both frameworks of breakup modeling for the case of binary  
174 breakage. The implementation of models with an incorporated daughter size distribution is direct and efficient,  
175 since it avoids the integration in Eq. 8.

#### 176 4. Numerical results

177 Following the case presented by Kumar & Ramkrishna (1996a), we assume pure breakage processes starting  
178 from a monodisperse initial condition

$$n(v, t_0) = \begin{cases} 0.05 & \text{if } v = 1 \text{ m}^3, \\ 0 & \text{otherwise.} \end{cases}$$

179 The numerical results are obtained for geometric grids, i.e., the class boundaries are calculated by  $v_{i+1} = sv_i$   
180 with  $s > 1$ . All results are compared with a reference solution obtained with the formulation of Kumar &  
181 Ramkrishna (1996a), denoted as "Reference" in subsequent figures. For each test case, the number density  
182 function

$$n(x_i, t) = \frac{N_i(t)}{v_{i+1} - v_i},$$

Case	Breakup rate	Daughter size distribution
1	$\Omega_t(v') = v'^2$	$\beta(v, v') = 2/v'$
2	$\Omega_t(v') = v'^2$	$\beta(v, v') = \frac{12}{v} \left(\frac{v}{v'}\right) \left(1 - \frac{v}{v'}\right)$
3	$\Omega_t(v') = 0.4 \frac{\epsilon^{1/3}}{(1 + \alpha_g)v'^{2/9}} \exp \left[ -0.08 \frac{\sigma(1 + \alpha_g)^2}{\rho_l \epsilon^{2/3} v'^{5/9}} \right]$	$\beta(v, v') = \frac{4.8}{v'} \exp \left[ -4.5 \left(\frac{2v - v'}{v'}\right)^2 \right]$
4	$\Omega_p(v, v') = C_1 \frac{1 - \alpha_g}{v'} \left(\frac{\epsilon}{d'^2}\right)^{1/3} \int_{\xi_{\min}}^1 \frac{(1 + \xi)^2}{\xi^{11/3}} \exp \left[ -\frac{12c_f \sigma}{C_2 \rho_l \epsilon^{2/3} d'^{5/3} \xi^{11/3}} \right] d\xi$ with $C_1 = 0.923$ , $C_2 = 2.0$ , $c_f = \left(\frac{v}{v'}\right)^{2/3} + \left(1 - \frac{v}{v'}\right)^{2/3} - 1$ , $\xi = \lambda/d'$ , $\xi_{\min} = 11.4\eta$	

Table 1: Test cases for validation

183 the time development of the total particle number

$$M_0(t) = \sum_{i=1}^M x_i^0 N_i(t),$$

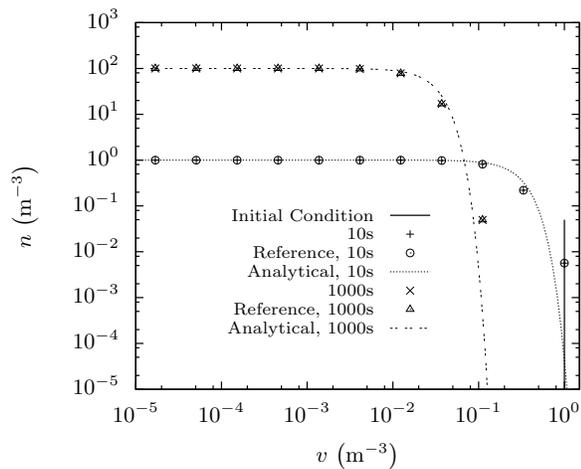
184 and the Sauter mean diameter

$$d_{32}(t) = \frac{\sum_{i=1}^M d_i^3 N_i(t)}{\sum_{i=1}^M d_i^2 N_i(t)}$$

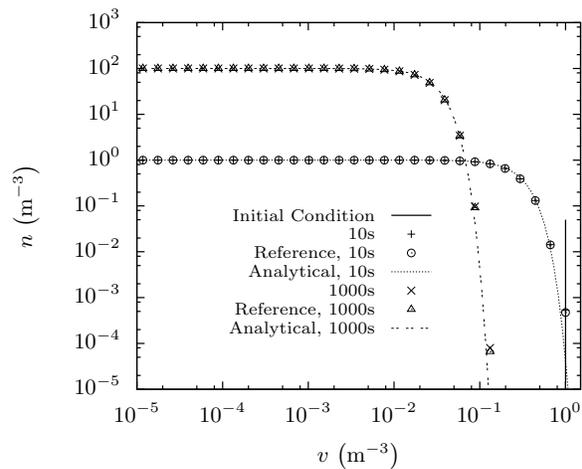
185 are presented.

