Subcritical instability in a Hartmann flow close to transitional Reynolds numbers

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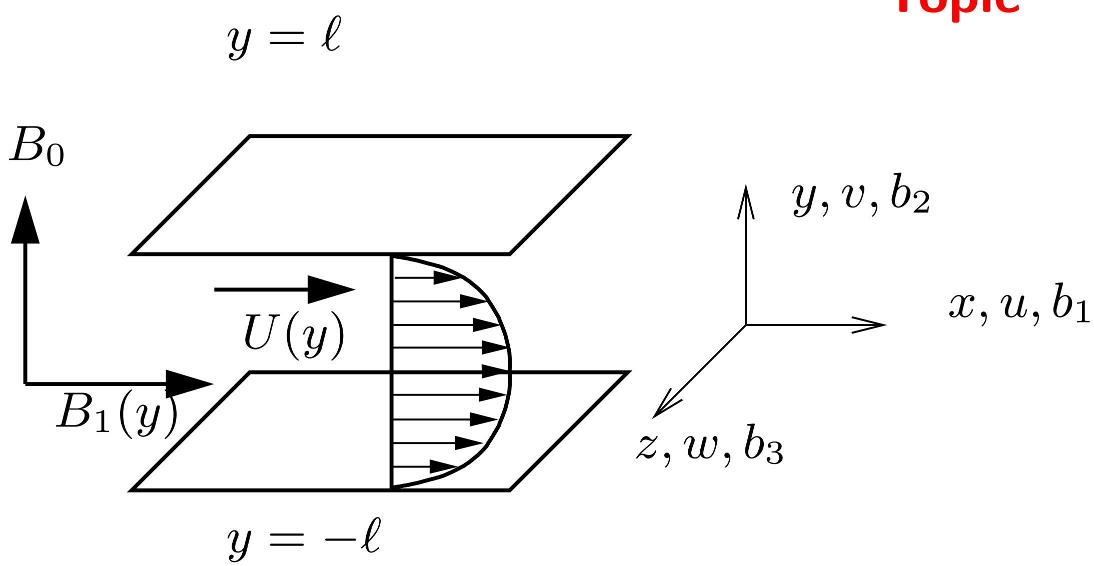
Summary

* Topic
* Modal Linear Stability
* Transient Growth and Optimal Perturbation
* Applications to the Hartmann Flow
* Concluding Remarks

Topic



Channel flow

Constant vertical magnetic field

Perfectly insulating walls

- ⋆ Hartmann number $Ha = \ell \ B_0 \sqrt{\sigma/\mu}$, magnetic Prandtl number : $Pr_m = \nu/\eta$
- ⋆ $Re = \bar{U} \ \ell/\nu$, $\quad Re_H = Re/Ha = \bar{U} \ \delta_H/\nu$
- ⋆ Modal (exponential) stability analysis (LST) : critical $Re_H \approx 30000 - 50000$
- ⋆ Experiments : Lazarus (1937), Moresco *et al* (2003) : $Re_H \approx 380$ at transition
- ⋆ Possible by-pass transition due to transient growth

Modal stability

⋆ Maxwell equations & Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} + (\vec{U}.\overrightarrow{\text{grad}}\,)\vec{u} + (\vec{u}.\overrightarrow{\text{grad}}\,)\vec{U} = -\overrightarrow{\text{grad}}\,(p + \mathbf{p_m}) + \frac{1}{Re}\Delta\vec{u} + \mathbf{Q_m}\{(\vec{\mathbf{B}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{b}} + (\vec{\mathbf{b}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{B}}\}, \quad (1)$$

$$\text{div } \vec{\text{u}} = 0, \quad (2)$$

$$\frac{\partial \vec{b}}{\partial t} + (\vec{\mathbf{U}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{b}} + (\vec{\mathbf{u}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{B}} = (\vec{\mathbf{B}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{u}} + (\vec{\mathbf{b}}.\overrightarrow{\text{grad}}\,)\vec{\mathbf{U}} + \frac{1}{Re_m}\Delta\vec{b}, \quad (3)$$

