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Inferring flows from magnetic field constraints

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Outline of the talk

- Inverting inversions
- **Solving linear inverse problems:**
  Contactless inductive flow tomography
- "Solving" non-linear inverse problems by "intuition":
  Flow optimization for the Riga dynamo experiment
- Solving non-linear inverse problems by evolutionary strategies:
  Oscillatory α^2-dynamos
- Closing the loop

Inverting inversions

- Main topic of the workshop is flow control by tailored (electro-)magnetic fields
- Navier-Stokes equation with Lorentz force:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \quad = \quad \frac{1}{\mu_0 \rho}(\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{\nabla p}{\rho} + \nu \Delta \mathbf{v} + \mathbf{f}_{other}$$

- Forward problem: $\mathbf{B} \implies \mathbf{v}$
- Inverse problem: $\mathbf{v} \implies \mathbf{B}$

Inverting inversions

- Magnetohydrodynamics is not always a one-way. For not too small magnetic Reynolds numbers $R_m := \mu\sigma v l$, we get a measurable induction of magnetic fields by the flow. Even self-excitation is possible (dynamo).

- Induction equation for magnetic field $\mathbf{B}$:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \Delta \mathbf{B}$$

- Forward problem: $\mathbf{v} \Longrightarrow \mathbf{B}$

- Inverse problem: $\mathbf{B} \Longrightarrow \mathbf{v}$

Inverting inversions

- First application: velocity reconstruction
  - Velocity measurement in hot and/or chemically aggressive melts is a notorious problem in basic research and in many industrial applications (metallurgy, crystal growth).
  - A contactless measurement technique is highly desirable.
  - Can we reconstruct the velocity field from induced magnetic fields?
- Second application: flow design for hydromagnetic dynamos
  - How to optimize a flow to get a dynamo at all?
  - How to tailor a flow to get interesting spectral features?

Contactless Inductive Flow Tomography (CIFT)

- Starting point (steady case): Expression for the electric current $\mathbf{j}$

$$\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B} - \nabla\varphi)$$

$\mathbf{v}$ - velocity; $\mathbf{B}$ - magnetic field
σ - electrical conductivity; φ - electric potential

- In general: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, i.e., it comprises the field $\mathbf{B}_0$ applied from outside, and the induced field $\mathbf{b}$.

CIFT: The integral equation approach to MHD

- Biot-Savart's law: $\mathbf{b}_1$ from primary and $\mathbf{b}_2$ from secondary currents:

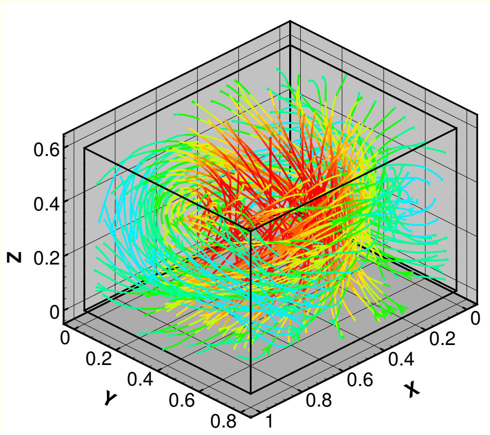
$$\mathbf{b}(\mathbf{r}) = \frac{\mu_0\sigma}{4\pi}\int_V \frac{(\mathbf{v}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}')) \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} dV' - \frac{\mu_0\sigma}{4\pi}\oint_S \frac{\varphi(\mathbf{s}')\mathbf{n}(\mathbf{s}') \times (\mathbf{r}-\mathbf{s}')}{|\mathbf{r}-\mathbf{s}'|^3} dS'$$

- Applying Green's theorem to the solution of the Poisson equation $\Delta\varphi = \nabla \cdot (\mathbf{v} \times \mathbf{B})$ gives an integral equation for φ at the boundary:

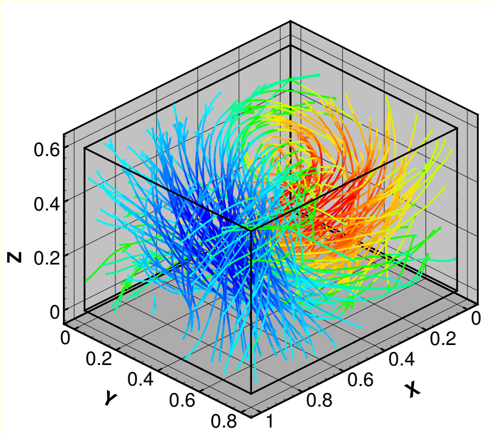
$$\varphi(\mathbf{s}) = \frac{1}{2\pi}\int_V \frac{(\mathbf{v}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}')) \cdot (\mathbf{s}-\mathbf{r}')}{|\mathbf{s}-\mathbf{r}'|^3} dV' - \frac{1}{2\pi}\oint_S \frac{\varphi(\mathbf{s}')\mathbf{n}(\mathbf{s}') \cdot (\mathbf{s}-\mathbf{s}')}{|\mathbf{s}-\mathbf{s}'|^3} dS'$$