186 The test cases, each applying a different breakup model, are summarized in Tab. 1. The first case, case  
187 1, uses a power-law kernel in combination with a uniform daughter size distribution. An analytical solution is  
188 given by Ziff & McGrady (1985). The daughter size distribution for the second test case, case 2, assumes a  
189 beta distribution, as used by Lee et al. (1987). The third case, case 3, employs the model from Coualoglou  
190 & Tavlarides (1977), which assumes an exponential function for the total breakup frequency and a normal  
191 function for the daughter size distribution. For case 1, 2 and 3, total breakup models with a separate daughter  
192 size distribution function are applied. These cases aim at a validation of the new formulation against an  
193 analytical solution and reference solutions based on the formulation of Kumar & Ramkrishna (1996a). It is  
194 expected that both formulations deliver a similar performance. In contrast to the first three cases, the last case,  
195 case 4, shows the benefits of the new formulation for partial breakup models with an incorporated daughter size  
196 distribution. The well-accepted model from Luo & Svendsen (1996) is applied for that purpose. As discussed  
197 in previous sections, the use of the new formulation in Eq. 13 has convincing advantages in this situation, since  
198 the model provides the partial breakup rate  $\Omega_p(v_i, v_j)$  directly. On the other hand, if the reference formulation  
199 is applied, an additional numerical integration is required when solving Eq. 8 in order to obtain the daughter  
200 size distribution  $\beta(v_i, v_j)$ . This increases the computational cost. The parameters used for the test cases are  
201  $\alpha_g = 0.05$ ,  $\rho_l = 997 \text{ kg m}^{-3}$ ,  $\rho_g = 1.0 \text{ kg m}^{-3}$ ,  $\sigma = 0.072 \text{ N m}^{-1}$ ,  $\epsilon = 0.001 \text{ m}^2 \text{ s}^{-3}$ , and  $\mu_l = 0.0008899 \text{ Pa s}^{-1}$ .

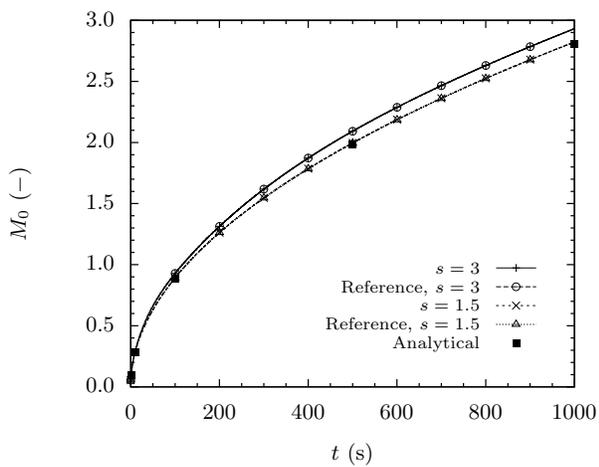
202 *Case 1.* Figure 3a and 3b show a comparison between the numerical and the analytical particle number density  
203  $n(v)$ . Note that for the class representing the smallest particle size, the lower boundary  $v_1$  is set to the value