$$\text{div } \vec{\text{b}} = 0, \quad (4)$$

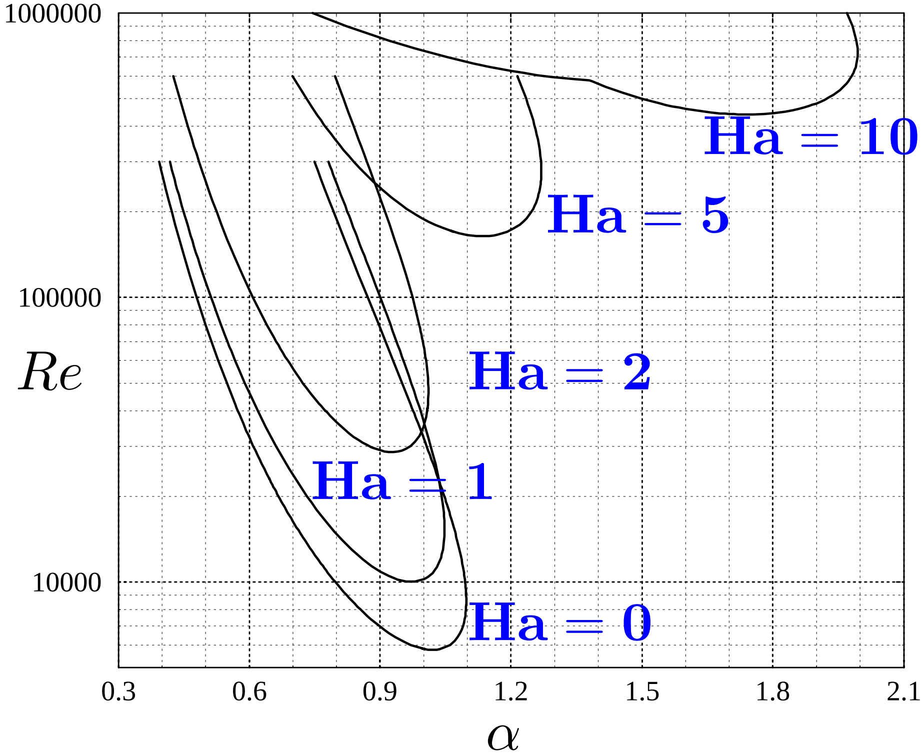
⋆ Coupling parameter $\mathbf{Q_m} = \frac{\mathbf{1}}{\mathbf{Re_H^2 Pr_m}}$ due to the Lorentz force

⋆ **Normal modes**: $q(x,y,z,t) = \hat{q}(y)\; e^{[i\alpha(x-ct)+i\beta z]}, \quad (\alpha,\beta) \in \mathbb{R}^2, \quad c \in \mathbb{C}$,

⋆ Reduced generalized **eigenvalue problem** : $[M]\;\hat{\Lambda} = \lambda\;[N]\hat{\Lambda}$

where $\hat{\Lambda} = [\hat{v}, \hat{b}_2, \hat{\xi}_2, \hat{j}_2], \qquad \lambda = -i\alpha c$ and $\vec{j} = \overrightarrow{\text{curl}}\;\vec{b}, \qquad \vec{\xi} = \overrightarrow{\text{curl}}\;\vec{u}$