The integral equation approach to MHD

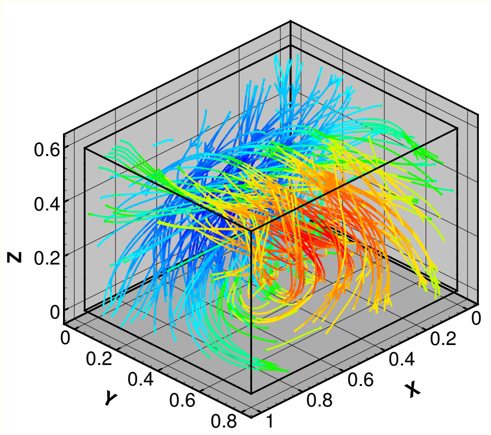
- Method can be used to treat hydromagnetic dynamos in finite domains.
- Example: dynamo with helical turbulence parameter α in a box



First eigenfield



Second eigenfield



Third eigenfield

CIFT: Linearization for small R_m

- Ratio of induced to applied field is governed by magnetic Reynolds number R_m

$$b/B_0 \sim R_m = \mu_0 \sigma v l$$

- In typical liquid metal applications: $R_m \sim 0.01 ... 0.1$
- Then, in $\mathbf{v} \times \mathbf{B}$ under the integrals, $\mathbf{B}$ can be replaced by $\mathbf{B}_0$
- We get a linear inverse problem similar to magnetoencephalography
- However: We are free to apply different external magnetic fields $\mathbf{B}_0$ to produce different current distributions from the same velocity field

CIFT: The forward problem

Basic question: How to take secondary currents into account without measuring the electric potential at the boundary?

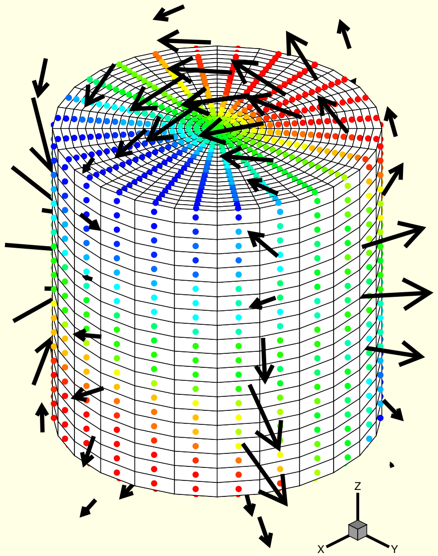
- Discretizations: NB measuring point, NP electric potential points at the boundary, NV degrees of freedom for the velocity. Then:

$$b_i^{(B_1,B_2)} = R_{in}^{(B_1,B_2)} v_n + S_{ik} \varphi_k^{(B_1,B_2)}, \quad \varphi_k^{(B_1,B_2)} = T_{kn}^{(B_1,B_2)} v_n + U_{kk'} \varphi_{k'}^{(B_1,B_2)}$$

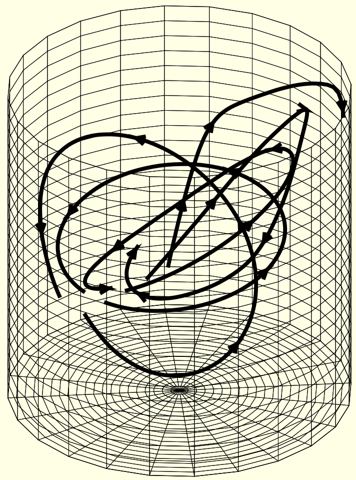
can be rewritten as:

$$b_i^{(B_1,B_2)} = R_{in}^{(B_1,B_2)} v_n + S_{ik'} ((I-U)^{-1,defl})_{k'k} T_{kn}^{(B_1,B_2)} v_n$$