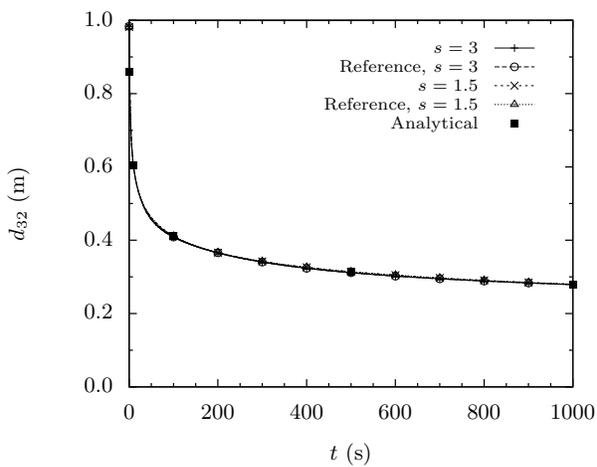
a) Number density,  $s=3$



b) Number density,  $s=1.5$



c) Number of particles



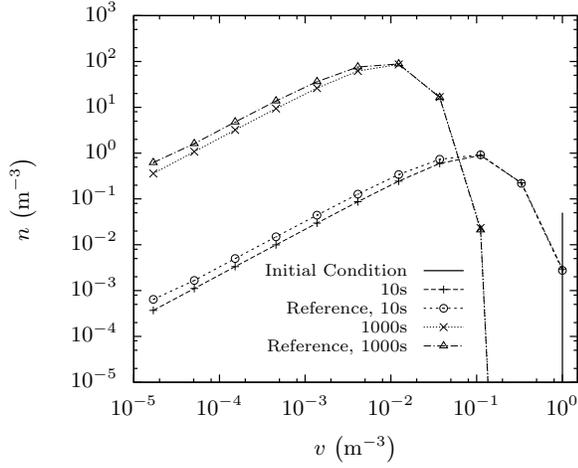
d) Sauter mean diameter

Figure 3: Results for Case 1

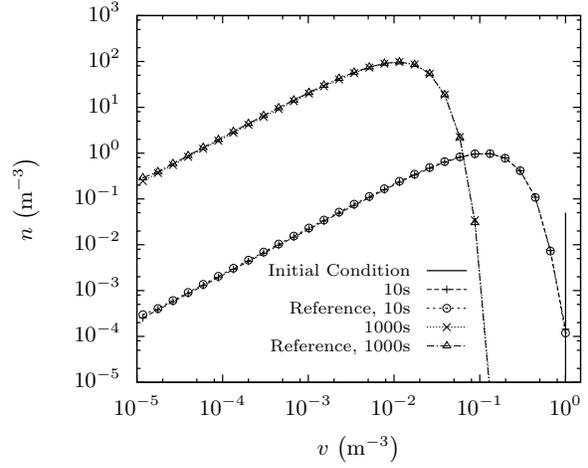
204 of the representative size  $x_1$ . It is obvious that the predictions for the power law kernel in combination with  
 205 a uniform daughter size distribution are in good agreement with the analytical solution even for the coarse  
 206 grid. However, a discretization error exists in both sets of numerical results for  $s = 3$  in the large particle  
 207 size range, where the initial number density gradient as well as the class width are large (see Fig. 3a). The  
 208 transient variation of  $M_0$  is shown in Fig. 3c. A slight over-prediction exists, if a coarse grid ( $s = 3$ ) is applied.  
 209 The discretization error vanishes as expected, if the resolution is increased by decreasing the value of  $s$ . The  
 210 numerical and the analytical results are in a good agreement at  $s = 1.5$  and the total particle number obtained  
 211 by the new and the reference formulation are identical for both values of  $s$ . The Sauter mean diameter  $d_{32}$  is  
 212 a key parameter for CFD simulations of particulate or bubbly flows, since it is used for many correlations for  
 213 the interfacial transfer of momentum, heat and mass. Its transient variation is shown in Fig. 3d. In a pure  
 214 breakage process, the mean size of particles decreases continuously. The effect of grid resolution is found to be  
 215 negligible in this case. The predictions for both fine and coarse grids coincide with the analytical results and  
 216 the reference solution.

217 *Case 2.* Unlike for the above uniform daughter size distribution, the probability of equal-sized breakage is the  
 218 highest in the beta distribution given for this case. Figure 4a and 4b show a comparison between the predictions  
 219 for the particle number density obtained by the reference and the new formulation. An analytical solution is  
 220 not available for this case. It is found that the results deviate from each other toward the small particle size end  
 221 for the coarse grid ( $s = 3$ ), as shown in Fig. 4a. In addition, a grid-dependency of the results obtained by using  
 222 the reference formulation is observed, while the new formulation behaves similar at both resolutions. The over-  
 223 prediction of particle number on coarse grids by the reference formulation is discussed in Kumar & Ramkrishna  
 224 (1996b) and Bayraktar (2014). With the refinement of the grid, the predictions of both formulations coincide  
 225 with each other, see Fig. 4b. The corresponding predictions for the total number of particles as well as Sauter  
 226 mean diameter for Case 2 are shown in Fig. 4c and Fig. 4d, respectively. Satisfying agreement is achieved for  
 227 the Sauter mean diameter, which is mainly governed by the particles in the large size range. Nevertheless, the  
 228 over-prediction of the total particle number by the Kumar & Ramkrishna (1996a) formulation at  $s = 3$  due to  
 229 the over-prediction of the birth rate of small particles is shown here as well.

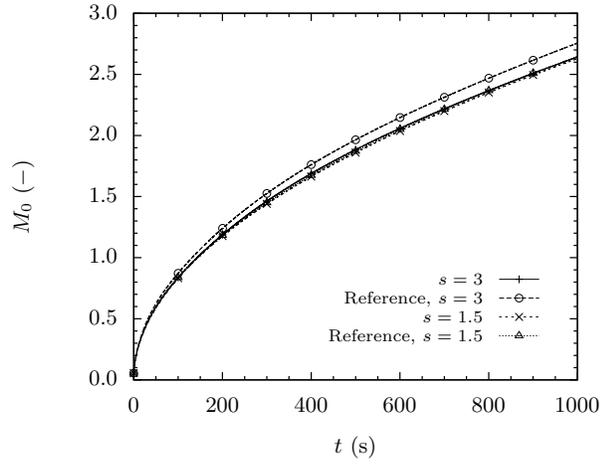
230 *Case 3.* In case 3 the breakup model from the work of Coulaloglou & Tavlarides (1977) is adopted. In contrast  
 231 to case 1 and 2, the breakup frequency is higher, which results in the effect that the majority of particles  
 232 is concentrated at the smaller particle size end after a relatively short time period. The parameters used to  
 233 determine the breakup rate are given at the beginning of Sect. 4. The number density function is plotted  
 234 in Fig. 5a and 5b. The deviation between the two sets of results for the coarse grid is evident here as well.  
 235 Yet they converge to one another for the finer grid. Figure 5c shows that in the first 30s, the total particle  
 236 number predicted by the reference and the new formulation are in quantitative agreement. With increasing time,  
 237 however, the results diverge from each other for the coarse grid, and the new formulation predicts higher values  
 238 than the reference formulation of Kumar & Ramkrishna (1996a). This deviation is caused by the treatment at  
 239 the lower bound of the first class. Since  $v_1$  is set to  $x_1$ , the birth rate of particles in the range  $[0, x_1]$  due to