Modal stability: neutral curves, mettre le Prm



2D waves

For Hartmann number > 5

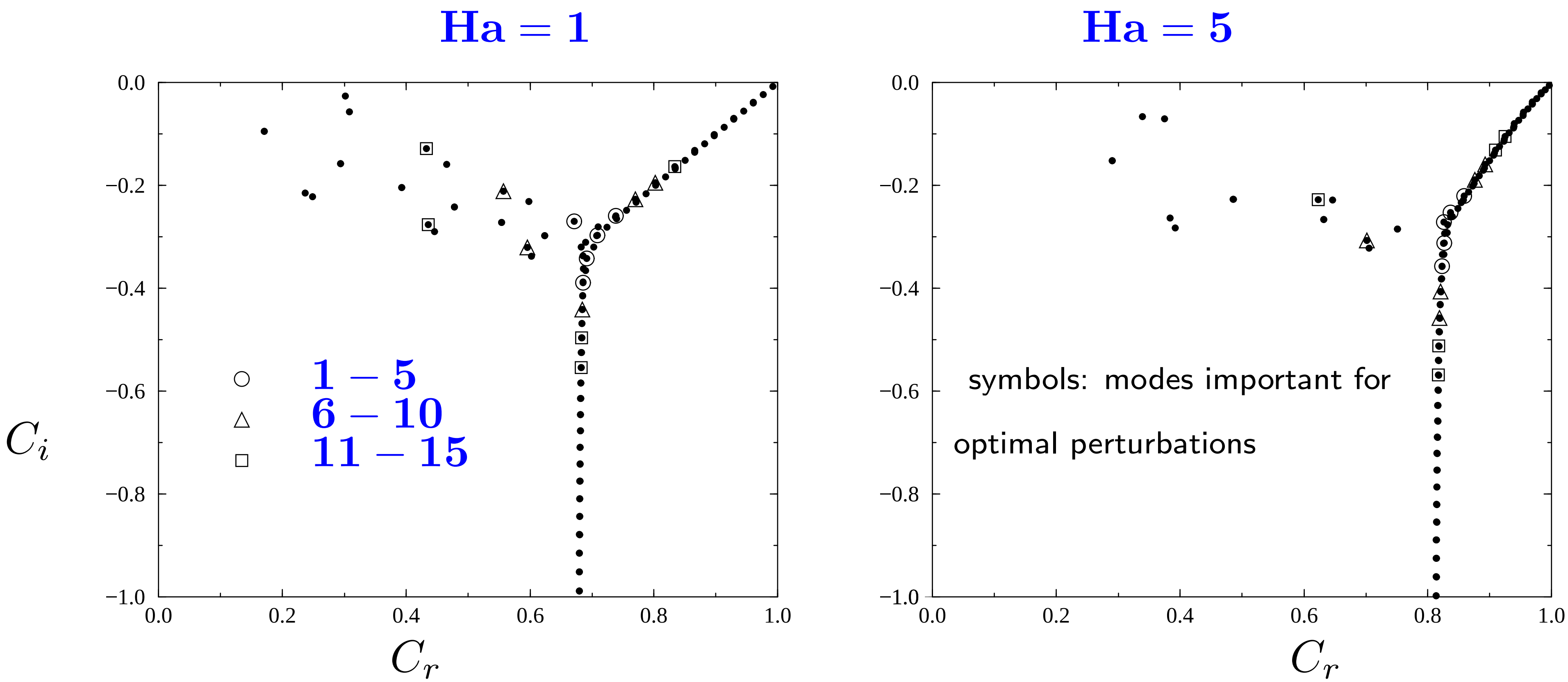
Critical Reynolds number > 100 000

$\Longrightarrow$ possible subcritical instability

- ⋆ Implementation in Matlab, Chebyshev polynomials
- ⋆ Critical Reynolds numbers : agreement with Takashima (1986) and all previous studies.
- ⋆ Small influence of wall electrical boundary conditions

Modal stability: eigenvalue spectrum

$$\alpha = 1.48,\ \beta = 0,\ Re = 5000$$



⋆ Less modes in the discontinuous branches, C_r of continuous branch grows with Ha.

⋆ $Ha > 10$: few discontinuous modes remain, modes concentrated at the branch intersection

Transient growth

⋆ References : Landahl, Trefethen, Butler & Farrell, Luchini, Henningson, Bottaro, Lingwood et al, Grard-Varet (MHD), Krasnov et al (MHD), ...

⋆ Linear growth in subcritical conditions due to the **non-normality of the linear stability operator**

⋆ $q(x, y, z, t) = \mathrm{e}^{i(\alpha x + \beta z)} \sum_{j=1}^{N} \kappa_j \; \hat{q}_j(y) \; \mathrm{e}^{\lambda_j t},$

$(\lambda_j, q_j(y))$: eigenmode j, κ_j : component j of the non-modal perturbation

⋆ **Energy gain** : $G(Re, Ha, Pr_m, \alpha, \beta, T) = \max_T \dfrac{E(T)}{E(0)},$ with $E(0) = 1$

Optimal perturbation

⋆ Butler and Farrell approach : $G = \zeta_{\max}$, the largest eigenvalue of

$\tilde{E}(T)\,\kappa - \zeta \tilde{E}(0)\,\kappa = 0, \qquad \kappa = (\kappa_j), \qquad \tilde{E}(.)$ matrix of the energy of the eigenmodes

⋆ The optimal perturbation maximizes the disturbance energy gain :

$$\tilde{G}(Re, Ha, Pr_m) = \sup_{\beta,\alpha,T} G(Re, Ha, Pr_m, T, \alpha, \beta)$$

⇒ **3D perturbations** $\alpha = 0$, $\beta \neq 0$ for Poiseuille flow :

at $t = 0$, $\quad u = 0$: vortex

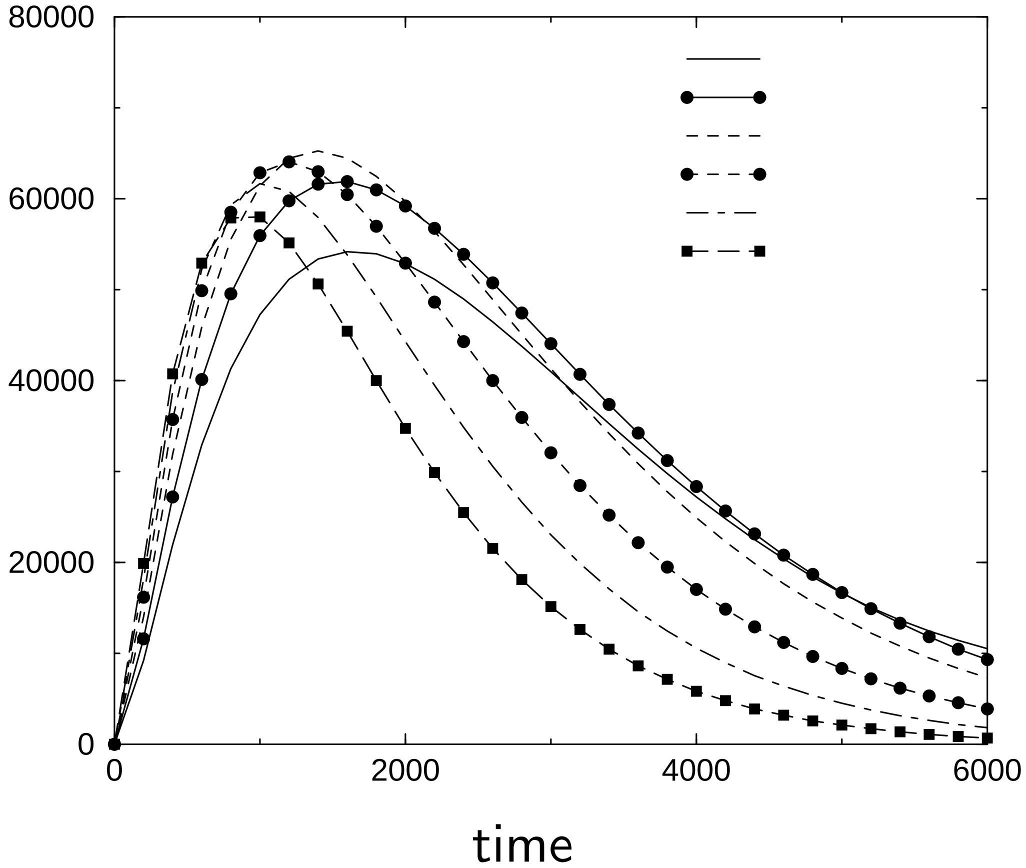
at $t = T_{opt}$, $\quad u >> v,\ u >> w$: streak

It is due to a 'lift-up' effect at the wall (Landhal).