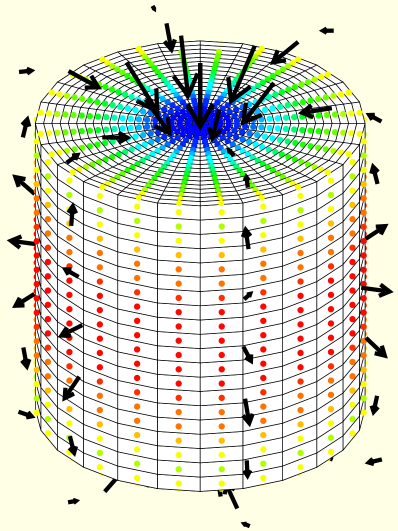
CIFT: The forward problem



Induced field for
$\mathbf{B_0} = B_0 \mathbf{e}_x$

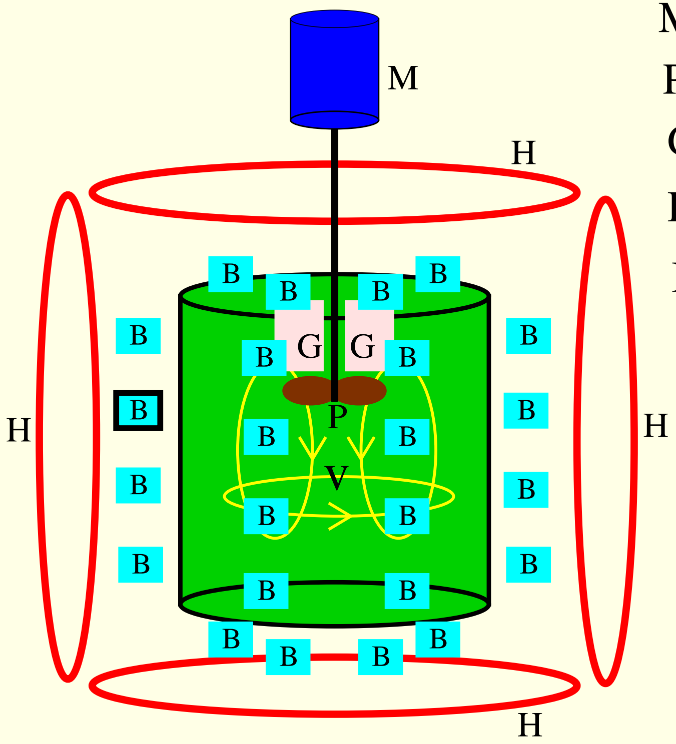


Flow with poloidal
and toroidal part



Induced field for
$\mathbf{B_0} = B_0 \mathbf{e}_z$

CIFT: Experimental set-up



M – Motor

P – Propeller

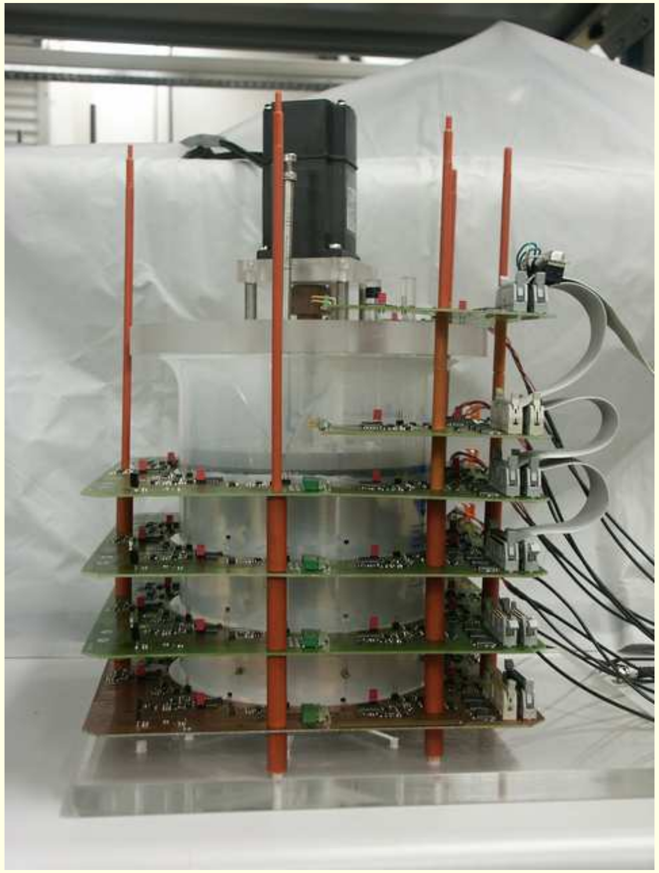
G – Guiding blades

H – Helmholtz coils

B – Hall sensors

CIFT: Experimental set-up

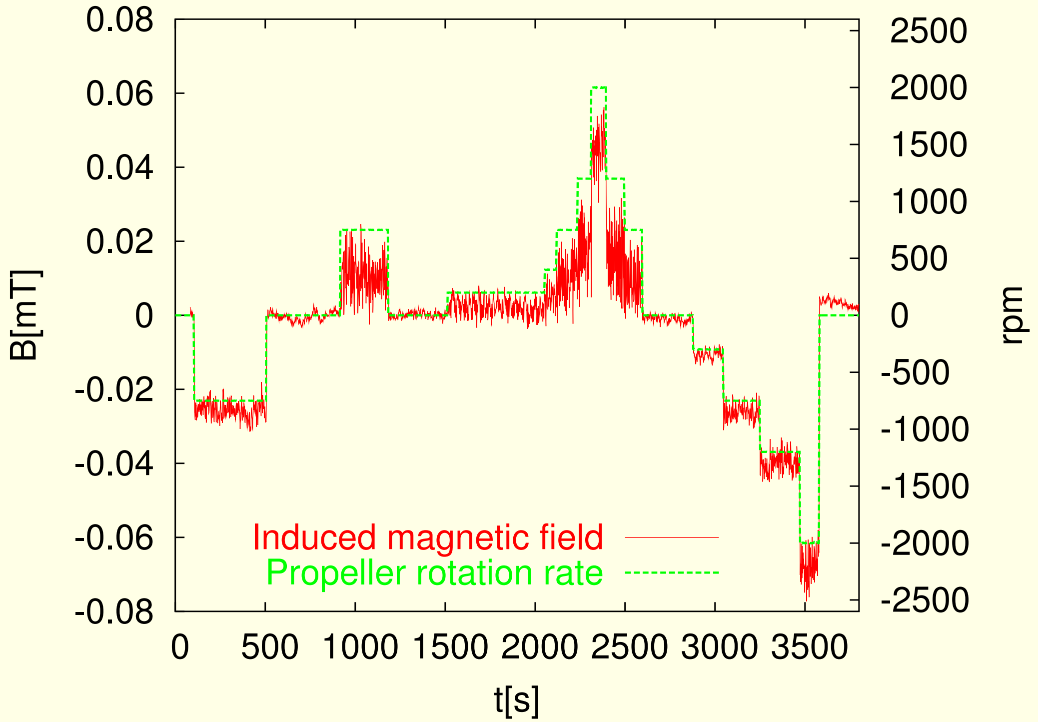
Part of the experiment with a propeller driven flow of InGaSn. The circuit boards contain the 48 Hall sensors and the A/D transformers



The main technical problem is the reliable determination of small induced fields on the back-ground of large applied fields.

CIFT: Experimental set-up

Radial component of induced magnetic field (red), and rotation rate of the propeller (green). The data are averaged over a period of 2 seconds.



CIFT: Inverse problem

Non-uniqueness of the problem is resolved by Tikhonov regularization