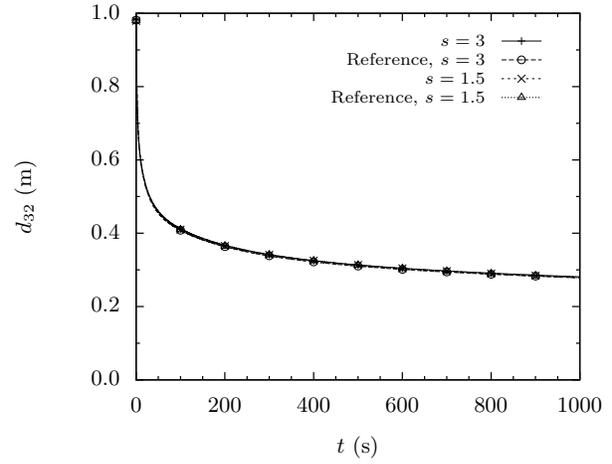
a) Number density,  $s=3$



b) Number density,  $s=1.5$

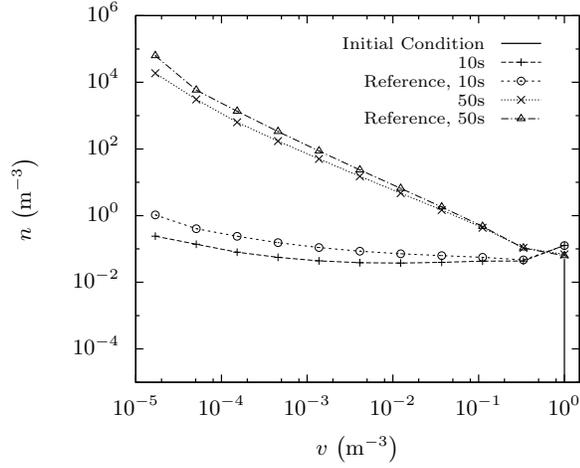


c) Number of particles

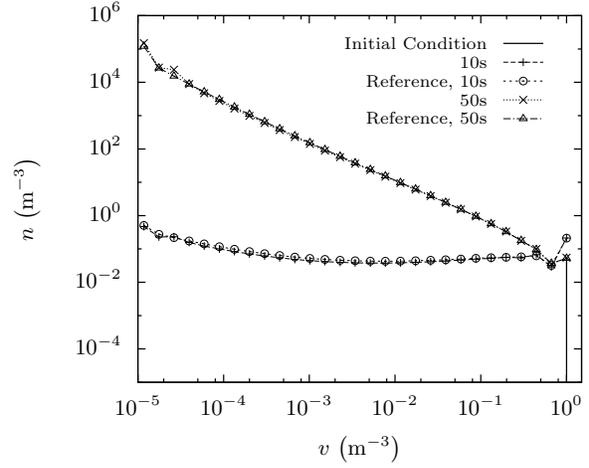


d) Sauter mean diameter

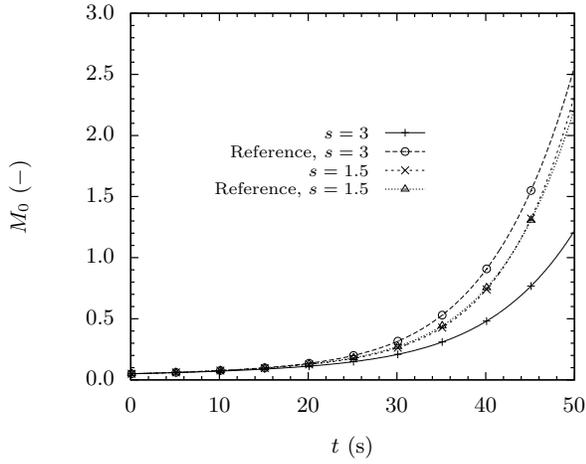
Figure 4: Results for Case 2



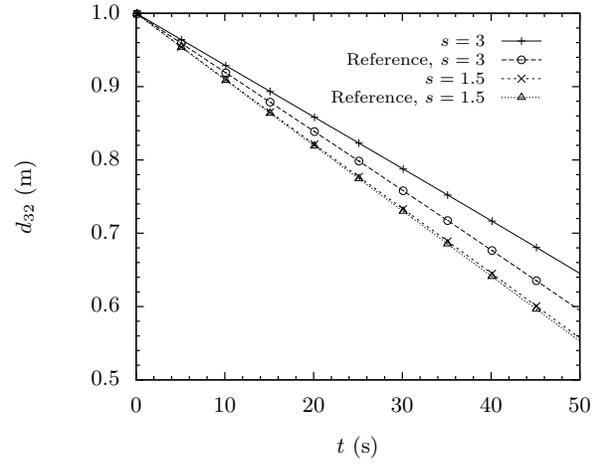
a) Number density,  $s=3$



b) Number density,  $s=1.5$



c) Number of particles



d) Sauter mean diameter

Figure 5: Results for Case 3

240 breakup of larger particles is not considered. The omitted parts,

$$\int_0^{x_1} N_j \Omega_t(x_j) \beta(v, x_j) dv$$

241 for the reference formulation, and

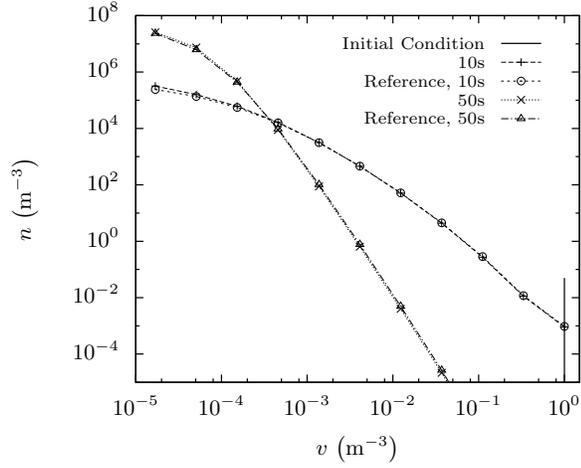
$$N_j \Omega_t(x_j) \beta(x_1, x_j) \Delta v_1(j)$$