⋆ For Poiseuillle flow, $\tilde{G}(Re) \propto Re^2$

Transient growth : Hartmann flow

$Re = 50000; Ha = 5$, varying $\beta : [2.5 - 5, \Delta\beta = 0.5]$, E_k, insulating walls



Large gain possible $(10^4 - 10^7)$ before damping

$\beta_{opt} = 3.07, \qquad \tilde{G}_{opt} \approx 65000$

Open questions for MHD case

- ⋆ What is the influence of the energy definition (magnetic and kinetic) ?
- ⋆ What is the right scaling for $\vec{b}$ or $\vec{j}$?
- ⋆ Is it possible to find a simple transition criterion from the linear stability analysis ?
- ⋆ What is the influence of boundary conditions : insulating or perfectly conducting walls ?
- ⋆ What is the behaviour of the magnetic perturbation $\vec{b}$ or the current density vector $\vec{j} = \overrightarrow{\mathrm{curl}}\ \vec{b}$?

Choice of the energy : optimal gains

$Re = 10000,\ \alpha = 0$

Ha	β	$\tilde{G}_{E_m}$	$\tilde{G}_{E_k}$
0	2.03	-	19589
1	2.1	172822	16967
3	2.6	76979	7019
5	3.7	21713	2590
10	7.4	2020.9	612

$$E(t) = E_k(t) + \ \epsilon_m \ E_m(t), \qquad \epsilon_m = 0 \text{ or } 1$$

$E_k(t)$: Kinetic energy of the velocity disturbance

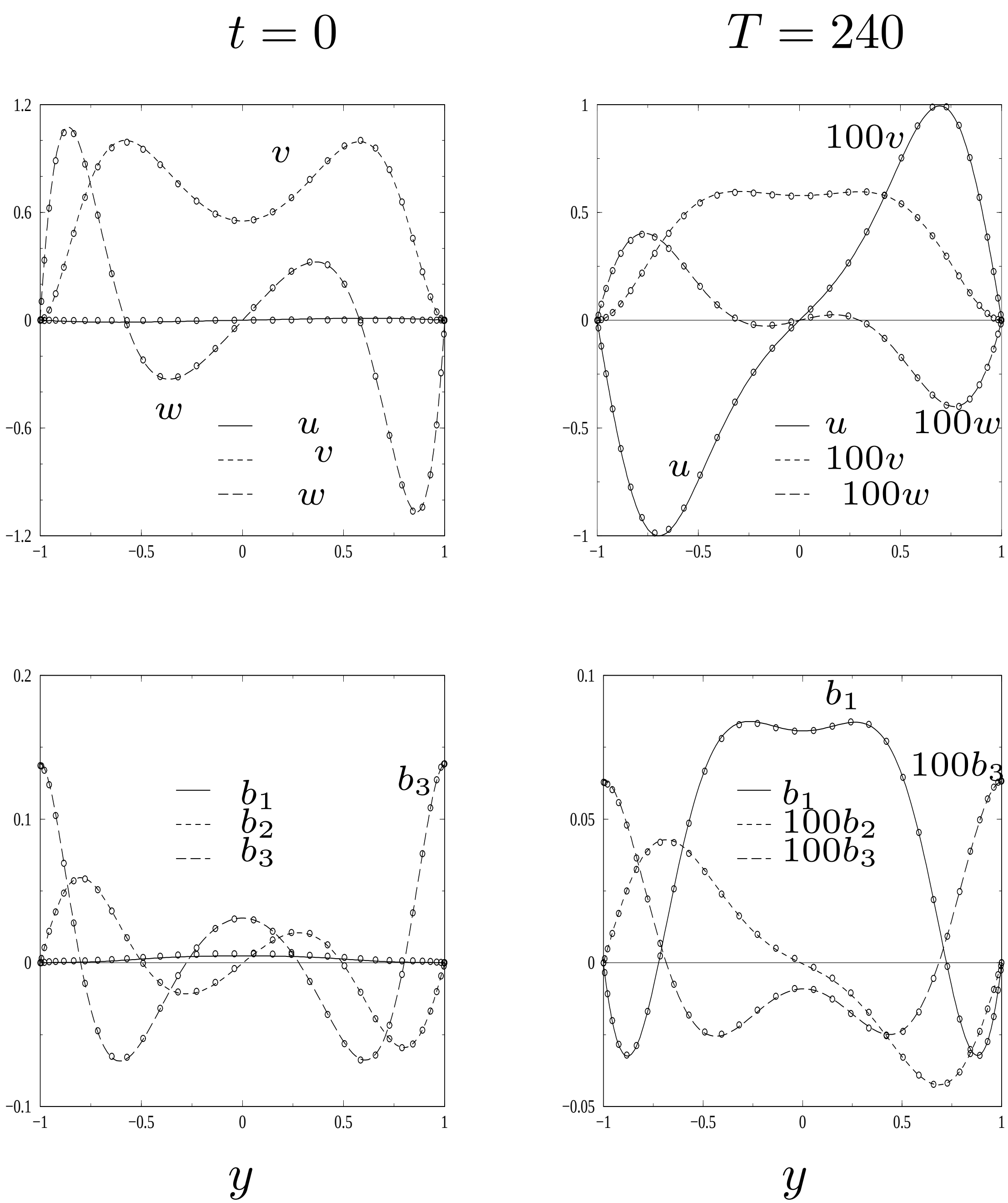
$E_m(t)$: Energy of the magnetic field disturbance