- Total functional to be minimized:

$$F[v] = F_{B_{0x}}[v] + F_{B_{0z}}[v] + F_{div}[v] + F_{pen}[v]$$

$F_{B_{0x}}[v]$	Mean square deviation of measured from modelled fields for $B_0\mathbf{e}_x$
$F_{B_{0z}}[v]$	Mean square deviation of measured from modelled fields for $B_0\mathbf{e}_z$
$F_{div}[v]$	Ensures continuity of the flow, $\nabla \cdot \mathbf{v} = \mathbf{0}$
$F_{pen}[v]$	Penalty functional, to keep kinetic energy of the flow small

CIFT: Inverse problem

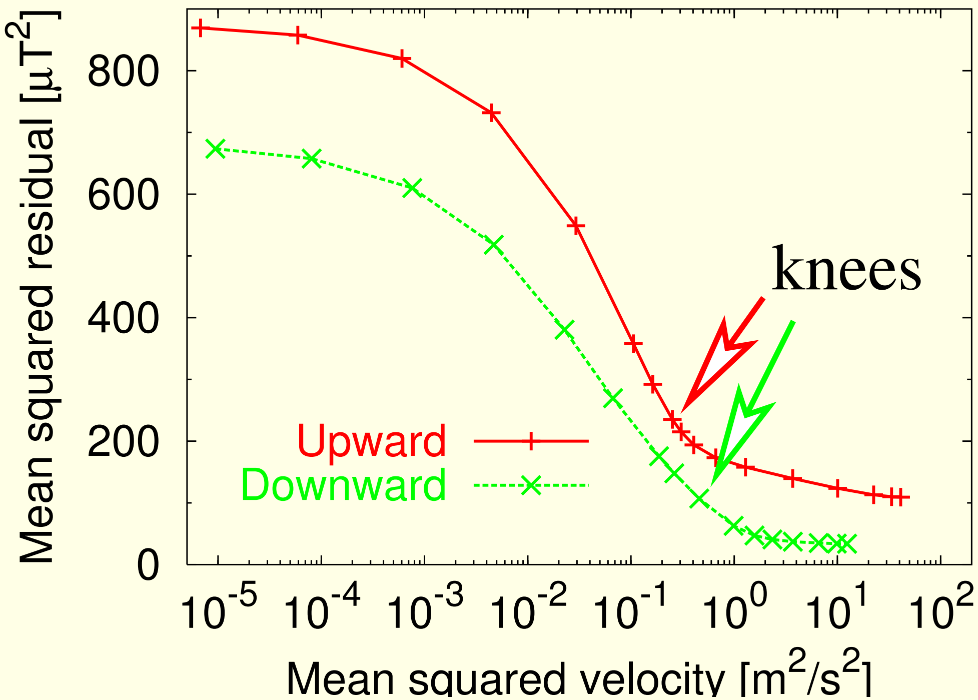
$$
\begin{aligned}
F_{B_{0,1}}[v] &= \frac{1}{\sigma_{B_1}^2} \sum_{i=1}^{NB} \left\{ {b_1}_i^{meas} - {b_1}_i^{mod}[v] \right\}^2 \\
F_{B_{0,2}}[v] &= \frac{1}{\sigma_{B_2}^2} \sum_{i=1}^{NB} \left\{ {b_2}_i^{meas} - {b_2}_i^{mod}[v] \right\}^2 \\
F_{div}[\mathbf{v}] &= \frac{1}{\sigma_{div}^2} \sum_{k=1}^{NV} \left[\Delta V (\nabla \cdot \mathbf{v})_k^2 \right] \\
F_{pen}[\mathbf{v}] &= \frac{1}{\sigma_{pen}^2} \sum_{k=1}^{NV} \left[\Delta V \mathbf{v}_k^2 \right]
\end{aligned}
$$