242 for the new one, are identical for constant  $\beta$  as in case 1 and of comparable magnitude for sufficiently large  
243 mother particle sizes  $x_j$  as in case 2. However, the difference becomes larger as the size of the breaking particles  
244 becomes smaller. The deviation depends on the features of the  $\beta$  function and the value of  $\Delta v_0(j)$ , and is  
245 responsible for the discrepancy between the methods. It should be mentioned though, that for a system with  
246 simultaneous coalescence and breakage, the discretization over the internal coordinate should ensure, that the  
247 number of particles in the first as well as the last size class remains zero. If this is the case, the above differences  
248 will be reduced. Furthermore, it is worth noting that limiting  $v_1$  to  $x_1$  is a necessity for the reference formulation,  
249 but not for the new formulation, which satisfies the overall balance between birth and death either way, provided  
250 that the death rate is calculated by Eq. 18. The temporal evolution of  $d_{32}$  is illustrated in Fig. 5d. In this  
251 case, the prediction by the new formulation also agrees well with the reference formulation if a sufficiently fine  
252 discretization is applied.

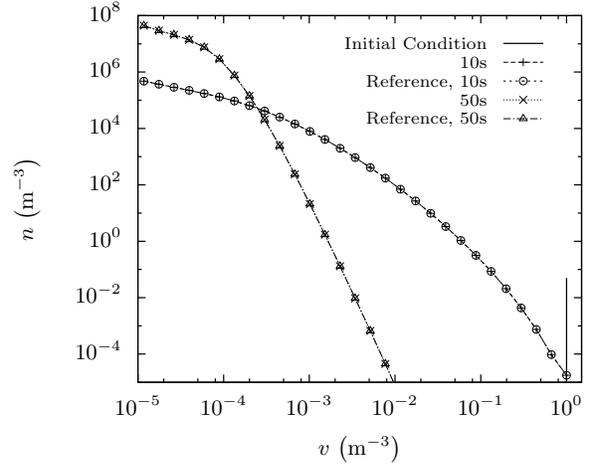
253 *Case 4.* The last case shows the behavior of the new and the reference formulation for partial breakup models  
254 like the model of Luo & Svendsen (1996). The predicted particle number density functions are shown in  
255 Fig. 6a and 6b, respectively. The two formulations deliver almost identical results for both grids. Figure  
256 6c shows the evolution of total particle number. It is shown that the prediction using the new formulation  
257 almost coincides with the reference solution. According to the Luo & Svendsen (1996) model, the daughter size  
258 distribution should obey a U-shape function, which gives the highest probability for unequal-sized breakage. A  
259 steep gradient appears in the birth rate of daughter particles with sizes approaching zero or the mother particle  
260 size, respectively. Because of that, a large number of nodes is required for the numerical integration of Eq. 8  
261 in these regions. Otherwise, the birth rate may be under-predicted. The temporal evolution of Sauter mean  
262 diameter in the first 50s is depicted in Fig. 6d, with very similar predictions as well. In a word, comparable  
263 results are obtained with both formulations. However, the implementation of partial breakup models in the  
264 new formulation given by Eq. 17 and 18 is direct and efficient without any numerical integrations.

## 265 5. Conclusions

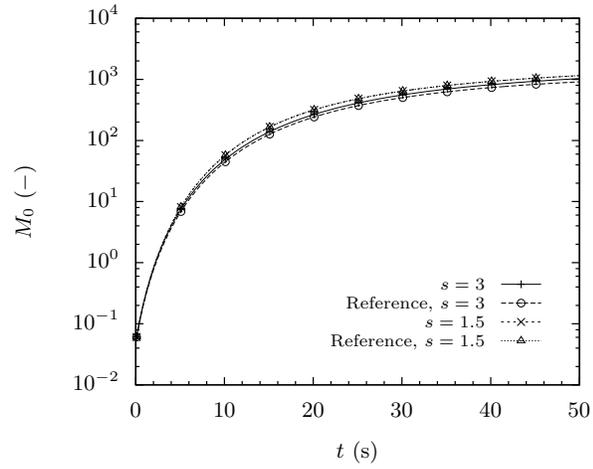
266 A discrete formulation of the PBE for binary breakage is presented. It is shown to be applicable for  
267 both frameworks of breakup modeling, i.e., total and partial breakup kernels with a separate or incorporated  
268 function for the daughter size distribution. Its validity is tested for various cases with different breakup models  
269 and daughter size distributions, using an analytical solution as well as reference solutions based on the work of  
270 Kumar & Ramkrishna (1996a). Two geometric grids are applied and the results for particle number density,  
271 total number and Sauter mean diameter predicted by the new and the reference formulation coincide for the finer