⋆ Optimal gains are larger with E_m and may cause numerical divergence for large Reynolds number ($Re > 100000$)

⋆ Gain :
$$\tilde{G}_{E_k+E_m} = \frac{(\max u_T)^2 \ ||\vec{\tilde{u}}||_T^2 + Pr_m^{-1} Re_H^{-2} (\max b_{1T})^2 \ ||\vec{\tilde{b}}||_T^2}{(\max v_0)^2 \ ||\vec{\tilde{u}}\ ||_0^2 + Pr_m^{-1} Re_H^{-2} (\max b_{20})^2 \ ||\vec{\tilde{b}}||_T^2},$$

⋆ $\tilde{G}_{E_k} \propto \dfrac{(\max u_T)^2}{(\max v_0)^2}, \qquad \tilde{G}_{E_m} \propto \dfrac{(\max b_{1T})^2}{(\max b_{20})^2} \propto \tilde{G}_{E_k} \times \dfrac{\hat{b}_1^2}{\hat{b}_{20}^2}, \qquad \tilde{G}_{E_k+E_m} \approx \tilde{G}_{E_k}.$

⋆ Kinetic energy is the best choice



Scaling of the optimal perturbation

Ha=5

velocity

$u \propto Re, \quad v \propto 1, \quad w \propto 1$

Profiles identical to Poiseuille flow case

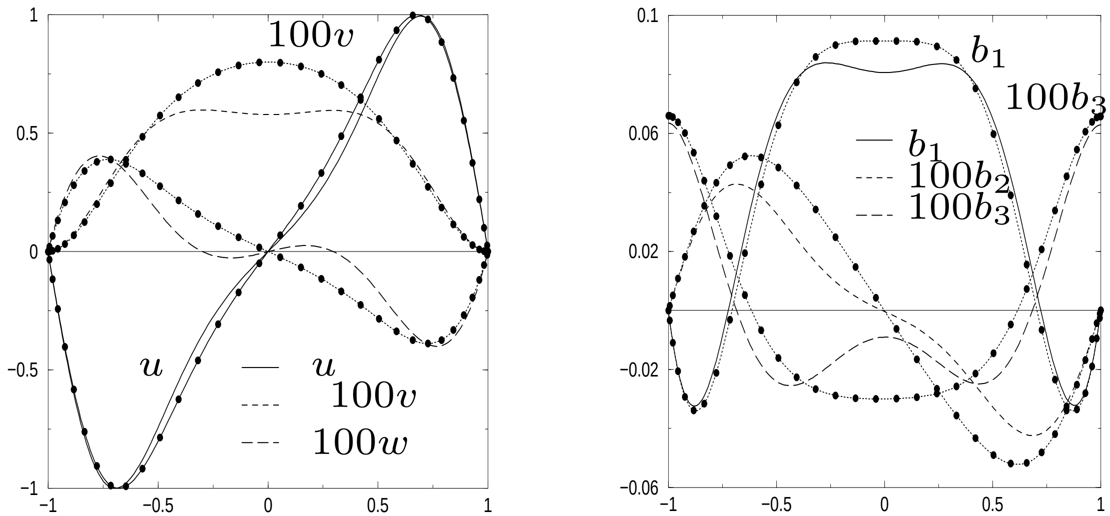
symbols: Re=50 000

lines: Re= 10 000

magnetic field

$b_1 \propto Re^2, \quad b_2 \propto Re, \quad b_3 \propto Re$

Optimal time

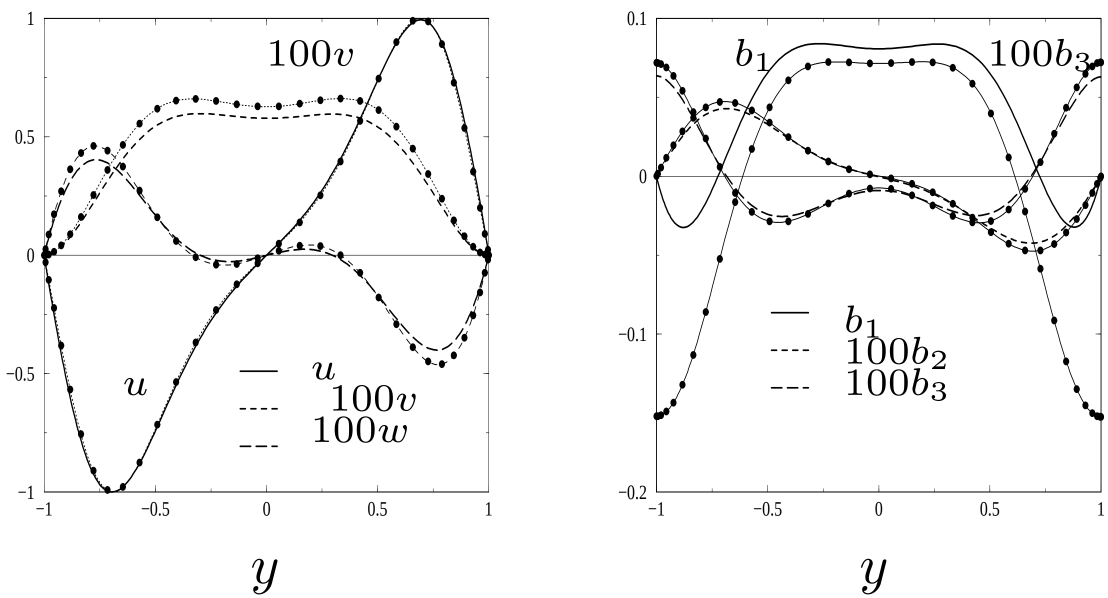


Choice of the energy

symbols : E_m

lines alone : E_k

Ha=5, Re=10000, $\alpha = 0$



boundary conditions