CIFT: Regularization issues

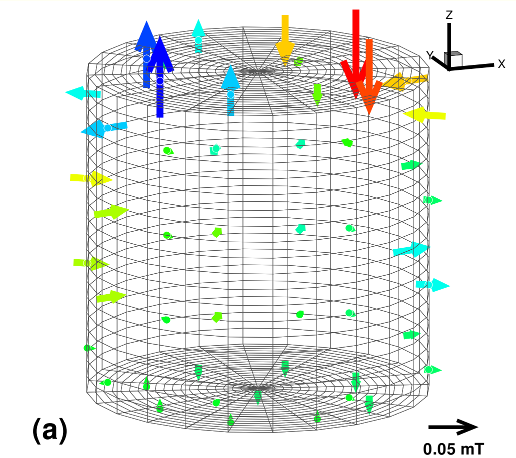
- Standard codes to solve the normal equations (Cholesky decomposition)
- Scaling σ^2_{pen}, the dependence of $F_B[v] = F_{B_1}[v] + F_{B_2}[v]$ on the mean kinetic energy can be plotted (Tikhonov's L-curve).
- A reasonable solution for $\mathbf{v}$ is found at the point with the highest curvature of the L-curve (the "knee").
- Automatic search for optimal regularization parameter still to be implemented.

CIFT: Experimental results

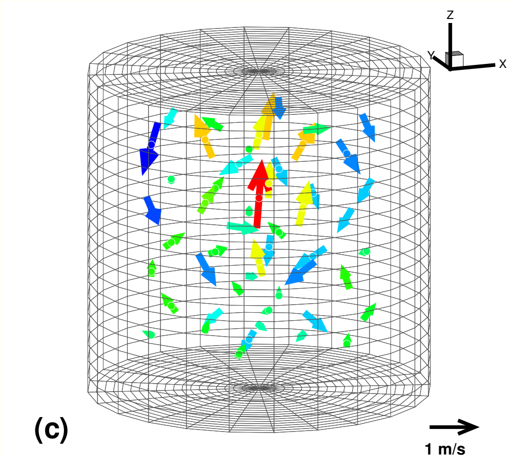
Tikhonov's L-curve, showing the rms in dependence on the mean kinetic energy. At the "knees" one gets a reasonable compromise between data fitting and minimum kinetic energy.