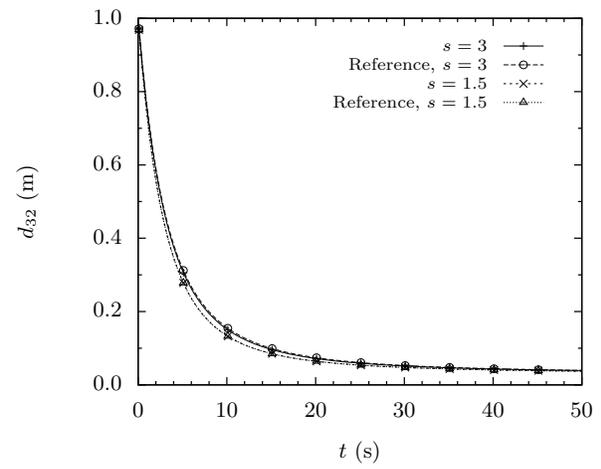
a) Number density,  $s=3$



b) Number density,  $s=1.5$



c) Number of particles



d) Sauter mean diameter

Figure 6: Results for Case 4

272 grid. Both formulations ensure the preservation of numbers and mass. The preservation of other moments may  
 273 be achieved by adapting the weight function  $Y_{ijk}$  (Eq. 14) accordingly, as presented by Kumar & Ramkrishna  
 274 (1996a). Furthermore, the formulation is flexible on the discretization and applicable for a constant or increasing  
 275 class width, i.e., uniform and geometric grids with section spacing factors  $s \geq 1$ . The advantage of the new  
 276 formulation is the possibility to directly implement partial breakup models (Luo & Svendsen, 1996; Liao et al.,  
 277 2015), which provide the breakup rate between a mother and a daughter particle directly. It avoids the extraction  
 278 of the daughter size distribution by means of numerical integration, which introduces considerable numerical  
 279 cost and is required for the reference formulation, making it more suitable for the coupling with CFD.

## Nomenclature

$B$	birth rate	$\text{m}^{-3} \text{m}^{-3} \text{s}^{-1}$
$D$	death rate	$\text{m}^{-3} \text{m}^{-3} \text{s}^{-1}$
$d$	bubble diameter	$\text{m}$
$M$	moment of number density distribution	$\text{m}^{-3}$
$n$	number density	$\text{m}^{-3} \text{m}^{-3}$
$N$	number concentration	$\text{m}^{-3}$
$s$	section spacing factor	—
$v, x$	particle volume, representative volume	$\text{m}^{-3}$

### *Greek letters*

$\alpha$	void fraction of particles	—
$\beta$	size distribution probability of daughter particles	$\text{m}^{-3}$
$\Delta v$	class width	$\text{m}^{-3}$
$\delta$	Dirac delta function	—
$\epsilon$	turbulent dissipation rate	$\text{m}^2 \text{s}^{-3}$
$\eta$	Kolmogorov length scale	$\text{m}$
$\mu$	liquid dynamic viscosity	$\text{Pa s}$
$\rho$	density	$\text{kg m}^{-3}$
$\sigma$	surface tension coefficient	$\text{N m}$
$\Omega_p$	partial breakup rate	$\text{s}^{-1} \text{m}^{-3}$
$\Omega_t$	total breakup rate	$\text{s}^{-1}$

### *Subscripts*

$g$	gas
$i, j, k$	indexes of size classes
$l$	liquid
$p$	partial
$t$	total
0	zeroth moment

### **References**

- Bannari, R., Kerdouss, F., Selma, B., Bannari, A., & Proulx, P. (2008). Three-dimensional mathematical modeling of dispersed two-phase flow using class method of population balance in bubble columns. *Computers & Chemical Engineering*, *32*, 3224–3237.
- Batterham, R. J., Hall, J. S., & Barton, G. W. (1981). Pelletizing kinetics and simulation of full scale balling circuits. In *3rd International Symposium on Agglomeration* (pp. A136–A150).
- Bayraktar, E. (2014). *Numerical aspects of population balance equations coupled to computational fluid dynamics*. Ph.D. thesis Technical University of Dortmund Dortmund, Germany.
- Bellot, J.-P., De Felice, V., Dussoubs, B., Jardy, A., & Hans, S. (2014). Coupling of CFD and PBE calculations to simulate the behavior of an inclusion population in a gas-stirring ladle. *Metallurgical and Materials Transactions B*, *45*, 13–21. <http://dx.doi.org/10.1007/s11663-013-9940-7>.
- Bove, S. (2005). *Computational fluid dynamics of gas-liquid flows including bubble population balances: Ph.D. thesis*. Ph.D. thesis Aalborg University Esbjerg Esbjerg, Denmark.
- Chen, P., Dudukovic, M. P., & Sanyal, J. (2005). Three-dimensional simulation of bubble column flows with bubble coalescence and breakup. *AIChE Journal*, *51*, 696–712.
- Cheung, S. C. P., Deju, L., Yeoh, G. H., & Tu, J. Y. (2013). Modeling of bubble size distribution in isothermal gas-liquid flows: Numerical assessment of population balance approaches. *Nuclear Engineering and Design*, *265*, 120–136.
- Coulaloglou, C. A., & Tavlarides, L. L. (1977). Description of interaction processes in agitated liquid-liquid dispersions. *Chemical Engineering Science*, *32*. [https://doi.org/10.1016/0009-2509\(77\)85023-9](https://doi.org/10.1016/0009-2509(77)85023-9).
- Engquist, B., Tornberg, A.-K., & Tsai, R. (2005). Discretization of dirac delta functions in level set methods. *Journal of Computational Physics*, *207*, 28–51. <https://doi.org/10.1016/j.jcp.2004.09.018>.
- Gelbard, F., & Tambour, Y. S. (1980). Sectional representations for simulating aerosol dynamics. *Journal of Colloid and Interface Science*, *76*, 541–556. [https://doi.org/10.1016/0021-9797\(80\)90394-X](https://doi.org/10.1016/0021-9797(80)90394-X).