symbols : conducting walls

lines alone : insulating walls

$b_1 \propto j_2 \neq 0$ at the wall

little influence on the optimal gain

Gain function

⋆ $\tilde{G}(Pr_m, Re, Ha) = C_{\max}(Pr_m, Ha)\ \ Re^2 = C_{\max}(Pr_m, Ha)\ Ha^2\ \ R_H^2.$

$Re = 10000,\ \alpha = 0,\ E_k$

Ha	$\beta_{\max}$	$C_{\max} Ha^2$ $\times 10^4$	R_H	$\tilde{G}_t$ $R_{Ht} = 380$
0	2.03	0	-	-
1	2.1	1.6967	10000	24.5
3	2.6	6.3171	3333.3	91.22
5	3.7	6.475	2000	93.49
10	7.4	6.12	1000	88.37
20	15	6.12	500	88.37

Ha > 3, $C_{\max} Ha^2 \approx 6\ \times 10^{-4}$

$$\mathbf{\tilde{G}(R_H) \approx 6\ \times 10^{-4}\ R_H^2}$$

From experiments, transition at $R_{Ht} = 380$

Gain given by the linear theory,
at the transition R_{Ht} number
is approximately constant :

$$\mathbf{\tilde{G}_t \approx 90}$$

⋆ $\mathbf{\tilde{G}_t \approx 90}$ could it be a new transition criteria ?

⋆ It should be verified for other MHD flows, and new experiments.

Scaling

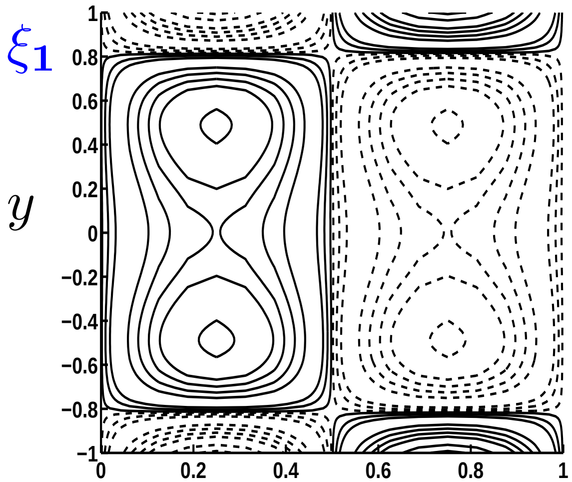
⋆ New scaling

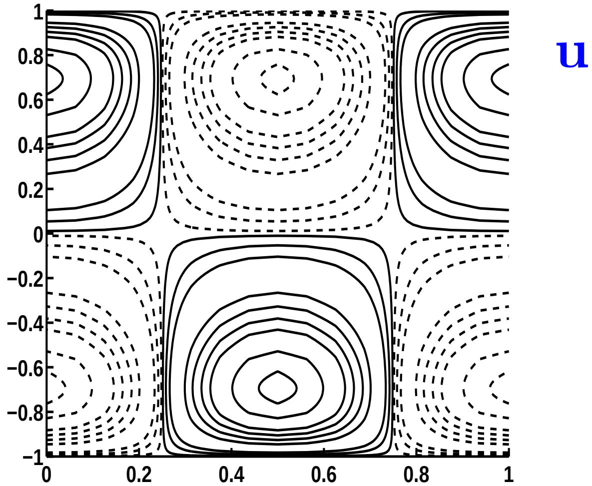
$$(x^*, y^*, z^*) = (R_H\ x, y, z),$$
$$(u^*, v^*, w^*) = (u, R_H\ v, R_H\ w),$$
$$(b_1^*, b_2^*, b_3^*) = (b_1/R_H, b_2, b_3).$$