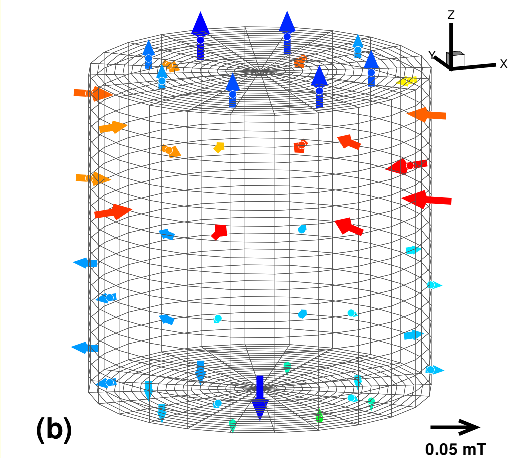
CIFT: Experimental results



Measured field for $\mathbf{B_0} = B_0\mathbf{e}_x$

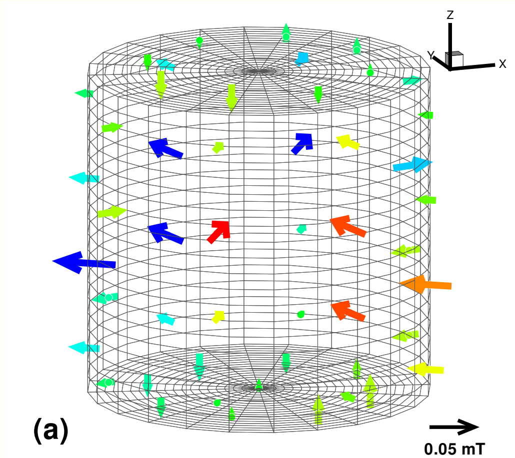


Reconstructed flow: upward pumping

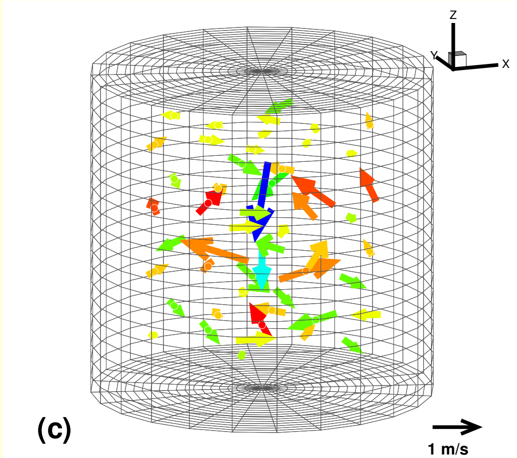


Measured field for $\mathbf{B_0} = B_0\mathbf{e}_z$

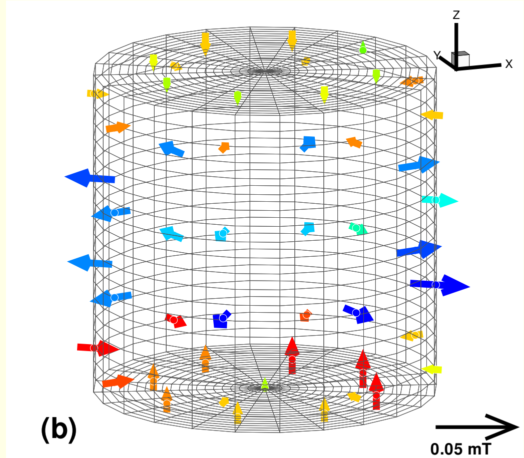
CIFT: Experimental results



Measured field for
$\mathbf{B_0} = B_0 \mathbf{e}_x$



Reconstructed flow:
downward pumping



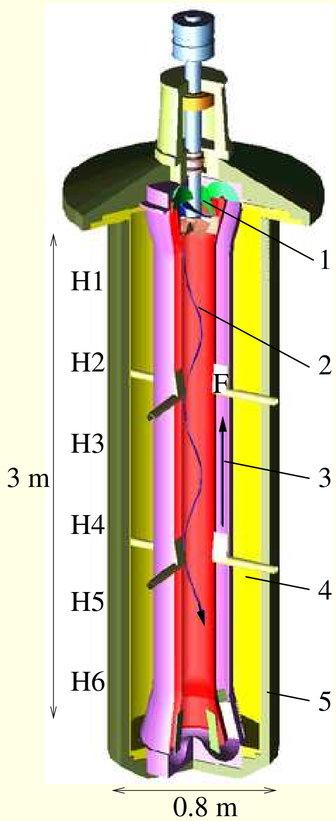
Measured field for
$\mathbf{B_0} = B_0 \mathbf{e}_z$

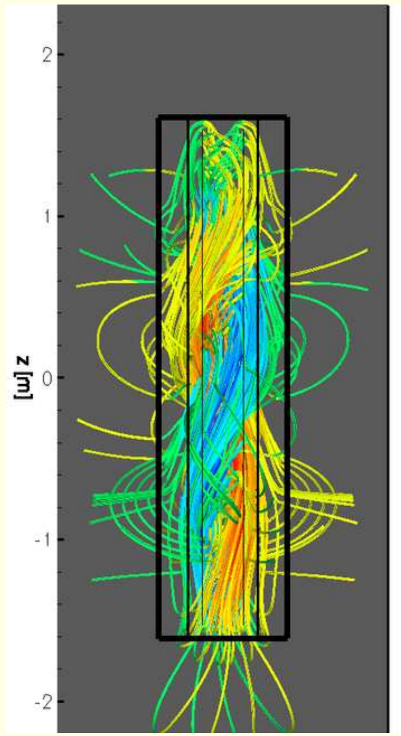
Flow optimization for the Riga dynamo experiment

- Study of the hydromagnetic dynamo effect in a large sodium facility (2 m^3 of liquid sodium, velocity up to 20 m/s).