- Goodson, M., & Kraft, M. (2004). Simulation of coalescence and breakage: An assessment of two stochastic methods suitable for simulating liquid-liquid extraction. *Chemical Engineering Science*, *59*, 3865–3881. <https://doi.org/10.1016/j.ces.2004.05.029>.
- Hagesaether, L., Jakobsen, H. A., & Svendsen, H. F. (2002). A model for turbulent binary breakup of dispersed fluid particles. *Chemical Engineering Science*, *57*, 3251–3267. [https://doi.org/10.1016/S0009-2509\(02\)00197-5](https://doi.org/10.1016/S0009-2509(02)00197-5).
- Hill, D., Wang, D., Gosman, A., & Issa, R. (1995). Numerical prediction of two-phase bubbly flow in a pipe. In *Proc. 2nd Int. Conf. on Multiphase Flow. Kyoto* (pp. MO3–1–MO3–6).
- Hounslow, M. J., Ryall, R. L., & Marshall, V. R. (1988). A discretized population balance for nucleation, growth, and aggregation. *AIChE Journal*, *34*, 1821–1832.
- Hulburt, H. M., & Katz, S. (1964). Some problems in particle technology: A statistical mechanical formulation. *Chemical Engineering Science*, *19*, 555.
- Jakobsen, H. A. (2008). *Chemical Reactor Modeling: Multiphase Reactive Flows*. (1st ed.). Heidelberg, Germany: Springer-Verlag Berlin Heidelberg. <https://doi.org/10.1007/978-3-540-68622-4>.
- Kruis, F. E., Wei, J., van der Zwaag, T., & Haep, S. (2012). Computational fluid dynamics based stochastic aerosol modeling: Combination of a cell-based weighted random walk method and a constant-number monte-carlo method for aerosol dynamics. *Chemical Engineering Science*, *70*, 109–120. <https://doi.org/10.1016/j.ces.2011.10.040>.
- Kumar, S., & Ramkrishna, D. (1996a). On the solution of population balance equations by discretization-I. a fixed pivot technique. *Chemical Engineering Science*, *51*, 1311–1342.
- Kumar, S., & Ramkrishna, D. (1996b). On the solution of population balance equations by discretization-II. a moving pivot technique. *Chemical Engineering Science*, *51*, 1333–1342.
- Laakkonen, M., Alopaeus, V., & Aittamaa, J. (2006). Validation of bubble breakage, coalescence and mass transfer models for gas-liquid dispersion in agitated vessel. *Chemical Engineering Science*, *61*, 218–228.
- Lee, C.-H., Erickson, L. E., & Glasgow, L. A. (1987). Dynamics of bubble size distribution in turbulent gas-liquid dispersions. *Chemical Engineering Communications*, *61*, 181–195. <https://doi.org/10.1080/00986448708912038>.
- Lehr, F., Millies, M., & Mewes, D. (2002). Bubble-size distributions and flow fields in bubble-columns. *AIChE Journal*, *48*, 2426.
- Li, Z., Kessel, J., Grünewald, G., & Kind, M. (2013). Coupled CFD-PBE simulation of nucleation in fluidized bed spray granulation. *Drying Technology*, *31*, 1888–1896. <https://doi.org/10.1080/07373937.2013.840649>.
- Liao, Y., Lucas, D., Krepper, E., & Schmidtke, M. (2011). Development of a generalized coalescence and breakup closure for the inhomogeneous MUSIG model. *Nuclear Engineering and Design*, *241*, 1024–1033.