⋆ Optimal gain

$$\tilde{G}(Pr_m, M, Re, T) = R_H^2 \quad \frac{||u^*||_T^2 + Pr_m^{-1}||b_1^*||_T^2}{||v^*||_0^2 + ||w^*||_0^2 + Pr_m^{-1}(||b_2^*||_0^2 + ||b_3^*||_0^2)}$$

Insulating wall, optimal velocity field



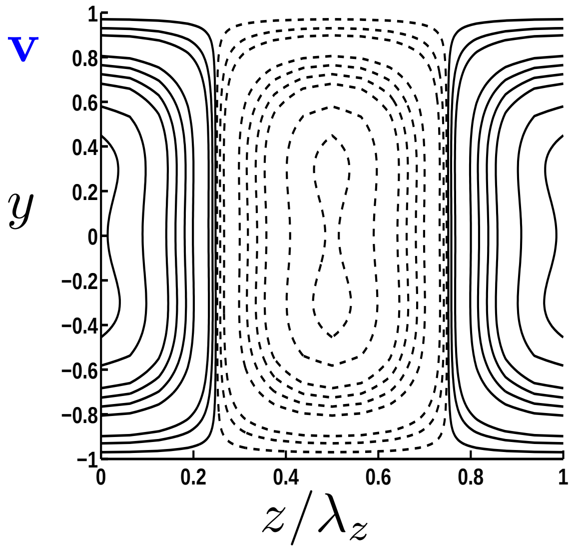


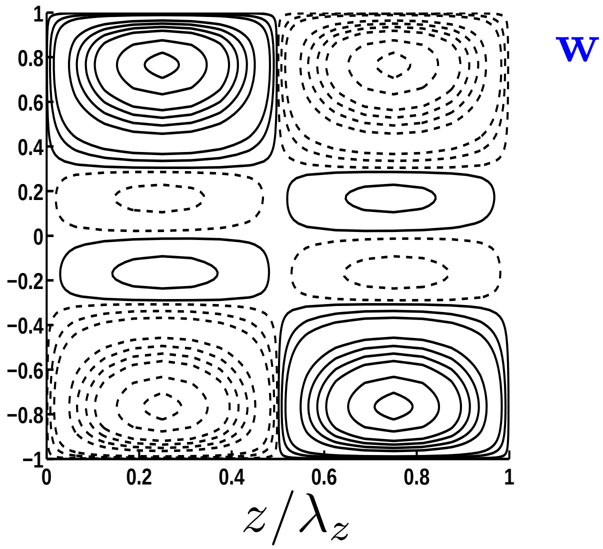
ξ_1 : streamwise vorticity

$\mathbf{u}$: streak

$\mathbf{v}, \mathbf{w}$: lift-up effect: at the wall up-wash and downwash velocity