- Goal: Optimizing the flow structure in order to achieve a critical R_m as small as possible

- Highly non-linear inverse problem. Even the forward problem is demanding, because of non-local boundary conditions for $\mathbf{B}$.





Flow optimization for the Riga dynamo experiment

- Rule of thumb: "Helicity is good for dynamo action"
- Let's maximize total helicity H

$$
\begin{aligned}
H &= 2\pi L \int_0^R \mathbf{v}(r) \cdot \nabla \times \mathbf{v}(r) r dr \\
&= 2\pi L \int_0^R \left[v_z(r) \frac{\partial v_\phi(r)}{\partial r} + \frac{v_z(r) v_\phi(r)}{r} - v_\phi(r) \frac{\partial v_z(r)}{\partial r} \right] r dr
\end{aligned}
$$

for given R_m

$$
R_m = \mu\sigma \sqrt{2 \int_0^R \left[v_z^2(r) + 2 v_\phi^2(r) \right] r dr}
$$

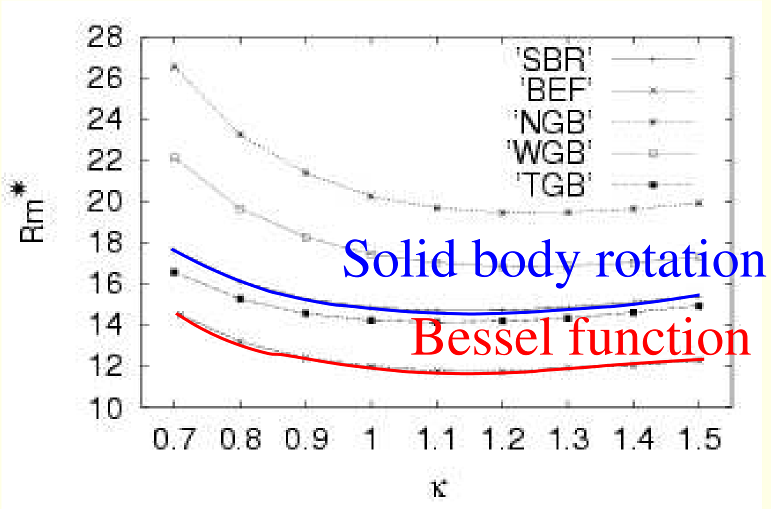
Flow optimization for the Riga dynamo experiment

- Solution of calculus of variation provides Bessel function profiles

$$v_\phi(r) = 1/\sqrt{2}\, J_1(2.4\, r)$$

$$v_z(r) = \kappa\, J_0(2.4\, r)$$

κ … pitch-parameter



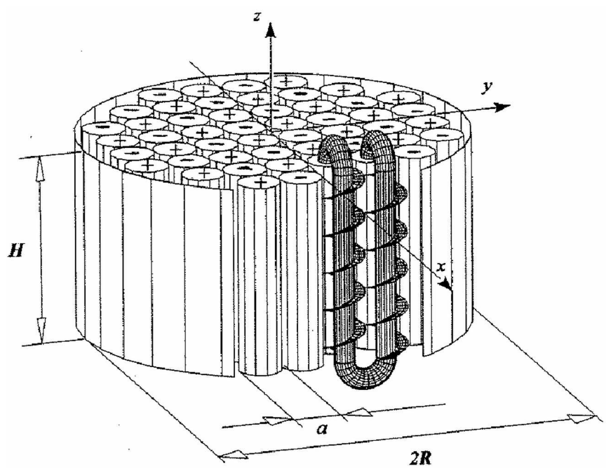
- This solution gives a decrease in R_m^{crit} of 20 % (compared to solid body rotation), that means a decrease of needed motor power by 50 % !

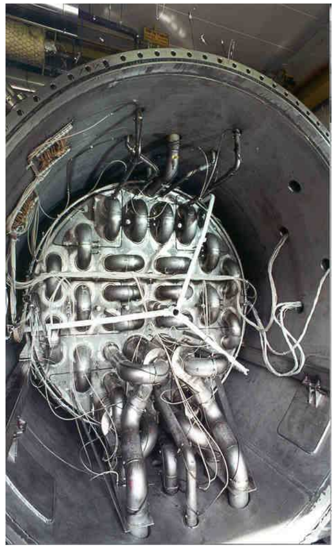
Oscillatory α^2 dynamos: Background and Motivation

- Mean-field dynamo theory: Electromotive force $\mathbf{v} \times \mathbf{B}$ due to large scale velocity field is replaced by mean electromotive force due to helical turbulence parameter (tensor) α: $\alpha\mathbf{B}$
- Oscillatory dynamos are interesting for
  - understanding oscillations of stellar magnetic fields
  - a possible explanation of Earth's magnetic field reversals
  - searching for a modified design of a Karlsruhe type dynamo which could exhibit oscillations or even reversals