- Liao, Y., Rzehak, R., Lucas, D., & Krepper, E. (2015). Baseline closure model for dispersed bubbly flow: Bubble-coalescence and breakup. *Chemical Engineering Science*, *122*, 336–349. <https://doi.org/10.1016/j.ces.2014.09.042>.
- Luo, H., & Svendsen, H. F. (1996). Theoretical model for drop and bubble-breakup in turbulent dispersions. *AIChE Journal*, *42*, 1225–1233.
- Madadi-Kandjani, E., & A., P. (2015). An extended quadrature-based moment method with log-normal kernel density functions. *Chemical Engineering Science*, *131*, 323–339.
- Marchal, P. (1988). Crystallization and precipitation engineering-I. an efficient method for solving population balance in crystallization with agglomeration. *Chemical Engineering Science*, *43*, 59–67.
- Martinez-Bazan, C., Montanes, J. L., & Lasheras, J. C. (1999). On the breakup of an air bubble injected into a fully developed turbulent flow. part 1. breakup frequency. *J. Fluid Mech*, *401*, 157–182.
- McCoy, B. J., & Madras, G. (2003). Analytical solution for a population balance equation with aggregation and fragmentation. *Chemical Engineering Science*, *58*, 3049–3051. [https://doi.org/10.1016/S0009-2509\(03\)00159-3](https://doi.org/10.1016/S0009-2509(03)00159-3).
- Meimaroglou, D., & Kiparissides, C. (2007). Monte carlo simulation for the solution of the bi-variate dynamic population balance equation in batch particulate systems. *Chemical Engineering Science*, *62*, 5295–5299. <https://doi.org/10.1016/j.ces.2006.11.032>.
- Metzger, L. (2016). The influence of mixing on fast precipitation processes – a coupled 3d cfd-pbe approach using the direct quadrature method of moments (dqmom). *Chemical Engineering Science*, .
- Pinar, Z. (2015). Analytical solution of population balance equation involving growth, nucleation and aggregation in terms of auxiliary equation method. *Applied Mathematics & Information Sciences*, *9*, 2467–2475.
- Ramkrishna, D. (1985). The status of population balances. *Rev. Chem. Eng.*, *3*, 49–95.
- Ramkrishna, D. (2000). *Population balances: Theory and applications to particulate systems in engineering*. Academic Press Inc.
- Ramkrishna, D., & Singh, M. R. (2014). Population balance modeling: current status and future prospects. *Annual Review of Chemical and Biomolecular Engineering*, *5*, 123–146. <https://doi.org/10.1146/annurev-chembioeng-060713-040241>.
- Randolph, A. D. (1964). A population balance for countable entities. *The Canadian Journal of Chemical Engineering*, *42*, 280–281. <https://doi.org/10.1002/cjce.5450420612>.
- Sen, M., Barrasso, D., Singh, R., & Ramachandran, R. (2014). A multi-scale hybrid cfd-dem-pbm description of a fluid-bed granulation process. *Processes*, *2*, 89–111. <https://doi.org/10.3390/pr2010089>.

- Singh, R. (2014). Adomian decomposition method for solving fragmentation and aggregation population balance equations. *Journal of Applied Mathematics and Computing*, (pp. 1–31). <https://doi.org/10.1007/s12190-014-0802-5>.
- Thomas, R. J. (2013). Particle size and pathogenicity in the respiratory tract. *Virulence*, *4*, 847–858. <https://doi.org/10.4161/viru.27172>.
- Tsouris, C., & Tavlarides, L. L. (1994). Breakage and coalescence models for drops in turbulent dispersions. *AIChE Journal*, *40*, 395–406. <https://doi.org/10.1002/aic.690400303>.
- Vanni, M. (1999). Discretization procedure for the breakage equation. *AIChE Journal*, *45*, 916–919. <https://doi.org/10.1002/aic.690450422>.
- Vanni, M. (2000). Approximate population balance equations for aggregation–breakage processes. *Journal of Colloid and Interface Science*, *221*, 143–160. <https://doi.org/10.1006/jcis.1999.6571>.
- Wang, T., & Wang, J. (2007). Numerical simulations of gas-liquid mass transfer in bubble columns with a cfd-pbm coupled model. *Chemical Engineering Science*, *62*, 7107–7118.
- Wang, T., Wang, J., & Jin, Y. (2003). A novel theoretical breakup kernel function for bubbles/droplets in a turbulent flow. *Chemical Engineering Science*, *58*, 4629–4637. <https://doi.org/10.1016/j.ces.2003.07.009>.
- Wick, A., Nguyen, T.-T., Laurent, F., Fox, R. O., & Pitsch, H. (2017). Modeling soot oxidation with the extended quadrature method of moments. *Proceedings of the Combustion Institute*, *36*, 789–797. <https://doi.org/10.1016/j.proci.2016.08.004>.
- Xing, C., Wang, T., Guo, K., & Wang, J. (2015). A unified theoretical model for breakup of bubbles and droplets in turbulent flows. *AIChE Journal*, *61*, 1391–1403. <https://doi.org/10.1002/aic.14709>.
- Yuan, C., Laurent, F., & Fox, R. O. (2012). An extended quadrature method of moments for population balance equations. *Journal of Aerosol Science*, *51*, 1–23. <https://doi.org/10.1016/j.jaerosci.2012.04.003>.
- Zhao, H., & Ge, W. (2007). A theoretical bubble breakup model for slurry beds or three-phase fluidized beds under high pressure. *Chemical Engineering Science*, *62*, 109–115. <https://doi.org/10.1016/j.ces.2006.08.008>.
- Ziff, R. M., & McGrady, E. D. (1985). The kinetics of cluster fragmentation and depolymerisation. *Journal of Physics A: Mathematical and General*, *18*, 3027.