Optimal time, $Re = 10000$, $Ha = 5$, $\alpha = 0$



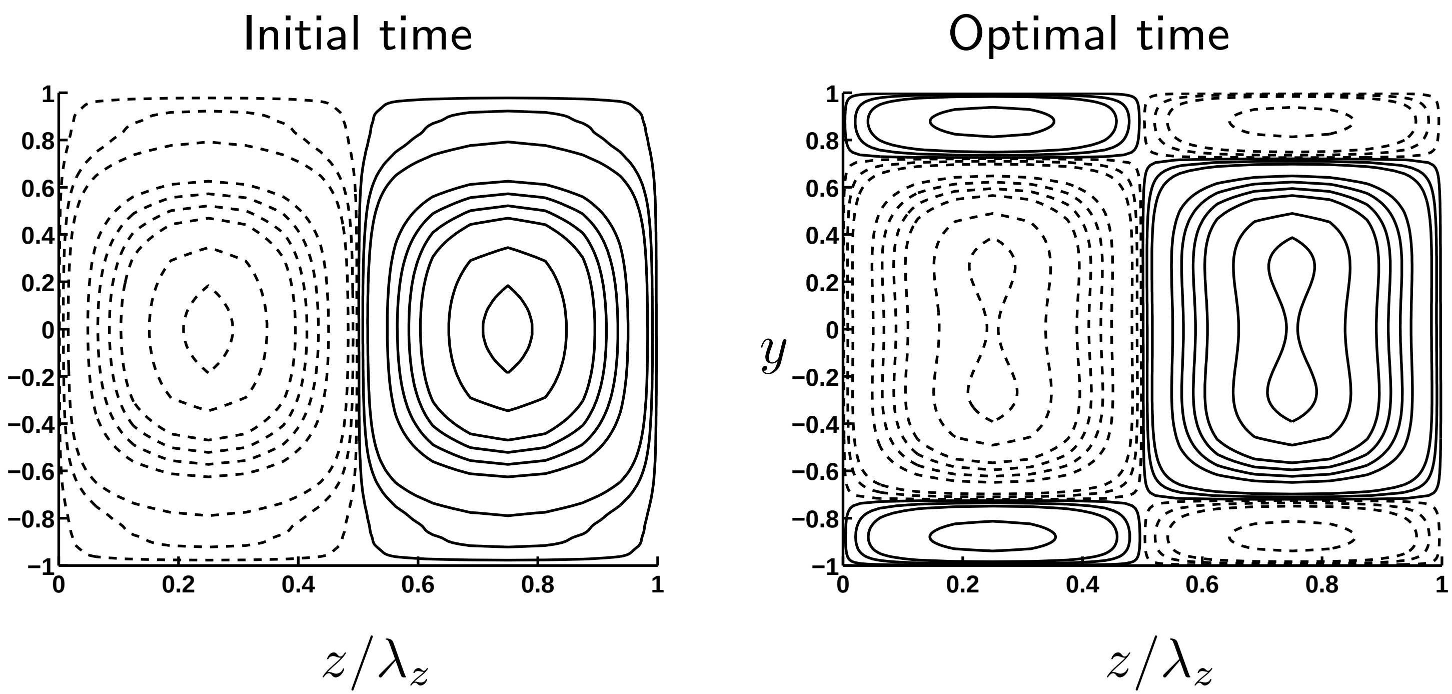


dashes lines : negative values

solid lines : positive values

$[0.01, 0.05, 0.1, 0.3, 0.4, 0.5, 0.6, 0.8, 0.95]$

Insulating wall, the normal current density j_2



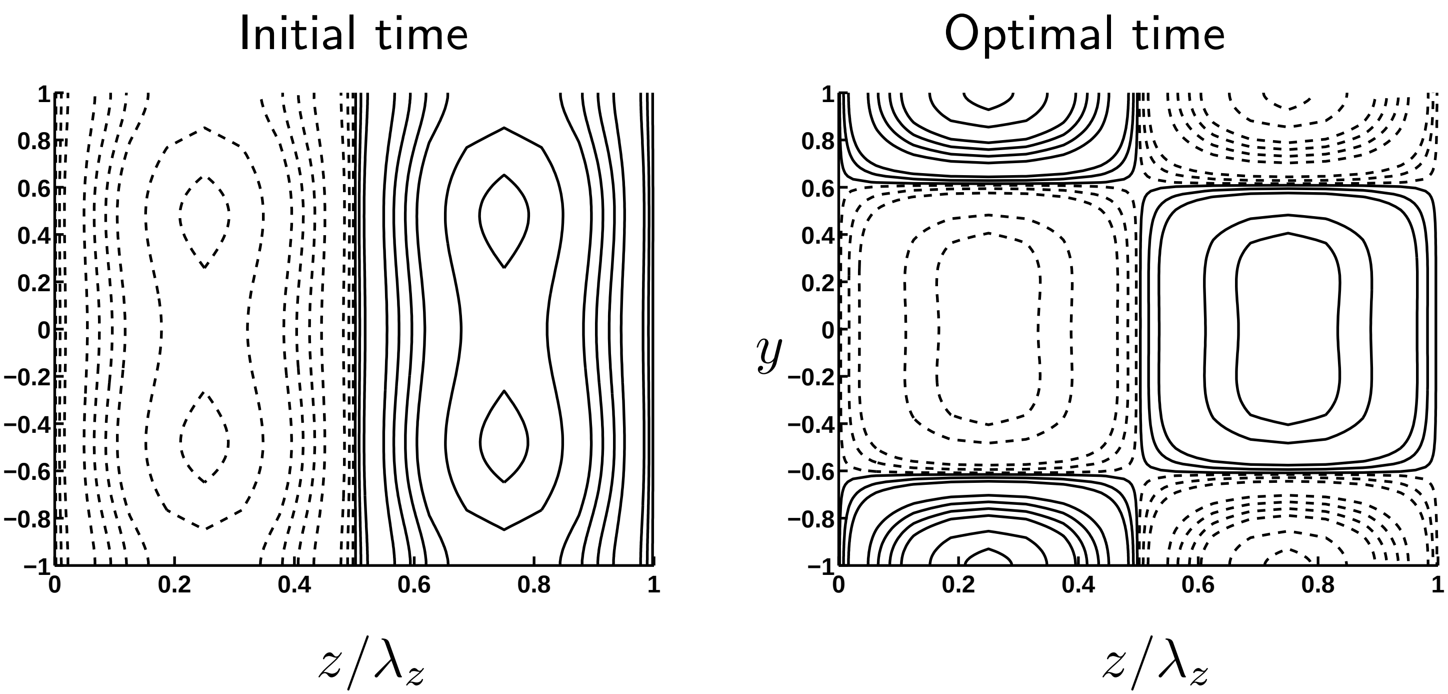
dashed lines : negative values

solid lines : positive values

$Re = 10000$, $Ha = 5$, $\alpha = 0$

- ⋆ The normal current j_2 is tangential to the wall, $j_2 \propto b_1$.
- ⋆ Small normal current in the Hartmann layer, close to the wall, with opposite sign.

Conducting wall, the normal density of current j_2



$Re = 10000,\ Ha = 5,\ \alpha = 0$

⋆ The current goes across the wall, but levels do not change much in the middle area of the channel

Concluding remarks

- ⋆ Large transient growth possible in the Hartmann flow, because Re_{crit} given by LST are large when $Ha > 5$
- ⋆ Energy gain $\tilde{G} \propto \ R_H^2$
- ⋆ For $Re_H = 380$, at transition, $\tilde{G}_t \approx 90$
- ⋆ The choice of the energy definition, and the small influence of wall conductivity properties have been investigated.
- ⋆ **Future** : optimal control of non modal instability by the magnetic field - wall action via a current for example.