Oscillatory α^2 dynamos: Background and Motivation

Karlsruhe Dynamo experiment: Realization of an (anisotropic) α^2 dynamo by means of 52 helical spin-generators





Oscillatory α^2 dynamos: Forward spectral problem

- Simplification for isotropic $\alpha(r)$: Reduction to coupled eigenvalue equation for defining scalars of poloidal (s) and toroidal (t) field components:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) + \frac{1}{\mu_0 \sigma} \Delta \mathbf{B}$$

$$\Downarrow$$

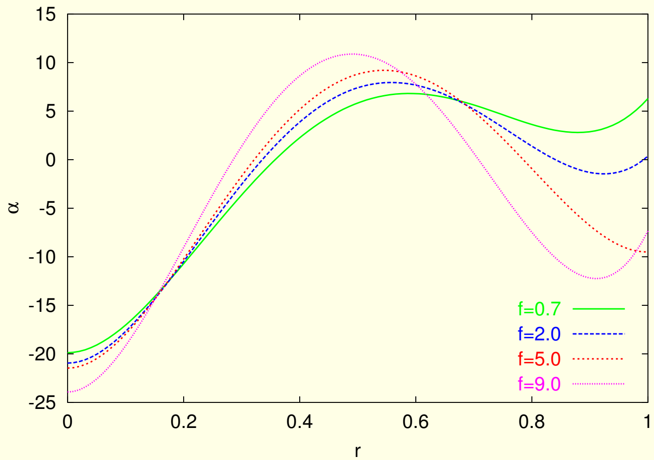
$$\begin{aligned}
\lambda_{l,n} s_l &= \frac{1}{r}\frac{d^2}{dr^2}(r s_l) - \frac{l(l+1)}{r^2} s_l + \alpha(r) t_l \;, \\
\lambda_{l,n} t_l &= \frac{1}{r}\frac{d}{dr}\left(\frac{d}{dr}(r t_l) - \alpha(r)\frac{d}{dr}(r s_l)\right) - \frac{l(l+1)}{r^2}(t_l - \alpha(r) s_l)
\end{aligned}$$

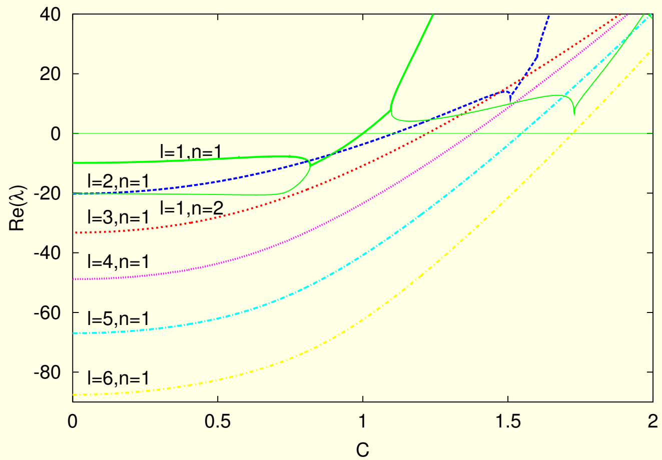
Oscillatory α^2 dynamos: Inverse spectral problem

- Method: Evolutionary strategy. A "population" of trial functions $\alpha(r)$ ("individuals") evolves according to biological principles.
- Demand: The mode with $(l = 1|n = 1, 2)$ should be marginal and oscillatory. All modes with higher l and n should have negative growth rates.
- Cost functional $F[\alpha]$ of a function $\alpha(r)$:

$$
\begin{aligned}
F[\alpha] \quad = \quad & (\Re(\lambda_{1,1}[\alpha]))^2 + (1 + \tanh(\Im(\lambda_{1,1}[\alpha]) - f))^{-1} \\
& + (1 + \tanh(-\Re(\lambda_{2,1}[\alpha])))^{-1} + (1 + \tanh(-\Re(\lambda_{3,1}[\alpha])))^{-1} \,.
\end{aligned}
$$

Oscillatory α^2 dynamos: A narrow "corridor" of solutions





Various profiles $\alpha(r)$ resulting for different demands on the oscillation frequency f.

Growth rates of the eigenfunctions with $(l = 1|n = 1, 2)$ and $(l = 2...6|n = 1)$. The marginal mode is indeed oscillatory.

Closing the loop

- Focus of this Workshop:
  $\mathbf{B}$ acts on $\mathbf{v}$
- Focus of this Talk:
  $\mathbf{v}$ induces $\mathbf{b}$
- Perspective: Control loop
  Acting with $\mathbf{B}$ on $\mathbf{v}$, Measuring the induced $\mathbf{b}$, Adjusting $\mathbf{B}$ etc, etc...